The FMM for 3D Helmholtz Equation

Nail Gumerov &
Ramani Duraiswami
UMIACS
[gumerov][ramani]@umiacs.umd.edu

Reference

N.A. Gumerov & R. Duraiswami

Fast Multipole Methods for Solution of the Helmholtz Equation in Three Dimensions

Academic Press, Oxford (2004)

(in process).

Content

- Helmholtz Equation
- Expansions in Spherical Coordinates
- Matrix Translations
- Complexity and Modifications of the FMM
- Fast Translation Methods
- Error Bounds
- Multiple Scattering Problem

Helmholtz Equation

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Helmholtz Equation

$$\nabla^2 \psi + k^2 \psi = 0$$

- Wave equation in frequency domain
  - Acoustics
  - Electromagneics (Maxwell equations)
  - Diffusion/heat transfer/boundary layers
  - Telegraph, and related equations
  - *k* can be complex
- Quantum mechanics
  - Klein-Gordan equation
  - Shroedinger equation
- Relativistic gravity (Yukawa potentials, *k* is purely imaginary)
- Molecular dynamics (Yukawa)
- Appears in many other models

Boundary Value Problems

- Dirichlet:

$$\psi|_S = 0,$$

- Neumann:

$$\left.\frac{\partial \psi}{\partial n}\right|_S = 0,$$

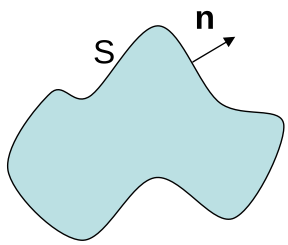
- Robin:

$$\left.\left(\frac{\partial \psi}{\partial n} + i\sigma\psi\right)\right|_S = 0.$$

- Sommerfield Radiation Condition (for external problems):

$$\psi = \psi_{in} + \psi_{scat}$$

$$\lim_{r\to\infty}\left[r\left(\frac{\partial \psi_{scat}}{\partial r} - ik\psi_{scat}\right)\right] = 0.$$



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Green's Function and Identity

Free space Green's function:

$$\nabla^2 G(\mathbf{x},\mathbf{y}) + k^2 G(\mathbf{x},\mathbf{y}) = -\delta(\mathbf{x}-\mathbf{y}),$$

$$G(\mathbf{x},\mathbf{y}) = \frac{\exp(ik|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}, \quad \mathbf{x},\mathbf{y} \in \mathbb{R}^3.$$

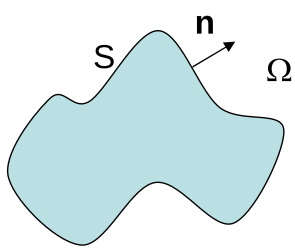
Green's formula:

$$\psi(\mathbf{y}) = \int_S \left[\psi(\mathbf{x}) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{x})} - G(\mathbf{x},\mathbf{y}) \frac{\partial \psi(\mathbf{x})}{\partial n(\mathbf{x})} \right] \mathrm{d}S(\mathbf{x}), \quad \mathbf{y} \in \Omega.$$

Boundary integral equation

$$\alpha\psi(\mathbf{y}) = \int_S \left(\psi(\mathbf{x}) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{x})} - G(\mathbf{x},\mathbf{y}) \frac{\partial \psi(\mathbf{x})}{\partial n(\mathbf{x})} \right) \mathrm{d}S(\mathbf{x}),$$

$$\alpha = \begin{cases} \frac{1}{2} & \mathbf{y} \text{ on a smooth part of the boundary} \\ \frac{\gamma}{4\pi} & \mathbf{y} \text{ at a corner on the boundary} \\ 1 & \mathbf{y} \text{ inside the domain} \end{cases}.$$



Distributions of Monopoles and Dipoles

Volume source distribution:

$$\psi(\mathbf{y}) = \sum_{j=1}^{N} Q_j G(\mathbf{x}_j,\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^3 \backslash \{\mathbf{x}_j\},$$

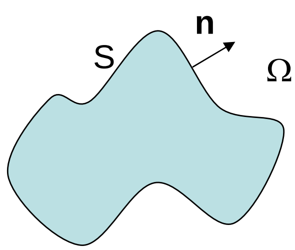
$$\psi(\mathbf{y}) = \int_{\overline{\Omega}} q(\mathbf{x}) G(\mathbf{x},\mathbf{y}) \mathrm{d}V(\mathbf{x}), \quad \mathbf{y} \in \Omega, \quad \overline{\Omega} \cap \Omega = \emptyset.$$

Single layer potential:

$$\psi(\mathbf{y}) = \int_S q_\sigma(\mathbf{x}) G(\mathbf{x},\mathbf{y}) \mathrm{d}S(\mathbf{x}), \quad \mathbf{y} \in \Omega, \quad S = \partial\Omega.$$

Double layer potential:

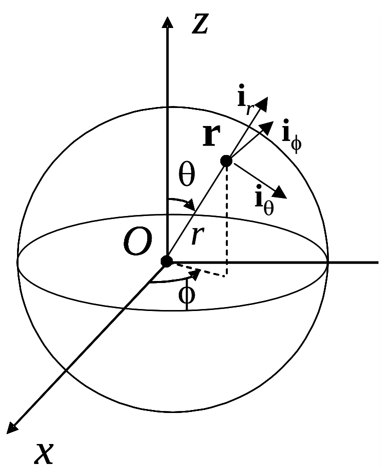
$$\psi(\mathbf{y}) = \int_S q_\mu(\mathbf{x}) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{x})} \mathrm{d}S(\mathbf{x}), \quad \mathbf{y} \in \Omega, \quad S = \partial\Omega.$$



Expansions in Spherical Coordinates

Spherical Basis Functions

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta.$$



Spherical Coordinates

Spherical Bessel Functions

Regular Basis Functions

$$R_n^m(\mathbf{r}) = j_n(kr)Y_n^m(\theta,\varphi),$$

Singular Basis Functions

$$S_n^m(\mathbf{r}) = h_n(kr)Y_n^m(\theta,\varphi).$$

Spherical Hankel Functions of the First Kind

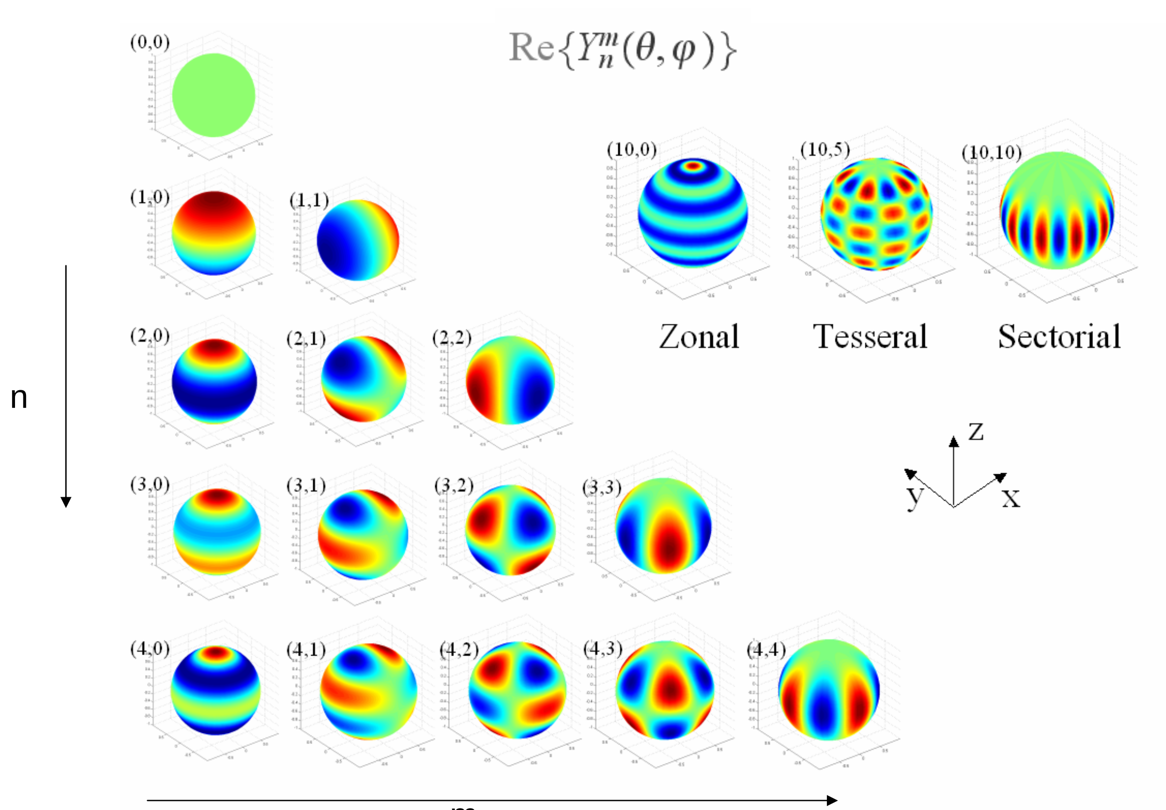
Spherical Harmonics

$$Y_n^m(\theta,\varphi) = (-1)^m\sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta)e^{im\varphi},$$

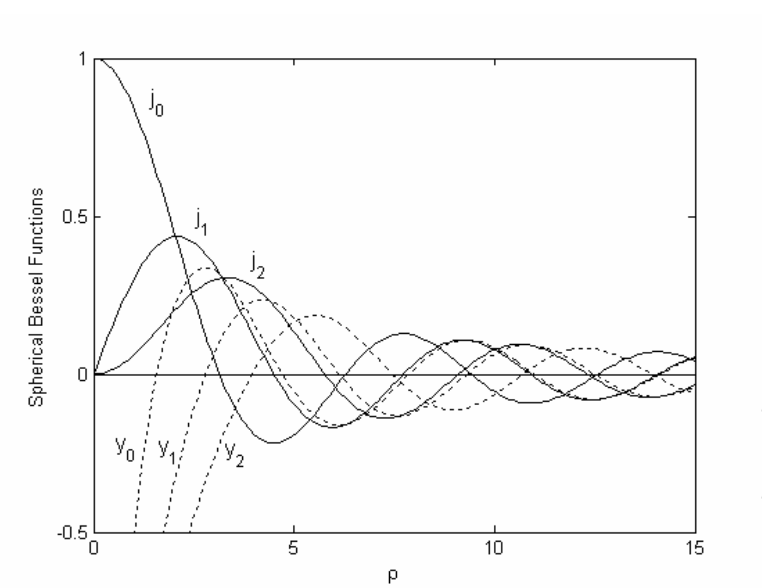
$$n = 0,1,2,\ldots; \qquad m = -n,\ldots,n.$$

Associated Legendre Functions

Spherical Harmonics



Spherical Bessel Functions



$$h_n(\rho) = j_n(\rho) + i y_n(\rho)$$

$$j_0(\rho) = \frac{\sin\rho}{\rho}, \quad j_1(\rho) = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho},$$

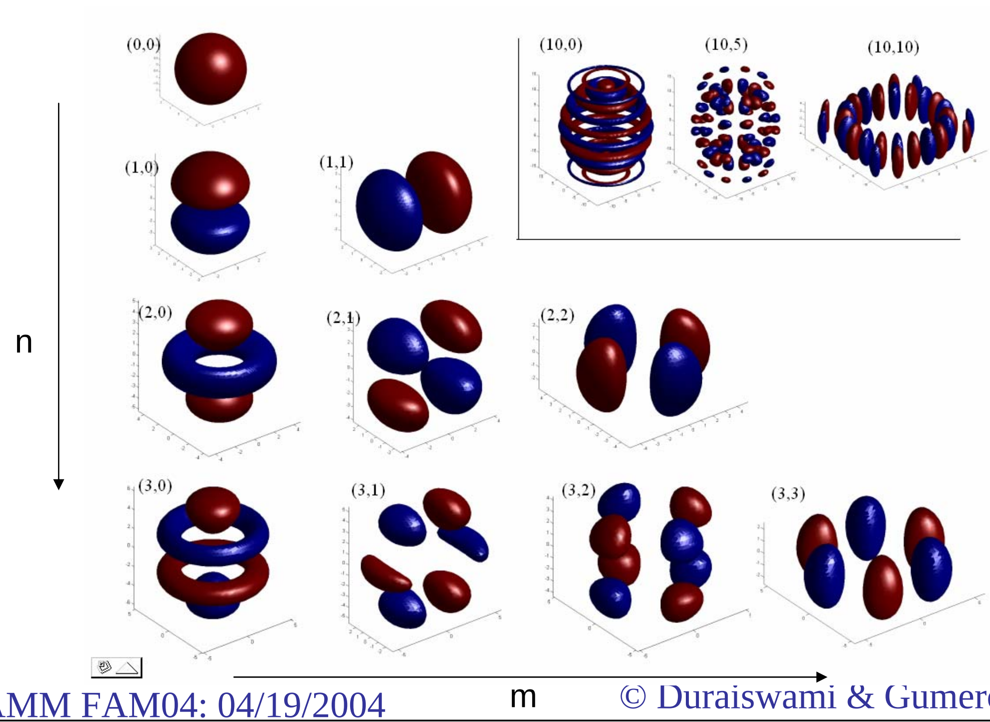
$$j_2(\rho) = \left(\frac{3}{\rho^3} - \frac{1}{\rho}\right)\sin\rho - \frac{3}{\rho^2}\cos\rho,$$

$$y_0(\rho) = -\frac{\cos\rho}{\rho}, \quad y_1(\rho) = -\frac{\cos\rho}{\rho^2} - \frac{\sin\rho}{\rho},$$

$$y_2(\rho) = \left(-\frac{3}{\rho^3} + \frac{1}{\rho}\right)\cos\rho - \frac{3}{\rho^2}\sin\rho.$$

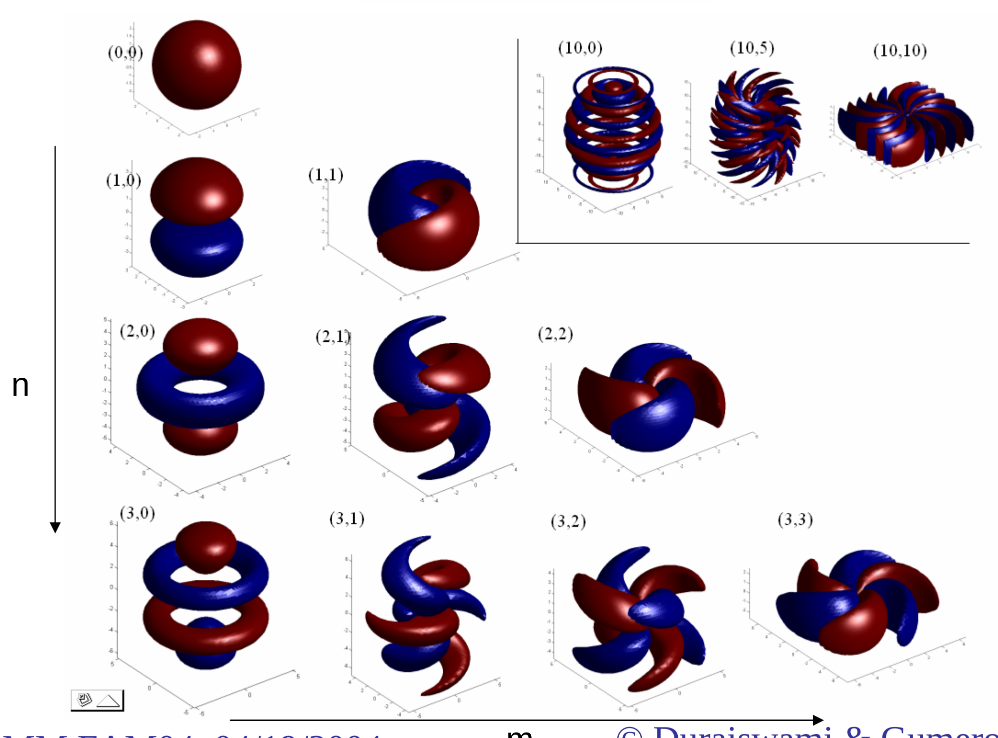
Isosurfaces For Regular Basis Functions

$$\mathrm{Re}\{R_n^m(\mathbf{r})\} = const$$



Isosurfaces For Singular Basis Functions

$$\mathrm{Re}\{S_n^m(\mathbf{r})\} = const$$



Expansions

$$\psi(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_n^m F_n^m(\mathbf{r}) = \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty} A_n^m F_n^m(\mathbf{r}), \quad F = S,R, \quad A_n^m \in \mathbb{C}.$$

Absolute and uniform convergence

$$\forall \epsilon > 0, \quad \exists p(\epsilon), \quad \left|\psi(\mathbf{r}) - \sum_{n=0}^{p-1}\sum_{m=-n}^{n} A_n^m F_n^m(\mathbf{r})\right| < \epsilon, \quad \forall \mathbf{r} \in \Omega,$$

and

$$\forall \epsilon > 0, \quad \exists p(\epsilon), \quad \sum_{n=p}^{\infty}\sum_{m=-n}^{n} |A_n^m F_n^m(\mathbf{r})| < \epsilon, \quad \forall \mathbf{r} \in \Omega.$$

Plane Wave expansion:

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 4\pi \sum_{n=0}^{\infty}\sum_{m=-n}^{n} i^n Y_n^{-m}(\theta_k, \varphi_k) R_n^m(\mathbf{r}),$$

$$\mathbf{k} = k\mathbf{s}, \quad \mathbf{s} = (\sin\theta_k \cos\varphi_k, \sin\theta_k \sin\varphi_k, \cos\theta_k).$$

Wave vector

Matrix Translations

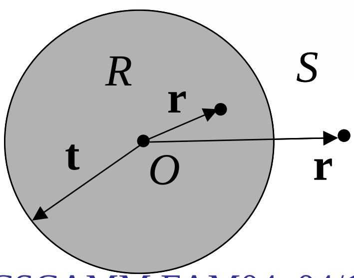
Reexpansions of Basis Functions

$$R_n^m(\mathbf{r}+\mathbf{t}) = \sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} (R|R)_{n'n}^{m'm}(\mathbf{t}) R_{n'}^{m'}(\mathbf{r}), \quad n = 0,1,2,\ldots, \quad m = -n,\ldots,n.$$

Reexpansion Matrices

$$S_n^m(\mathbf{r}+\mathbf{t}) = \sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} \begin{cases} (S|R)_{n'n}^{m'm}(\mathbf{t}) R_{n'}^{m'}(\mathbf{r}), & |\mathbf{r}| < |\mathbf{t}| \\ (S|S)_{n'n}^{m'm}(\mathbf{t}) S_{n'}^{m'}(\mathbf{r}), & |\mathbf{r}| > |\mathbf{t}| \end{cases},$$

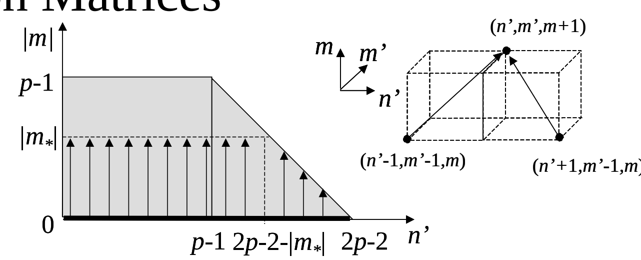
$$n = 0,1,2,\ldots, \quad m = -n,\ldots,n.$$

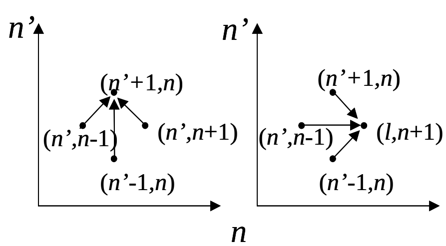


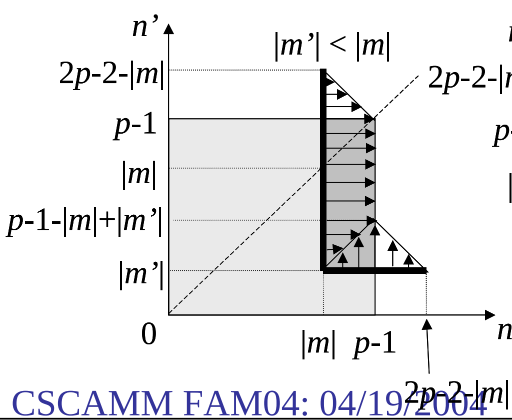
Recursive Computation of Reexpansion Matrices

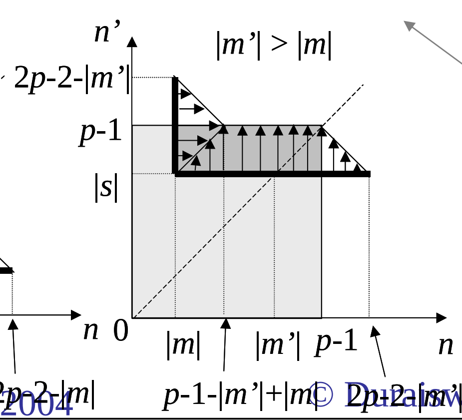
Gumerov & Duraiswami, *SIAM J. Sci. Stat. Comput.* 25(4), 1344-1381, 2003.

p[4] elements of the truncated reexpansion matrices can be computed for O(p[4]) operations recursively









$$(E|F)_{n'|m|}^{m'm}$$

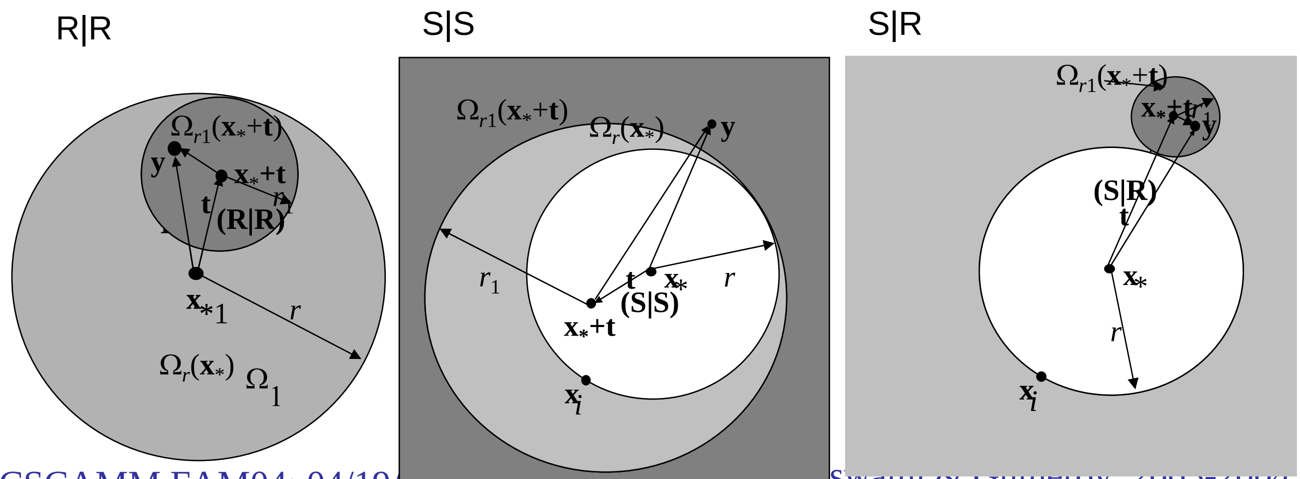
$$(E|F)_{n'n}^{m'm}$$

Translations

$$\psi(\mathbf{y}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_n^m(\mathbf{x}_{*1}) E_n^m(\mathbf{y}-\mathbf{x}_{*1}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_n^m(\mathbf{x}_{*2}) F_n^m(\mathbf{y}-\mathbf{x}_{*2}),$$

$$C_n^m(\mathbf{x}_{*2}) = \sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} (E|F)_{nn'}^{mm'}(\mathbf{t}) C_{n'}^{m'}(\mathbf{x}_{*1}), \quad \mathbf{t} = \mathbf{x}_{*2} - \mathbf{x}_{*1}$$

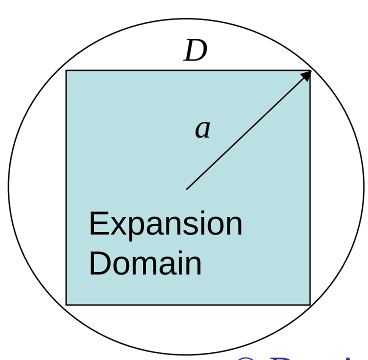
$$E,F = S,R, \quad n = 0,1,\ldots, \quad m = -n,\ldots,n.$$



Problem:

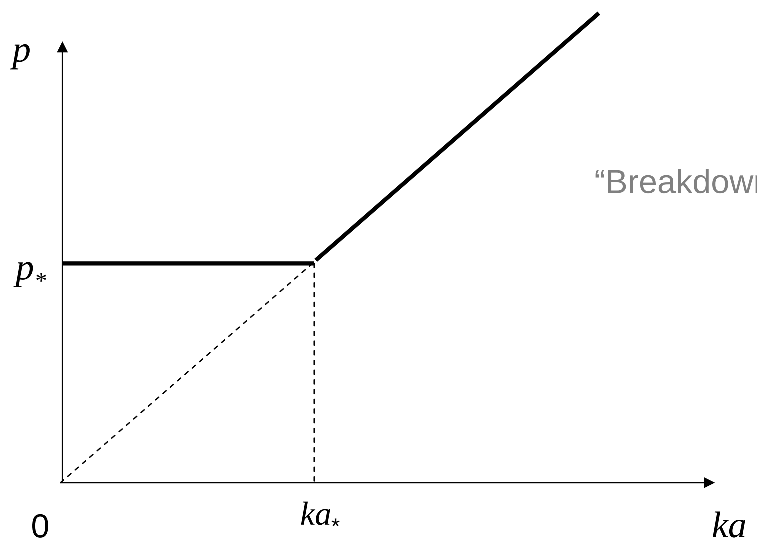
- For the Helmholtz equation absolute and uniform convergence can be achieved only for

  $p > ka$. For large ka the FMM with constant p is
  - very expensive (comparable with straightforward methods);
  - inaccurate (since keeps much larger number of terms than required, which causes numerical instabilities).



$2a=3^{1/2}D$

Model of Truncation Number Behavior for Fixed Error



In the multilevel FMM we associate its own p_l with each level *l*:

$$P_l = p_l^2$$

"Breakdown level"

$$kD_{l_{b*}} = \kappa_b, \quad l_{b*} = \log_2 \frac{kD_0}{\kappa_b}$$

$$P_l = A(kD_l)^2, \quad l \leqslant l_{b*}$$

$$P_l = A\kappa_b^2 = P_*, \quad l \geqslant l_{b*}$$

Box size at level *l*

Complexity of Single Translation

Translation exponent

$$CostTrans(P_l) = CP_l^{\nu} = Cp_l^{2\nu}, \quad l = 2, \ldots, l_{\max}.$$

Spatially Uniform Data Distributions

$$N_l \sim 8^{-l} N, \quad l_{\max} \sim \frac{1}{3} \log N$$

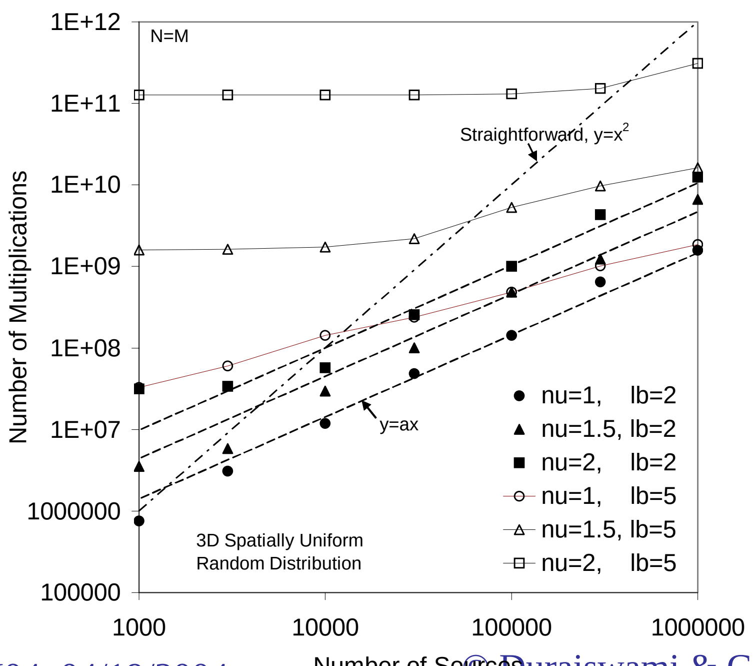
$$p_l \sim 2^{-l} kD_0,$$

$$N_{oper} \sim (kD_0)^{2\nu} \sum_{l=2}^{l_{\max}} 2^{-2\nu l} 8^l = (kD_0)^{2\nu} \sum_{l=2}^{l_{\max}} 2^{(3-2\nu)l}.$$

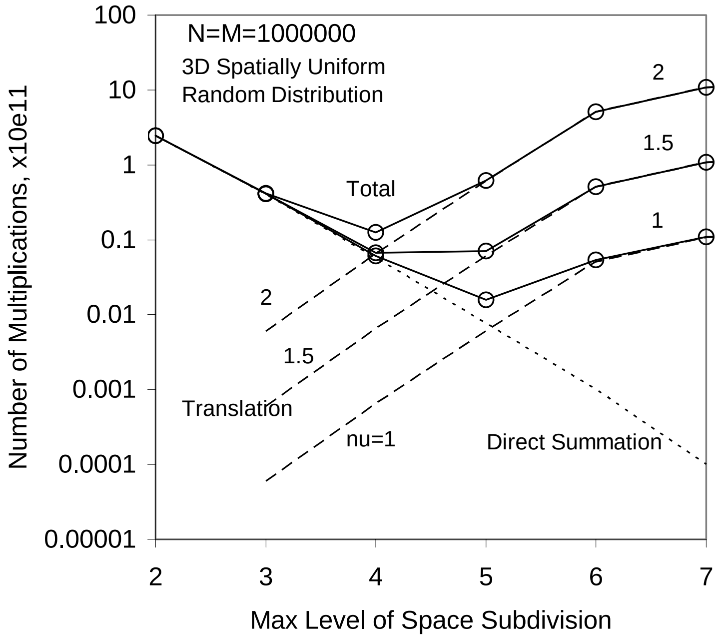
- $\nu < 1.5:\quad ComplexityFMM \sim (kD_0)^{2\nu} 2^{(3-2\nu)l_{\max}} \sim (kD_0)^{2\nu} N^{1-2\nu/3}$
- $\nu = 1.5:\quad ComplexityFMM \sim (kD_0)^{2\nu} l_{\max} \sim (kD_0)^{2\nu} \log N$
- $\nu > 1.5:\quad ComplexityFMM \sim (kD_0)^{2\nu}$

Constant!

Complexity of the Optimized FMM for Fixed kD_0 and Variable N



Optimum Level for Low Frequencies



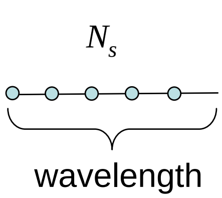
Volume Element Methods

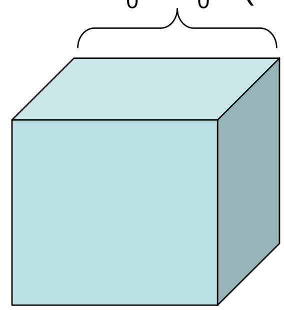
$$N = \left(\frac{N_s}{2\pi} kD_0\right)^3, \quad kD_0 \sim N^{1/3}$$

- $\nu < 1.5: \quad ComplexityFMM \sim (kD_0)^{2\nu} 2^{(3-2\nu)l_{\max}} \sim (kD_0)^{2\nu} N^{1-2\nu/3} \sim N$
- $\nu = 1.5: \quad ComplexityFMM \sim (kD_0)^{2\nu} l_{\max} \sim (kD_0)^{2\nu} \log N \sim N \log N$
- $\nu > 1.5: \quad ComplexityFMM \sim (kD_0)^{2\nu} \sim N^{2\nu/3} \gg N \log N.$

Critical Translation Exponent!

$D_0 = D_0\, k/(2\pi)$ wavelengths = $N^{1/3}$ sources





computational domain

What Happens if Truncation Number is Constant for All Levels?

$$N_{oper} \sim (kD_0)^{2\nu} \sum_{l=2}^{l_{\max}} 8^l = (kD_0)^{2\nu} \sum_{l=2}^{l_{\max}} 2^{3l} \sim (kD_0)^{2\nu} 2^{3l_{\max}} \sim (kD_0)^{2\nu} N \sim N^{1+2\nu/3}.$$

- $\nu < 1.5: \quad N \ll ComplexityFMM \ll N^2$
- $\nu = 1.5: \quad ComplexityFMM \sim N^2$
- $\nu > 1.5: \quad ComplexityFMM \sim N^{1+2\nu/3} \gg N^2$

"Catastrophic Disaster of the FMM"

Surface Data Distributions

$$N_l \sim 4^{-l} N, \quad l_{\max} \sim \frac{1}{2} \log N$$

$$p_l \sim 2^{-l} kD_0,$$

$$N_{oper} \sim (kD_0)^{2\nu} \sum_{l=2}^{l_{\max}} 2^{-2\nu l} 4^l = (kD_0)^{2\nu} \sum_{l=2}^{l_{\max}} 2^{(2-2\nu)l}.$$

- $\nu = 1: \quad ComplexityFMM \sim (kD_0)^{2\nu} l_{\max} \sim (kD_0)^{2\nu} \log N$
- $\nu > 1: \quad ComplexityFMM \sim (kD_0)^{2\nu}$

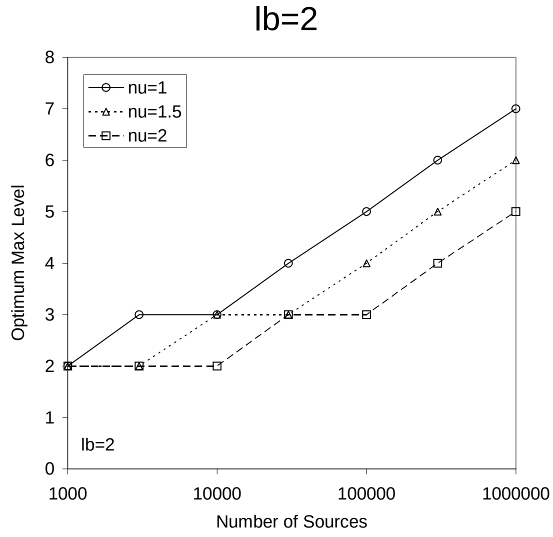
Boundary Element Methods:

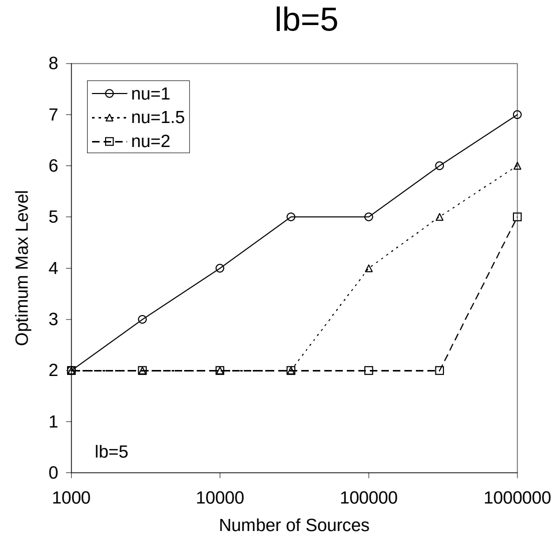
$$N = \left(\frac{N_s}{2\pi} kD_0 \right)^2, \quad kD_0 \sim N^{1/2}$$

- $\nu = 1: \quad ComplexityFMM \sim (kD_0)^{2\nu} l_{\max} \sim (kD_0)^{2\nu} \log N \sim N \log N$
- $\nu > 1: \quad ComplexityFMM \sim (kD_0)^{2\nu} \sim N^{\nu} \gg N \log N.$

Critical Translation Exponent!

Optimum Level of Space Subdivision





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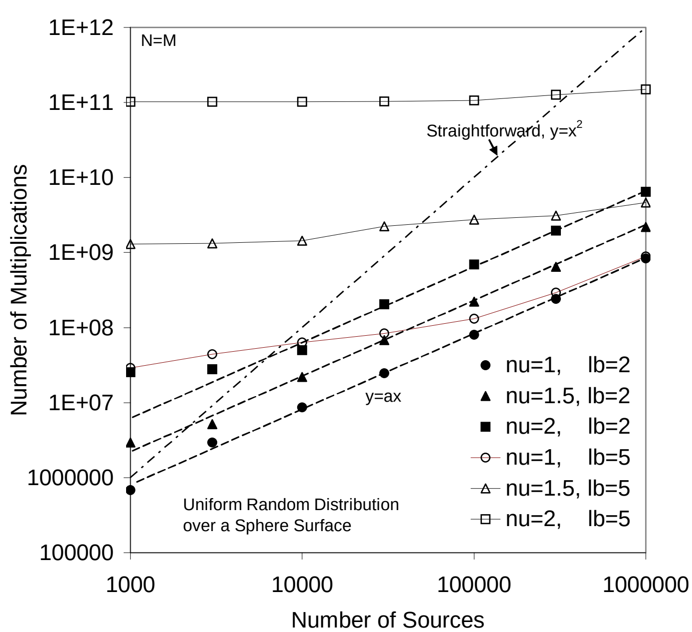
Performance of the MLFMM for Surface Data Distributions



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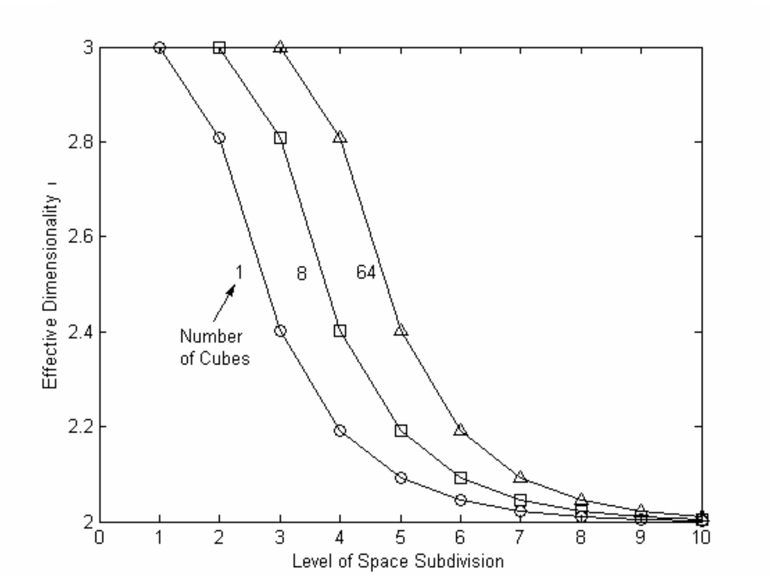
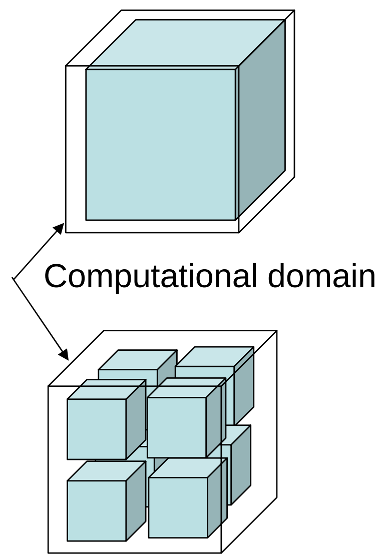
Effective Dimensionality of the Problem

$$d_{eff}(l) = \log_2 \frac{N_{non-empty}(l)}{N_{non-empty}(l-1)}, \quad l = 1, 2, \ldots$$





Fast Translation Methods

Translation Methods

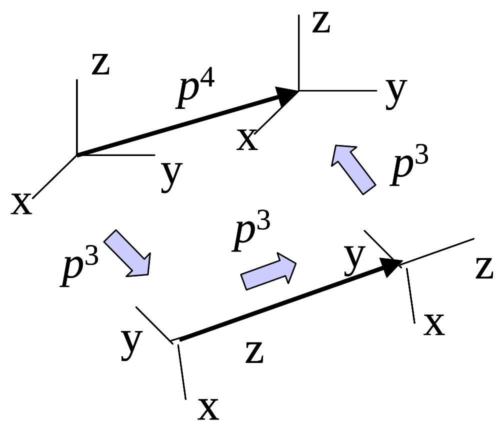
- $O(p^5)$: Matrix Translation with Computation of Matrix Elements Based on Clebsch-Gordan Coefficients;
- $O(p^4)$ (Low Asymptotic Constant): Matrix Translation with Recursive Computation of Matrix Elements
- $O(p^3)$ (Low Asymptotic Constants):
  - Rotation-Coaxial Translation Decomposition with Recursive Computation of Matrix Elements;
  - Sparse Matrix Decomposition;
- $O(p^2 \log^\beta p)$
  - Rotation-Coaxial Translation Decomposition with Structured Matrices for Rotation and Fast Legendre Transform for Coaxial Translation;
  - Translation Matrix Diagonalization with Fast Spherical Transform;
  - Asymptotic Methods;
  - Diagonal Forms of Translation Operators with Spherical Filtering.

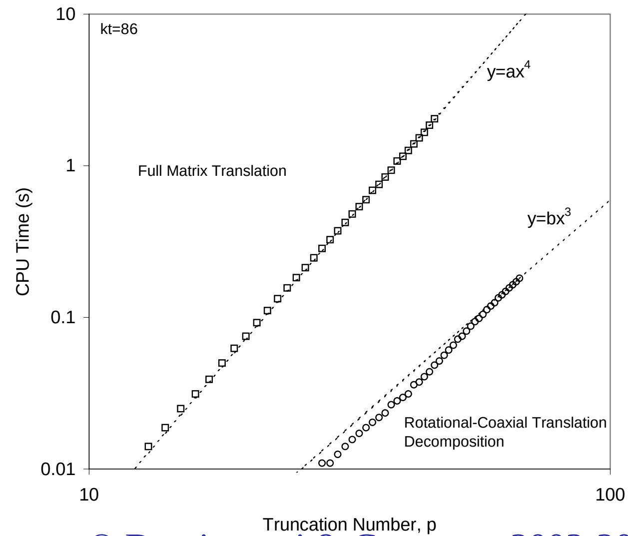
$O(p^3)$ Methods

Rotation - Coaxial Translation Decomposition (Complexity O(p^3))

From the group theory follows that general translation can be reduced to

$$(\mathbf{F}|\mathbf{E})(\mathbf{t}) = \mathbf{Rot}(Q^{-1})(\mathbf{F}|\mathbf{E})_{(coax)}(t)\mathbf{Rot}(Q), \quad F,E = S,R.$$





Sparse Matrix Decomposition

$$(\mathbf{R}|\mathbf{R})(\mathbf{t}) = (\mathbf{S}|\mathbf{S})(\mathbf{t}) = \sum_{n=0}^{\infty} \frac{(kt)^n}{n!}\mathbf{D}_\mathbf{t}^n = e^{kt\mathbf{D}_\mathbf{t}} = \Lambda_r(kt, -i\mathbf{D}_\mathbf{t})$$

$$(\mathbf{S}|\mathbf{R})(\mathbf{t}) = \Lambda_s(kt, -i\mathbf{D}_\mathbf{t})$$

$$\Lambda_r(kt, -i\mathbf{D}_\mathbf{t}) = \sum_{n=0}^{\infty}(2n+1)i^n j_n(kt)P_n(-i\mathbf{D}_\mathbf{t})$$

$$\Lambda_s(kt, -i\mathbf{D}_\mathbf{t}) = \sum_{n=0}^{\infty}(2n+1)i^n h_n(kt)P_n(-i\mathbf{D}_\mathbf{t}).$$

Matrix-vector products with these matrices computed recursively

$$(\mathbf{D}_\mathbf{t}\mathbf{C})_n^m = \frac{1}{2t}\left[(t_x + it_y)\left(C_{n-1}^{m+1}b_n^m - C_{n+1}^{m+1}b_{n+1}^{-m-1}\right) + (t_x - it_y)\left(C_{n-1}^{m-1}b_n^{-m} - C_{n+1}^{m-1}b_{n+1}^{m-1}\right)\right]$$
$$+ \frac{t_z}{t}(a_n^m C_{n+1}^m - a_{n-1}^m C_{n-1}^m), \qquad m = 0, \pm 1, \pm 2, \ldots, \quad n = |m|, |m|+1, \ldots$$

O($p^2 \log^\beta p$) Methods

Fast Rotation Transform

Diagonal matrices

$$\mathbf{Rot}(Q(\alpha,\beta,\gamma)) = \mathbf{Rot}\left(A\left(\gamma+\frac{\pi}{2}\right)\right)\mathbf{Rot}\left(B\left(\frac{\pi}{2}\right)\right)\mathbf{Rot}(A(\beta))\mathbf{Rot}\left(B\left(\frac{\pi}{2}\right)\right)\mathbf{Rot}\left(A\left(\frac{\pi}{2}-\alpha\right)\right).$$

Euler angles

Block-Toeplitz matrices

$$\mathbf{Rot}\left(B\left(\frac{\pi}{2}\right)\right) = \mathbf{L}\mathbf{K}^T\mathbf{M}\mathbf{K}\mathbf{L}$$

Diagonal matrices

Complexity: O($p^2\log p$)

Fast Coaxial Translation

$$(\mathbf{R}|\mathbf{R})^{(p,p')}_{(coax)}(t) = (\mathbf{S}|\mathbf{S})^{(p,p')}_{(coax)}(t) = \mathbf{i}^{(p)}\underline{\mathbf{L}}^{(p)}\mathbf{W}\Lambda_r^{(p+p'-1)}(kt)\left(\underline{\mathbf{L}}^{(p')}\right)^T\overline{\mathbf{i}^{(p')}},$$

$$(\mathbf{S}|\mathbf{R})^{(p,p')}_{(coax)}(t) = \mathbf{i}^{(p)}\underline{\mathbf{L}}^{(p)}\mathbf{W}\Lambda_s^{(p+p'-1)}(kt)\left(\underline{\mathbf{L}}^{(p')}\right)^T\overline{\mathbf{i}^{(p')}}.$$

Legendre and transposed Legendre matrices

Diagonal matrices

Fast multiplication of the Legendre and transposed Legendre matrices can be performed via the forward and inverse FAST LEGENDRE TRANSFORM (FLT) with complexity $O(p^2\log^2 p)$

Healy et al *Advances in Computational Mathematics* **21**: 59-105, 2004.

Diagonalization of General Translation Operator

$$(\mathbf{E}|\mathbf{F})^{(p,p')}(\mathbf{t}) = \mathbf{i}^{(p)}\mathbf{Y}^{(p)}\mathbf{W}\Lambda^{(p+p'-1)}(\mathbf{t})\left(\overline{\mathbf{Y}^{(p')}}\right)^T\overline{\mathbf{i}^{(p')}}.$$

Matrices for the forward and inverse and Spherical Transform

Diagonal matrices

FAST SPHERICAL TRANSFORM (FST) can be performed with complexity $O(p^2\log^2 p)$

Healy et al *Advances in Computational Mathematics* **21**: 59-105, 2004.

Method of Signature Function (Diagonal Forms of the Translation Operator)

$$\psi(\mathbf{r}) = \frac{1}{4\pi}\int_{S_u} e^{ik\mathbf{s}\cdot\mathbf{r}}\Psi(\mathbf{s})\mathrm{d}S(\mathbf{s}),$$ Regular Solution

$$\psi^{(p)}(\mathbf{r}) = \frac{1}{4\pi}\int_{S_u} \Lambda_s^{(p)}(\mathbf{r};\mathbf{s})\Psi(\mathbf{s})\mathrm{d}S(\mathbf{s}),$$ Singular Solution

$$\Lambda_r(\mathbf{r};\mathbf{s}) = \sum_{n=0}^{\infty}(2n+1)i^n j_n(kr)P_n\left(\frac{\mathbf{r}\cdot\mathbf{s}}{r}\right)$$

$$\Lambda_s^{(p)}(\mathbf{r};\mathbf{s}) = \sum_{n=0}^{p-1}(2n+1)i^n h_n(kr)P_n\left(\frac{\mathbf{r}\cdot\mathbf{s}}{r}\right).$$

$$\widehat{\Psi}(\mathbf{s}) = (\mathcal{S}|\mathcal{S})(\mathbf{t})[\Psi(\mathbf{s})] = (\mathcal{R}|\mathcal{R})(\mathbf{t})[\Psi(\mathbf{s})] = e^{ik\mathbf{s}\cdot\mathbf{t}}\Psi(\mathbf{s}),$$

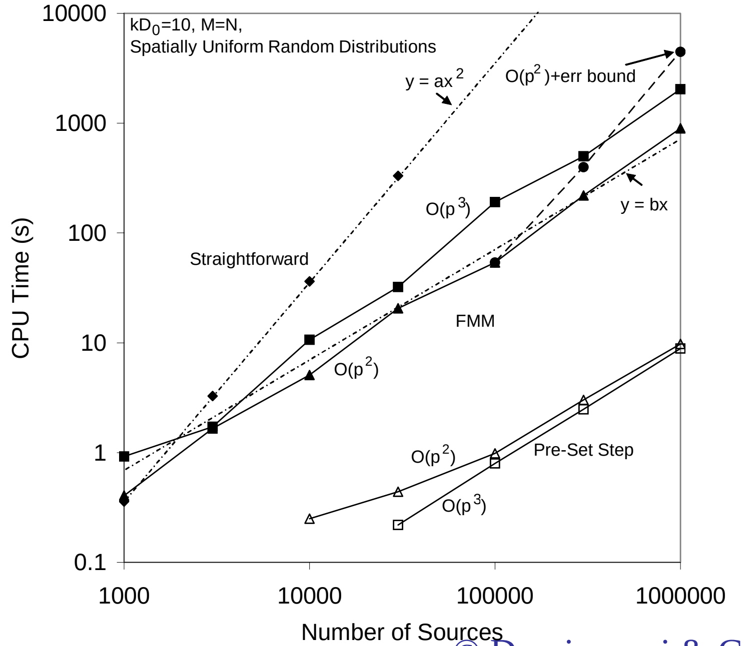
$$\widehat{\Psi}_{(p)}(\mathbf{s}) = (\mathcal{S}|\mathcal{R})(\mathbf{t})[\Psi(\mathbf{s})] = \Lambda_s^{(p)}(\mathbf{t};\mathbf{s})\Psi(\mathbf{s}).$$

Final Summation and Initial Expansion

$$\psi(\mathbf{r}) = \frac{1}{4\pi}\sum_{j=0}^{N_c-1} w_j e^{ik\mathbf{s}_j\cdot\mathbf{r}}\Psi(\mathbf{s}_j) + \epsilon_c, \quad \mathbf{s}_j \in S_u,$$

$$G(\mathbf{r}-\mathbf{r}_s) \rightleftarrows \Psi_{(0)}(\mathbf{s}_j;\mathbf{r}_s-\mathbf{r}_*) = \frac{ik}{4\pi}e^{-ik\mathbf{s}_j\cdot(\mathbf{r}_s-\mathbf{r}_*)}$$

The FMM with Band-Unlimited Signature Functions (O(p^2) method)



Deficiencies

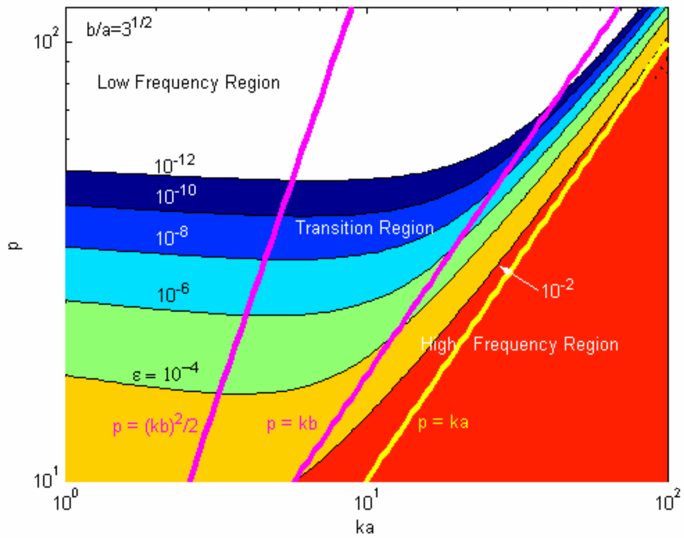
- Low Frequencies;
- High Frequencies;
- Constant p;
- Instabilities after two or three levels of translations.

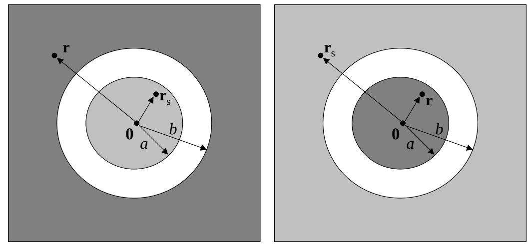
Methods to Fix:

- Use of Band-limited functions;
- Error control via band-limits;
- Requires filtering procedures (complexity $O(p^2 \log^2 p)$ or $O(p^2 \log p)$) with large asymptotic constants;
- The length of the representation is changed via interpolation/anterpolation procedures.

Error Bounds

Source Expansion Errors





Low frequencies:

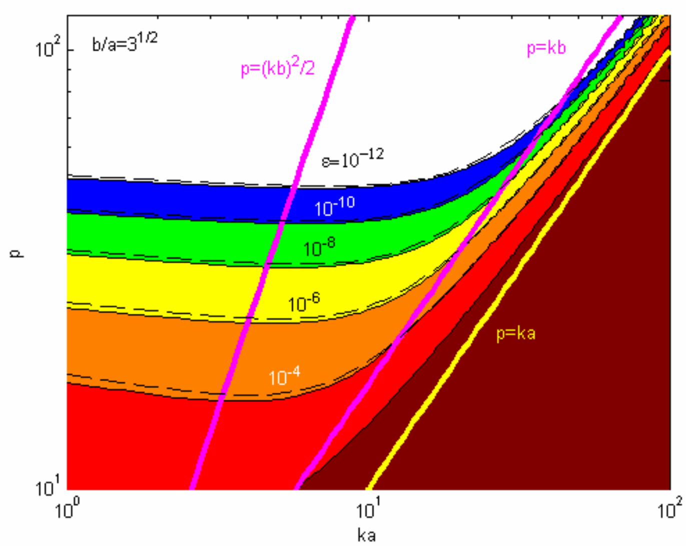
$$p = -\frac{\ln\left[\epsilon ka(1-\sigma^{-1})^{3/2}\right]}{\ln\sigma} - 1.$$

High frequencies:

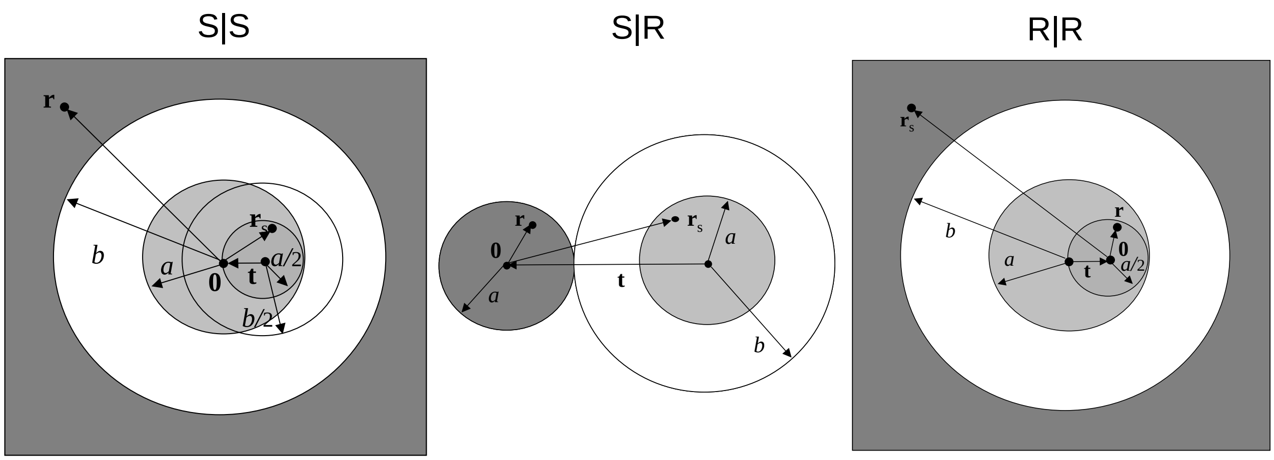
$$p = ka + \frac{1}{2}\left(3\ln\frac{1}{\epsilon\sigma}\right)^{2/3}(ka)^{1/3}.$$

Approximation of the Error

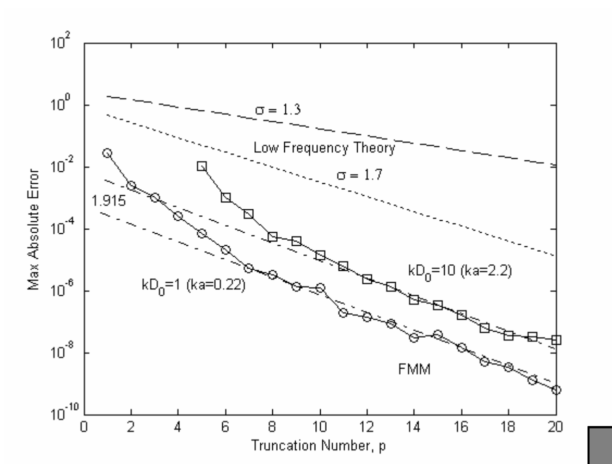
$$p = \left\{\left[\frac{1}{\ln\sigma}\ln\frac{1}{\epsilon ka(1-\sigma^{-1})^{3/2}} + 1\right]^4 + \left[ka + \frac{1}{2}\left(3\ln\frac{1}{\epsilon\sigma}\right)^{2/3}(ka)^{1/3}\right]^4\right\}^{1/4}$$

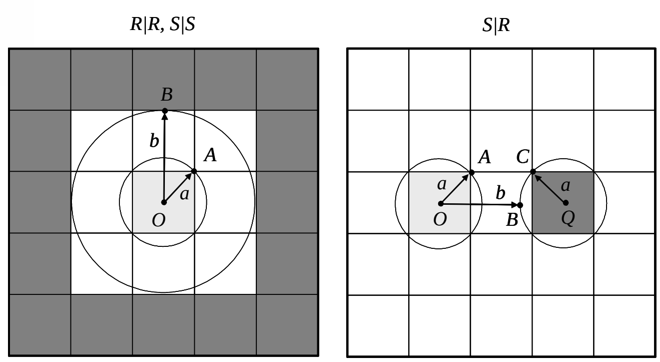


We proved that for source summation problems the truncation numbers can be selected based on the above chart when using translations with rectangularly truncated matrices

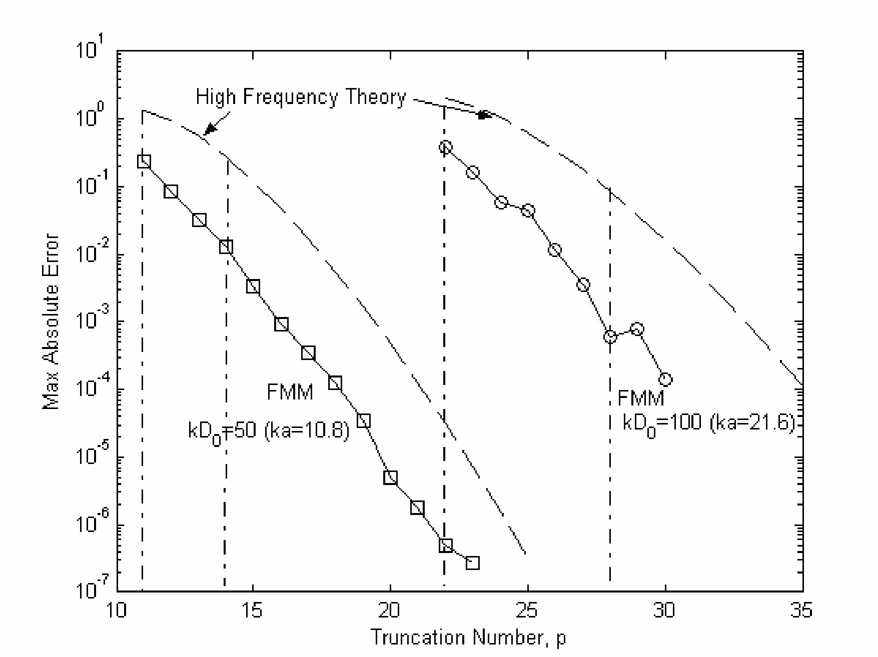


Low Frequency FMM Error





High Frequency FMM Error

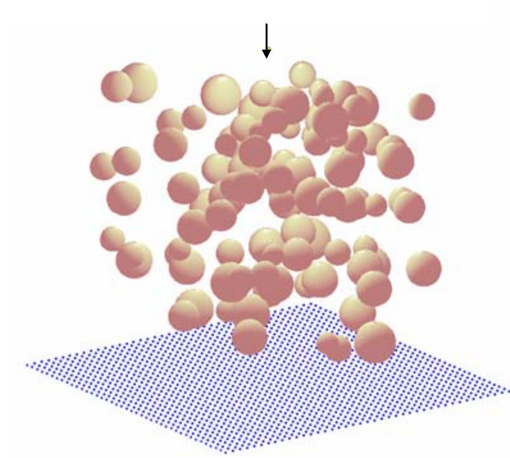


Multiple Scattering Problem

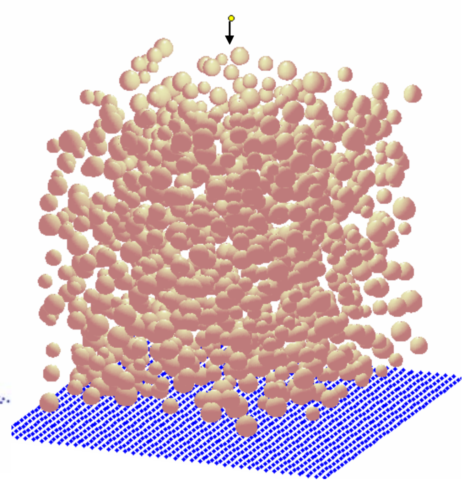
Problem

Boundary Conditions:

$$|\mathbf{r}-\mathbf{r}'_q| = a_q \; : \quad \frac{\partial \psi(\mathbf{r})}{\partial n_q} + i\sigma_q \psi(\mathbf{r}) = 0, \quad q = 1,\ldots,N.$$



100 random spheres



1000 random spheres

Scattered Field Decomposition

$$\psi_{scat}(\mathbf{r}) = \sum_{p=1}^{N} \psi_p(\mathbf{r}), \quad \lim_{r\to\infty} r\left(\frac{\partial \psi_p}{\partial r} - ik\psi_p\right) = 0, \quad p = 1,\ldots,N.$$

Singular Basis Functions — Hankel Functions

$$\psi_p(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_n^{(p)m} S_n^m(\mathbf{r}-\mathbf{r}'_p), \qquad S_n^m(\mathbf{r}) = h_n(kr) Y_n^m(\theta,\varphi).$$

Expansion Coefficients — Spherical Harmonics

$$\mathbf{A} = \left(A_0^0, A_1^{-1}, A_1^0, A_1^1, A_2^{-2}, A_2^{-1}, A_2^0, A_2^1, A_2^2, \ldots\right)^T,$$

Vector Form:

$$\psi_p(\mathbf{r}) = \overline{\mathbf{A}^{(p)}} \cdot \mathbf{S}(\mathbf{r}-\mathbf{r}'_p).$$

dot product

Solution of Multiple Scattering Problem

"Effective" Incident Field

$$\psi(\mathbf{r}) = \psi_q(\mathbf{r}) + \psi_{in}(\mathbf{r}) + \psi^{(q)}_{other}(\mathbf{r}) = \psi_q(\mathbf{r}) + \psi^{(q)(in)}_{eff}(\mathbf{r}),$$

$$\psi^{(q)}_{other}(\mathbf{r}) = \sum_{p \neq q} \overline{\mathbf{A}^{(p)}} \cdot \mathbf{S}(\mathbf{r}_p) = \overline{\mathbf{B}^{(q)}} \cdot \mathbf{R}(\mathbf{r}_q), \quad \psi^{(q)(in)}_{eff}(\mathbf{r}) = \overline{\mathbf{E}^{(q)}_{eff}} \cdot \mathbf{R}(\mathbf{r}_q).$$

Coupled System of Equations:

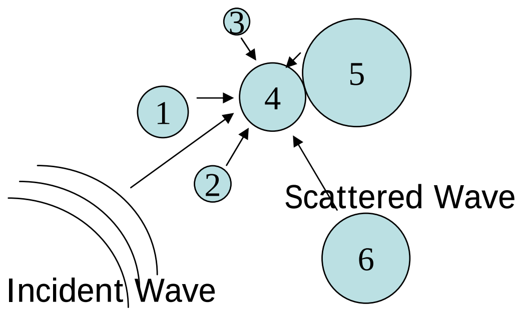
$$\mathbf{A}^{(q)} = \mathbf{T}^{(q)}\mathbf{E}^{(q)}_{eff},$$

$$\mathbf{B}^{(q)} = \sum_{p \neq q} (\mathbf{S}|\mathbf{R})(\mathbf{r}'_q - \mathbf{r}'_p)\mathbf{A}^{(p)},$$

$$\mathbf{E}^{(q)}_{eff} = \mathbf{E}^{(in)}(\mathbf{r}'_q) + \mathbf{B}^{(q)},$$

$$q = 1, \ldots, N$$

(S|R)-Translation Matrix



Reflection Method & Krylov Subspace Method (GMRES)

Reflection (Simple Iteration) Method:

$$\mathbf{A}^{(q)}_j = \mathbf{T}^{(q)}\left[\mathbf{E}^{(in)}(\mathbf{r}'_q) + \mathbf{B}^{(q)}_j\right],$$

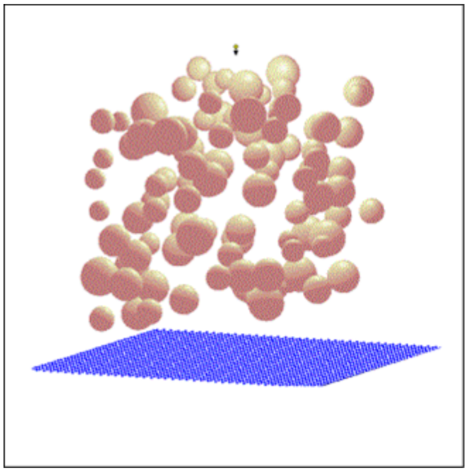
$$\mathbf{B}^{(q)}_{j+1} = \sum_{p \neq q} (\mathbf{S}|\mathbf{R})(\mathbf{r}'_q - \mathbf{r}'_p)\mathbf{A}^{(p)}_j,$$

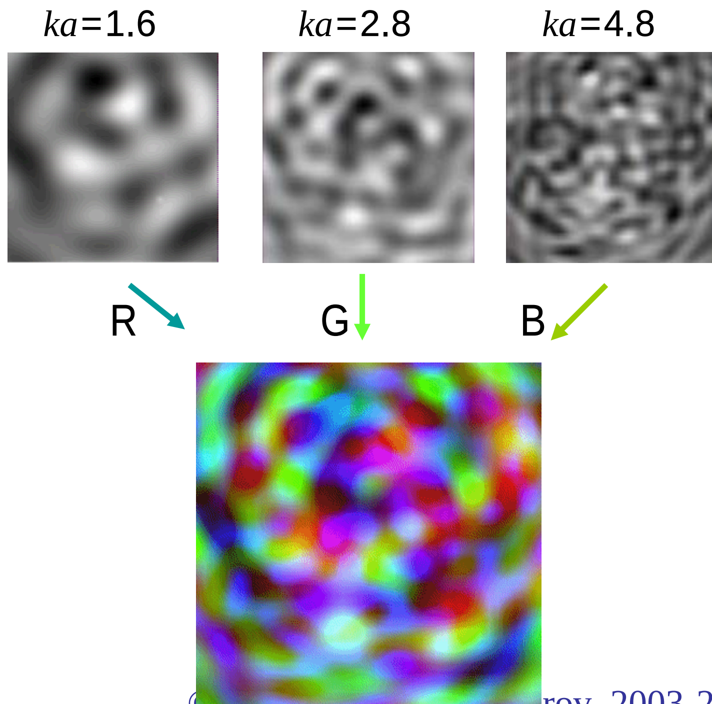
$$\left|\mathbf{A}^{(q)}_j - \mathbf{A}^{(q)}_{j+1}\right| < \epsilon, \quad q = 1, \ldots, N.$$

General Formulation (used in GMRES)

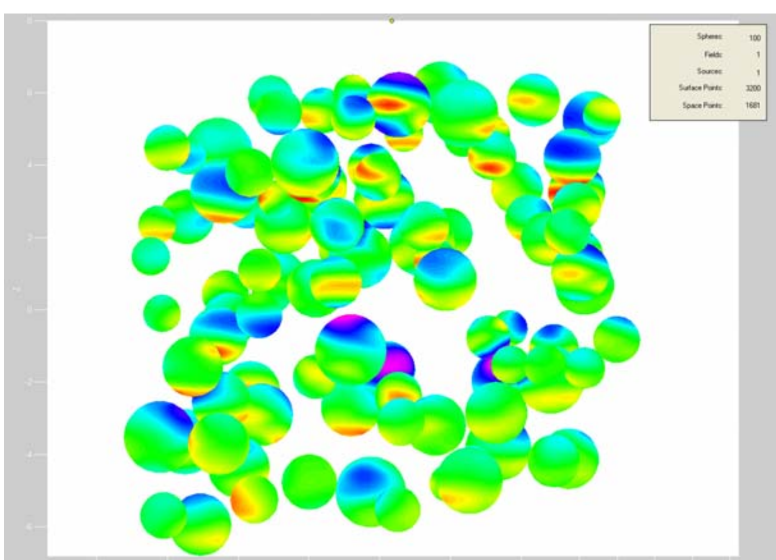
$$\left[\mathbf{I} - \mathbf{T}^{(q)} \sum_{p \neq q} (\mathbf{S}|\mathbf{R})(\mathbf{r}'_q - \mathbf{r}'_p)\right]\mathbf{A}^{(q)} = \mathbf{T}^{(q)}\mathbf{E}^{(in)}(\mathbf{r}'_q).$$

100 random spheres (MLFMM)

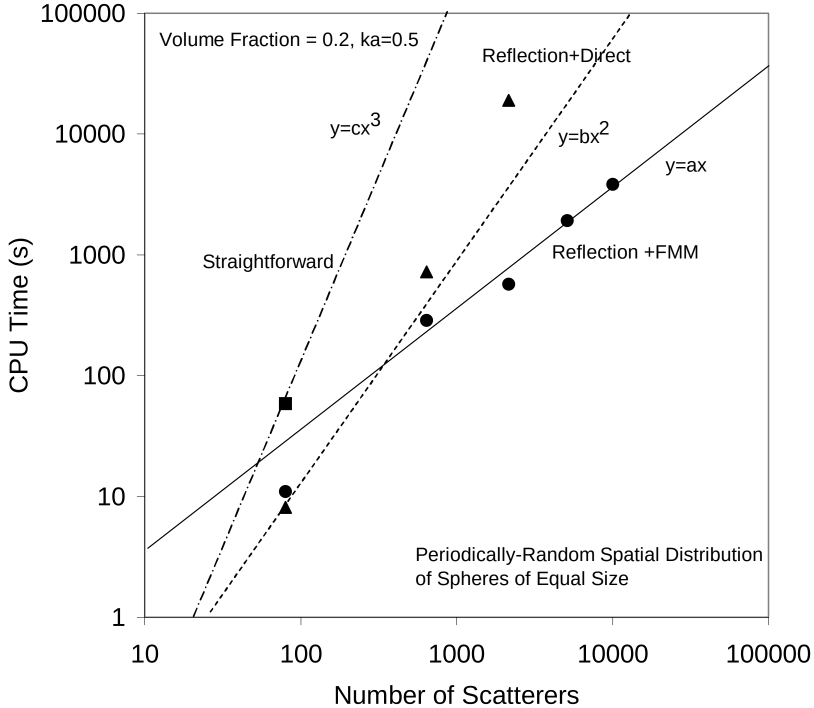




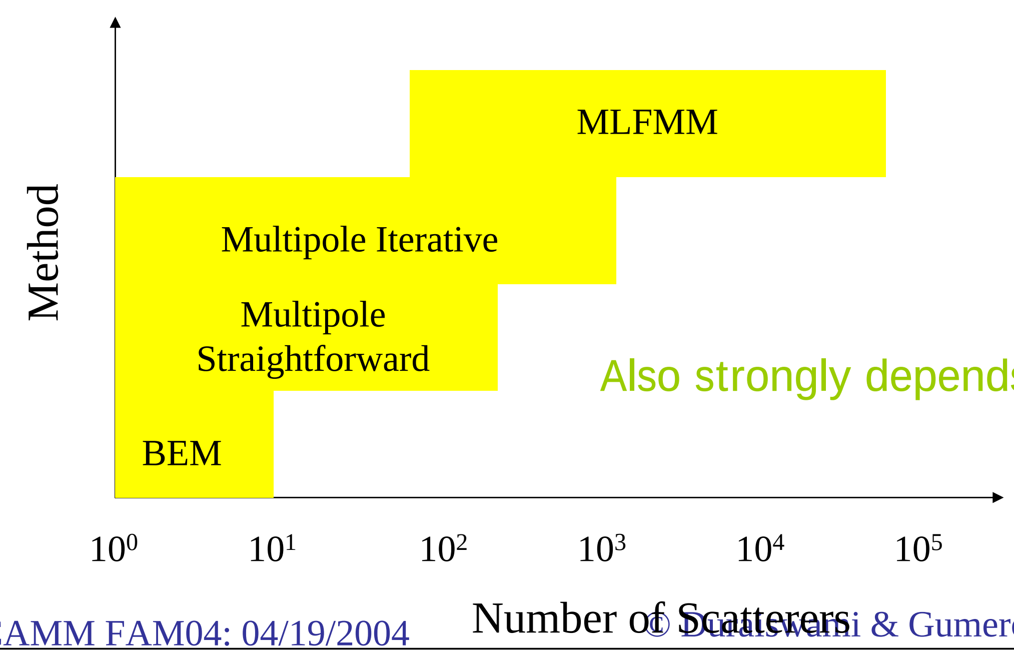
Surface Potential Imaging



Performance Test



Computable Problems on Desktop PC



Also strongly depends on ka !

More About This Problem
in Our Talk Next Week