EnKF and Catastrophic filter divergence

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The set-up for EnKF

We have a **deterministic model**

$$\frac{dv}{dt} = F(v) \quad \text{with } v_0 \sim N(m_0, C_0) \ .$$

We will denote $v(t) = \Psi_t(v_0)$.

We want to **estimate** $v_j = v(jh)$ for some $h > 0$ and $j = 0, 1, \ldots, J$ given the **observations**

$$y_{j+1} = Hv_{j+1} + \xi_{j+1} \quad \text{for } \xi_{j+1} \text{ iid } N(0, \Gamma).$$

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We estimate using an **ensemble** of particles $\{u^{(k)}\}_{k=1}^{K}$. Each particle is a statistical **representative** of the **posterior**.

For each particle, we have an **artificial observation**

$$y_{j+1}^{(k)} = y_{j+1} + \xi_{j+1}^{(k)} \quad , \quad \xi_{j+1}^{(k)} \text{ iid } N(0, \Gamma).$$

We update each particle using the **Kalman update**

$$u_{j+1}^{(k)} = \Psi_h(u_j^{(k)}) + G(u_j)\left(y_{j+1}^{(k)} - H\Psi_h(u_j^{(k)})\right) ,$$

where $G(u_j)$ is the **Kalman gain** computed using the **forecasted ensemble covariance**

$$\widehat{C}_{j+1} = \frac{1}{K}\sum_{k=1}^{K}(\Psi_h(u_j^{(k)}) - \overline{\Psi_h(u_j)})^T(\Psi_h(u_j^{(k)}) - \overline{\Psi_h(u_j)}) .$$

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Filter divergence

It has been observed ($\star$) that the ensemble can **blow-up** (ie. reach machine-infinity) in **finite time**, even when the model has nice bounded solutions.

This is known as **catastrophic filter divergence**.

It is suggested in ($\star$) that this is caused by numerically integrating a stiff-system. Our aim is to "prove" this.

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Discrete time results

We make a **"dissipativity"** assumption on F. Namely that

$$F(\cdot) = A \cdot + B(\cdot, \cdot) \qquad (\dagger)$$

with A linear elliptic and B bilinear, satisfying certain estimates and symmetries.

Eg. 2d-Navier-Stokes, Lorenz-63, Lorenz-96.

Theorem (AS,DK)

If $H = I$ and $\Gamma = \gamma^2 I$, then there exists constant β, K such that

$$\mathbf{E}|u_j^{(k)}|^2 \leq e^{2\beta jh}\mathbf{E}|u_0^{(k)}|^2 + 2K\gamma^2 \left(\frac{e^{2\beta jh} - 1}{e^{2\beta h} - 1}\right)$$

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The EnKF equations look like a discretization

Recall the ensemble update equation

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$$= \Psi_h(u_j^{(k)}) + \widehat{C}_{j+1}H^T(H^T\widehat{C}_{j+1}H + \Gamma)^{-1}\left(y_{j+1}^{(k)} - H\Psi_h(u_j^{(k)})\right)$$

Subtract $u_j^{(k)}$ from both sides and divide by h

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Continuous-time limit

If we set $\Gamma = h^{-1}\Gamma_0$ and substitute $y_{j+1}^{(k)}$, we obtain

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But we know that

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We end up with

$$\frac{u_{j+1}^{(k)} - u_j^{(k)}}{h} = \frac{\Psi_h(u_j^{(k)}) - u_j^{(k)}}{h} - C(u_j)H^T\Gamma_0^{-1}H(u_j^{(k)} - v_j)$$
$$+ C(u_j)H^T\Gamma_0^{-1}\left(h^{-1/2}\xi_{j+1} + h^{-1/2}\xi_{j+1}^{(k)}\right) + O(h)$$

This looks like a **numerical scheme** for

$$\frac{du^{(k)}}{dt} = F(u^{(k)}) - C(u)H^T\Gamma_0^{-1}H(u^{(k)} - v) \qquad (\bullet)$$
$$+ C(u)H^T\Gamma_0^{-1/2}\left(\frac{dW^{(k)}}{dt} + \frac{dB}{dt}\right).$$

Rmk. The extra dissipation term **only sees differences in observed space** and **only dissipates in the space spanned by ensemble**.

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Continuous-time results

Theorem (AS,DK)

Suppose the model v *satisfies* (†) *and* $\{u^{(k)}\}_{k=1}^{K}$ *satisfy* (•). *Let*

$$e^{(k)} = u^{(k)} - v \ .$$

If $H = I$ *and* $\Gamma = \gamma^2 I$*, then there exists constant* β, K *such that*

$$\mathbf{E} \sum_{k=1}^{K} |e^{(k)}(t)|^2 \leq \mathbf{E} \sum_{k=1}^{K} |e^{(k)}(0)|^2 \exp(\beta t) \ .$$

Summary + Future Work

(1) Writing down an SDE/SPDE allows us to see the **important quantities** in the algorithm.

(2) Does not "prove" that filter divergence is a numerical phenomenon, but is a decent starting point.

(1) Improve the condition on H.

(2) If we can **measure** the important quantities, then we can test the performance during the algorithm.

(3) Suggests new EnKF-like algorithms, for instance by discretising the stochastic PDE in a more **numerically stable** way.

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