Smoluchowski Navier-Stokes Systems

Peter Constantin
Mathematics, U. of Chicago

CSCAMM, April 18, 2007

Outline:

1. Navier-Stokes

2. Onsager and Smoluchowski

3. Coupled System

Fluid: Navier Stokes Equation

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + \nabla \cdot \sigma$$
$$\nabla \cdot u = 0$$

The tensor $\sigma_{ij}(x,t)$: added stress.

Sufficient for regularity, if σ smooth

$$\int_0^T \|u\|^2_{L^\infty(dx)} dt < \infty$$

Necessary for applications

$$\int_0^T \|\nabla u\|_{L^\infty(dx)} dt < \infty$$

2D, Bounded stress

Theorem 1 *Let* $\sigma \in L^\infty(dtdx)$. *Let* $u_0 \in L^2(dx)$. *There exists a unique weak solution of the forced 2D NS eqns, with*

$$u \in L^\infty(dt)(L^2(dx)) \cap L^2(dt)(W^{1,2}(dx))$$

Moreover,

$$\int_0^T \|\nabla u\|_{L^q(dx)}^{\frac{q}{q-1}} dt < \infty, \quad \forall\, q \geq 2$$

$$\int_0^T \|u\|_{L^\infty(dx)}^{p} dt < \infty, \quad \forall\, p < 2.$$

Open questions

$$\int_0^T \|u\|^2_{L^\infty(dx)} dt < \infty \Rightarrow \int_0^T \|\nabla u\|_{L^\infty(dx)} < \infty \quad ?$$

$$\int_0^T \|u\|^2_{L^\infty(dx)} dt < \infty \quad ?$$

$$\int_0^T \|\nabla u\|_{L^\infty(dx)} dt < \infty \quad ?$$

Partial regularity?

Navier-Stokes with nearly singular forces

$$\partial_t u + u \cdot \nabla_x u - \nu \Delta_x u + \nabla_x p = \mathrm{div}_x \sigma, \quad \nabla_x \cdot u = 0$$

Theorem 2 *Let u be a solution of the $2D$ Navier-Stokes system with divergence-free initial data $u_0 \in W^{1,2}(\mathbb{R}^2) \cap W^{1,r}(\mathbb{R}^2)$. Let $T > 0$ and let the forces $\nabla \cdot \sigma$ obey*

$$\sigma \in L^1(0,T; L^\infty(\mathbb{R}^2)) \cap L^2(0,T; L^2(\mathbb{R}^2))$$
$$\nabla \cdot \sigma \in L^1(0,T; L^r(\mathbb{R}^2)) \cap L^2(0,T; L^2(\mathbb{R}^2))$$

with $r > 2$. Then

$$\int_0^T \|\nabla u(t)\|_{L^\infty} dt \le K \log_*(B)$$

and also

$$\frac{1}{M}\sum_{q=1}^{M}\int_0^T \|\Delta_q \nabla u(t)\|_{L^\infty} dt \leq K$$

with K *depending on* T*, norms of* σ *and the initial velocity, but not on gradients of* σ *nor* M*, and* B *depending on norms of the spatial gradients of* σ*.*

$$u \sim \sum_q \Delta_q(u)$$

Littlewood-Paley decomposition.

$$\mathcal{F}(\Delta_q(u))(k) \neq 0 \Leftrightarrow k \sim 2^q$$

Particles in Equilibrium

• $f(p)dp$: probability density of director p. • Mean field interaction potential

$$(\mathcal{K}f)(p) = \int_M K(p,q)f(q)dq$$

• Free energy:

$$\mathcal{E} = \int_M (f \log f - \frac{1}{2} f\mathcal{K}f)dp$$

$$\frac{\delta \mathcal{E}}{\delta f} = 0$$

Onsager equation

$$\log f(m) = \int_M K(m,p) f(p) dp - \log Z$$

Z a normalizing constant.

$$f = Z^{-1} e^{\mathcal{K} f}$$

- Nonlinear, nonlocal.

General Onsager Equation

•Conjecture: The prolate nematic state is the generic stable limit.

Partition function

$$Z(f,b) = \int_M e^{b\mathcal{K}f} dm$$

with b concentration parameter. Define, for $\phi : M \to \mathbb{R}$,

$$[\phi](f,b) = (Z(f,b))^{-1} \int_M \phi(m) e^{b\mathcal{K}f} dm.$$

$$K(m,p) = \sum_{j=1}^{\infty} \mu_j \phi_j(m) \phi_j(p)$$

ϕ_j real, complete, orthonormal in $L^2(M)$,

$$\mathcal{K}\phi_j = \mu_j \phi_j$$

Expand f:

$$v_j(f) = \int_M f(p)\phi_j(p)dp.$$

Onsager's equation

$$f = Z^{-1} e^{b\mathcal{K}f}$$

is equivalent to the system

$$v_j(f) = [\phi_j](f, b).$$

Onsager solution is a critical point of the free energy

$$\mathcal{F}(v, b) = \log Z(v, b) - b \sum_{j=1}^{\infty} \mu_j \frac{v_j^2}{2}$$

Differentiation: For any function $\phi(p)$

$$\frac{\partial[\phi]}{\partial v_i} = b\mu_i \left\{[\phi\phi_i] - [\phi][\phi_i]\right\}$$

Therefore the Hessian $\frac{\partial^2\mathcal{F}}{\partial v_i \partial v_j}$ is

$$\mathcal{H}_{ij} = b^2\mu_i\mu_j[\xi_i\xi_j] - b\mu_i\delta_{ij}$$

with $\xi_j = \phi_j - [\phi_j]$. For b small the isotropic state $v = 0$ is stable.

$$\lim_{b\to\infty}[\phi](v,b) = \phi(p(v))$$

Morse lemma.

Smoluchowski (Nonlinear Fokker-Planck) Equation

$$\partial_t f + u \cdot \nabla_x f + \mathrm{div}_g(Gf) = \frac{1}{\tau}\Delta_g f$$

$$G = \frac{1}{\tau}\nabla_g \mathcal{K} f + W,$$

The $(0,1)$ tensor field W is:

$$W(x,m,t) =$$
$$= \left(\sum_{i,j=1}^{3} c_\alpha^{ij}(m) \frac{\partial u_i}{\partial x_j}(x,t) \right)_{\alpha=1,\ldots,d.}$$

Example, rod-like particles:

$$W(x,m,t) = (\nabla_x u(x,t))m - ((\nabla_x u(x,t))m \cdot m)m.$$

Macro-Micro Effect: from first principles, in principle...

Energetics

$$\sigma_{ij}(x) = -\epsilon \int_M \left(\text{div}_g c^{ij} + c^{ij} \cdot \nabla_g \mathcal{K} f(x,m) \right) f(x,m) dm$$

Micro-Macro Effect: from Energetics !

- $f = Z^{-1} e^{\mathcal{K} f} \Rightarrow \sigma = 0$
- $\mathcal{K} = 0,\ W = (\nabla_x u)m - m((\nabla_x u)m \cdot m) \Rightarrow \sigma = \epsilon \int (3n \otimes n - 1) dm$

$$\begin{cases} \gamma_{ij}^{(1)} = -\epsilon \mathrm{div}_g c^{ij} + \lambda \delta_{ij} \\ \gamma_{ij}^{(2)}(m,n) = -\epsilon c_\alpha^{ij}(m) g^{\alpha\beta}(m) \partial_\beta K(m,n) + \mu \delta_{ij} \end{cases}$$

$$\sigma(x,t) = \sigma^{(1)}(x,t) + \sigma^{(2)}(x,t) + \dots$$

where

$$\sigma_{ij}^{(1)}(x,t) = \int_M \gamma_{ij}^{(1)}(m) f(x,m,t) dm$$

$$\sigma_{ij}^{(2)}(x,t) = \int\limits_{M\times M} \gamma_{ij}^{(2)}(m,n) f(x,m,t) f(x,n,t) dm dn.$$

Theorem 3 *Liapunov functional*

$$E(t) = \frac{1}{2}\int |u|^2 dx +$$
$$+\epsilon \int \left\{ f \log f - \frac{1}{2}(\mathcal{K}f)f \right\} dx dm.$$

If (u, f) *is a smooth solution then*

$$\frac{dE}{dt} = -\nu \int |\nabla_x u|^2 dx -$$
$$-\frac{\epsilon}{\tau}\int \int_M f \left|\nabla_g \left(\log f - \mathcal{K}f\right)\right|^2 dm dx.$$

If the smooth solution is time independent, then $u = 0$ *and* f *solves the Onsager equation*

$$f = Z^{-1} e^{\mathcal{K}f}.$$

Time dependent Stokes and Nonlinear Fokker-Planck in 3D

$$\partial_t f + u\cdot\nabla_x f + \mathrm{div}_g(Wf) + \frac{1}{\tau}\mathrm{div}_g(f\nabla_g(\mathcal{K}f)) = \epsilon\Delta_g f$$
$$\partial_t u - \nu\Delta_x u + \nabla_x p = \mathrm{div}_x\sigma + F, \quad \nabla_x\cdot u = 0.$$

Theorem 4 *Assume u_0 is divergence-free and belongs to $W^{2,r}(\mathbb{T}^3)$, $r>3$, assume that f_0 is positive, normalized, and $f_0 \in L^\infty(dx;\mathcal{C}(M)) \cap \nabla_x f_0 \in L^r(dx;H^{-s}(M))$, $s \le \frac{d}{2}+1$. Then the solution exists for all time and*

$$\|u\|_{L^p[(0,T);W^{2,r}(dx)]} < \infty,$$
$$\|\nabla_x f\|_{L^\infty[(0,T);L^r(dx;H^{-s}(M))]} < \infty$$

for any $p > \frac{2r}{r-3}$, $T>0$ $\tau \le \infty$, $\epsilon \ge 0$.

Global existence, NSE and Nonlinear Fokker-Planck 2D

Theorem 5 *(C-Masmoudi) Let* $u_0 \in \left(W^{\alpha,r} \cap L^2\right)(\mathbb{R}^2)$ *be divergence-free, and* $f_0 \in W^{1,r}(H^{-s}(M))$*, with* $r > 2$, $\alpha > 1$, $s \leq \frac{d}{2}+1$ *and* $f_0 \geq 0$, $\int_M f_0 dm \in (L^1 \cap L^\infty)(\mathbb{R}^2)$*. Then the coupled NS and nonlinear Fokker-Planck system in 2D has a global solution* $u \in L^\infty_{loc}(W^{1,r}) \cap L^2_{loc}(W^{2,r})$ *and* $f \in L^\infty_{loc}(W^{1,r}(H^{-s}))$*. Moreover, for* $T > T_0 > 0$*, we have* $u \in L^\infty((T_0,T); W^{2-0,r})$*.*

No a priori bound.

$$\sup_k \lambda_k^{\alpha - \frac{1}{k}\int_0^t \|\nabla_x S_{k-1}(u(s))\|_{L^\infty} ds} \|\Delta_k(u)(t)\|_{L^p}$$

Basic Fokker-Planck Lemma

$$\partial_t f + u \cdot \nabla_x f \sim (\nabla_x u)\text{div}_g(cf) + \text{div}_g(f\nabla_g(\mathcal{K}f))$$

$$\partial_t N + u \cdot \nabla_x N \le c\,|\nabla_x u|\, N + c\,|\nabla_x \nabla_x u|$$

pointwise at (x, t), with

$$N(x,t) = \sqrt{\int_M \left|(\mathbf{I} - \Delta_g)^{-\frac{s}{2}} \nabla_x f(x,m,t)\right|^2 dm}$$

$s > \frac{d}{2} + 1.$

Growth

$$\frac{d}{dt}n \leq c(gn + G)$$

$$n(t) = \|N(\cdot, t)\|_{L^r(dx)\cap L^2(dx)}, \quad g(t) = \|\nabla u(\cdot, t)\|_{L^\infty}$$

$$G(t) = \|\nabla_x \nabla_x u(\cdot, t)\|_{L^r(dx)\cap L^2(dx)}.$$

$$n(t) \leq \left\{ n(0) + \int_0^t G(s)ds \right\} e^{c\int_0^t g(s)ds}$$

Bounds

$$\int_0^t n = I$$

$$\int_0^t g \le K \log_* I$$

$$\int_0^t G \le K I \log_* I$$

$$\nabla_x u = e^{t\Delta} \nabla_x u(0) + \int_0^t e^{(t-s)\Delta} \Delta \mathbb{H} \{(u \otimes u) - \sigma\} \, ds$$

Linear piece

$$\int_0^t e^{(t-s)\Delta} \Delta \mathbb{H}(D)(\sigma(s))ds$$

$$\|\mathbb{H}(D)\sigma\|_{L^\infty} \leq C\|\sigma\|_{L^\infty} \log_* B$$

$$\left\| \int_0^t e^{(t-s)\Delta} \Delta \phi(s)ds \right\|_{L^\infty} \leq C\|\phi\|_{L^\infty} \log_* B$$

together: one log, not two

Nonlinear piece

$$U(t) = \int_0^t e^{(t-s)\Delta} \Delta \mathbb{H}(D)((u \otimes u)(s))ds.$$

Idea = Chemin and Masmoudi: take time integral first

$$\sum_{q \geq 3} \Delta_q(U) = C(t) + I(t)$$

$$C(t) = \sum_{q \geq 3} \int_0^t e^{(t-s)\Delta} \Delta \mathbb{H}(D) \Delta_q \left(\sum_{|p-p'| \leq 2} \Delta_p(u(s)) \otimes \Delta_{p'}(u(s)) \right) ds$$

$$\int_0^T \|C(t)\|_{L^\infty} dt \leq cE_1$$

$$\|\Delta_q((\Delta_p u(s))\otimes(\Delta_{p'}(u(s)))\|_{L^\infty} \le c2^{2q}\|\Delta_q((\Delta_p u(s))\otimes(\Delta_{p'}(u(s)))\|_{L^1}$$
$$\le c2^{(2q-2p)}\|\nabla\Delta_p u(s)\|_{L^2}\|\nabla\Delta_{p'}u(s)\|_{L^2}$$

$p \ge q-2$, $|p-p'| \le 2$, $p' = p+j$, $j \in [-2,2]$

$$\|C(t)\|_{L^\infty} \le \sum_{j=-2}^{2}\sum_{p=1}^{\infty} c\int_0^t \|\Delta_p\nabla u(s)\|_{L^2}\|\Delta_{p+j}\nabla u(s)\|_{L^2}$$
$$\times\left\{\sum_{q=3}^{p+2} e^{-2^{2(q-1)}(t-s)}2^{2q}2^{2(q-p)}\right\}ds$$

$$\int_0^T \|C(t)\|_{L^\infty}dt \le$$
$$c\sum_{j=-2}^{2}\sum_{p=1}^{\infty}\int_0^T \|\Delta_p(\nabla u(s))\|_{L^2}\|\Delta_{p+j}(\nabla u(s))\|_{L^2}\sum_{q=3}^{p+2}2^{2(q-p)}ds$$
$$\le c\sum_{j=-2}^{2}\int_0^T\sum_{p=1}^{\infty}\|\Delta_p(\nabla u(s))\|_{L^2}\|\Delta_{p+j}(\nabla u(s))\|_{L^2}ds.$$

$$I(t) = \sum_{q \geq 3} \int_0^t e^{(t-s)\Delta} \Delta \mathbb{H}(D) \Delta_q \left(\sum_{|p-p'| \geq 3} \Delta_p(u) \otimes \Delta_{p'}(u) \right) ds$$

$$\int_0^T \|I(t)\|_{L^\infty} dt \leq c E_1 \log_* \left(\frac{c(1+T)R_1}{\epsilon} \right) + \epsilon$$

$$I(t) = I_1(t) + I_2(t)$$

$$I_1(t) = \sum_{q\geq 3} \int_0^t e^{(t-s)\Delta} \Delta \mathbb{H}(D) \Delta_q \left(\sum_{p \geq p'+3} \Delta_p(u(s)) \otimes \Delta_{p'}(u(s)) \right) ds$$

$$I_2(t) = \sum_{q\geq 3} \int_0^t e^{(t-s)\Delta} \Delta \mathbb{H}(D) \Delta_q \left(\sum_{p' \geq p+3} \Delta_p(u(s)) \otimes \Delta_{p'}(u(s)) \right) ds$$

$$I_1(t) = \sum_{j=-2}^{2} \sum_{q\geq 3} \int_0^t e^{(t-s)\Delta} \Delta \mathbb{H}(D) \Delta_q (J_q(s)) ds$$

with

$$J_q(s) = \Delta_{q+j}(u(s)) \otimes S_{q+j-3}(u(s)).$$

$q \leq M$:

$$\begin{aligned} \|\Delta_q(J_q(s))\|_{L^\infty} &\leq c \|S_{q+j-3}(u(s))\|_{L^\infty} \|\Delta_{q+j}(u(s))\|_{L^\infty} \\ &\leq c \left[\|u(s)\|_{L^2} + \sqrt{M+2} \|\nabla u(s)\|_{L^2} \right] \|\Delta_{q+j}(u(s))\|_{L^\infty} \end{aligned}$$

$$\|S_{q+j}(u(s))\|_{L^\infty} \le c\left(\|u(s)\|_{L^2} + \sqrt{q+j}\|\nabla u(s)\|_{L^2}\right).$$

Bernstein's inequality: $\|\Delta_{q+j}u(s)\|_{L^\infty} \le c\|\Delta_{q+j}\nabla u(s)\|_{L^2}$

$\|\Delta_q(J_q(s))\|_{L^\infty} \le c\left(\|u(s)\|_{L^2} + \sqrt{M+2}\|\nabla u(s)\|_{L^2}\right)\|\Delta_{q+j}\nabla(u(s))\|_{L^2}$

$q \ge M$:

$\|\Delta_q(J_q(s))\|_{L^\infty} \le c\left(\|u(s)\|_{L^2} + \|\Delta u(s)\|_{L^2}\right)2^{-q}\|\Delta_{q+j}\Delta(u(s))\|_{L^2}$

$$\int_0^T \|I_1(t)\|_{L^\infty}dt \le A + B$$

with

$$A = c\sum_{j=-2}^{2}\sum_{q=3}^{M}\int_0^T \|\Delta_q(J_q(s))\|_{L^\infty}\left(\int_s^T 2^{2q}e^{-(t-s)2^{2(q-1)}}dt\right)ds$$

and

$$B = c \sum_{j=-2}^{2} \sum_{q=M}^{\infty} \int_0^T \|\Delta_q(J_q(s))\|_{L^\infty} \left(\int_s^T 2^{2q} e^{-(t-s)2^{2(q-1)}} dt \right) ds$$

A:

$$\begin{aligned} A &\le c \int_0^T \left(\|u(s)\|_{L^2} + \sqrt{M+2} \|\nabla u(s)\|_{L^2} \right) \left(\sum_{q=3}^{M} \|\Delta_{q+j} \nabla u(s)\|_{L^2} \right) ds \\ &\le c \int_0^T \left(\|u(s)\|_{L^2} + \sqrt{M+2} \|\nabla u(s)\|_{L^2} \right) \left(\sqrt{M} \|\nabla u(s)\|_{L^2} \right) ds. \end{aligned}$$

We have therefore

$$A \le cME_1$$

B:

$$B \le c \int_0^T \left(\|u(s)\|_{L^2} + \|\Delta u(s)\|_{L^2} \right) \sum_{q \ge M} \left(2^{-q} \|\Delta_{q+j} \Delta(u(s))\|_{L^2} \right) ds$$

and therefore

$$B \leq c2^{-M} \int_0^T \|u(s)\|_{H^2}^2 ds = c2^{-M} E_2$$

where

$$E_2 = \int_0^T \|u(t)\|_{H^2}^2 dt.$$

$$B \leq c2^{-M}(1+T)\left\{\|\sigma\|_{L^2(0,T;H^1)}^2 + \|u(0)\|_{H^1}^2\right\} = c2^{-M}(1+T)R_1$$

$$M = \log_* \left(\frac{c(1+T)R_1}{\epsilon}\right)$$

Coming attractions:

- A priori bound in 2D
- Stochastic system: existence and bounds
- Traveling waves, standing waves (with Berestycki, Ryzhik, Tsogka)

References

- P. Constantin, Nonlinear Fokker-Planck Navier-Stokes Systems, CMS, (2005)

- P. Constantin, C. Fefferman, E. Titi, A. Zarnescu, Regularity of coupled two-dimensional Nonlinear Fokker-Planck and Navier-Stokes systems, CMP (2007), to appear.

- P. Constantin, N. Masmoudi, Global well-posedness for a Smoluchowski equation coupled with Navier-Stokes equations in 2D, preprint 2007.

More References

- P. Constantin, I. Kevrekidis, E. S. Titi, Remarks on a Smoluchowski Equations, Discrete and Continuous Dyn. Syst, (2004)
- P. Constantin. I Kevrekidis, E. S. Titi Asymptotic States of a Smoluchowski Equation, Arch. Rational Mech. Analysis, (2004)
- P. Constantin, E.S. Titi, J. Vukadinovic, Dissipativity and Gevrey Regularity of a Smoluchowski Equation, Indiana U. Math J., (2004)
- P. Constantin, J. Vukadinovic, Note on the number of steady states for a 2D Smoluchowski equation, Nonlinearity (2004).

References for Stochastic representation

- G. Iyer, Thesis, U of C

- P. Constantin, G. Iyer, A stochastic representation of the 3D incompressible NSE, Comm. Pure Appl. Math., (2007) to appear.

- P. Constantin, G. Iyer, Stochastic Lagrangian transport and generalized relative entropies, CMS, (2006).

- P. Constantin, An Eulerian-Lagrangian approach for incompressible fluids: local theory. Journal of AMS **14** (2001), 773-783.

- P. Constantin, An Eulerian-Lagrangian approach to the Navier-Stokes equations, Comm. Math. Phys. **216** (2001), 663-686.

- K. Ohkitani, P. Constantin, Numerical study of the Eulerian-Lagrangian formulation of the NSE, Phys. fluids **15** (2003) 3251-3254.