High Frequency Scattering by Convex Polygons

Stephen Langdon

University of Reading, UK

Joint work with: **Simon Chandler-Wilde, Steve Arden**

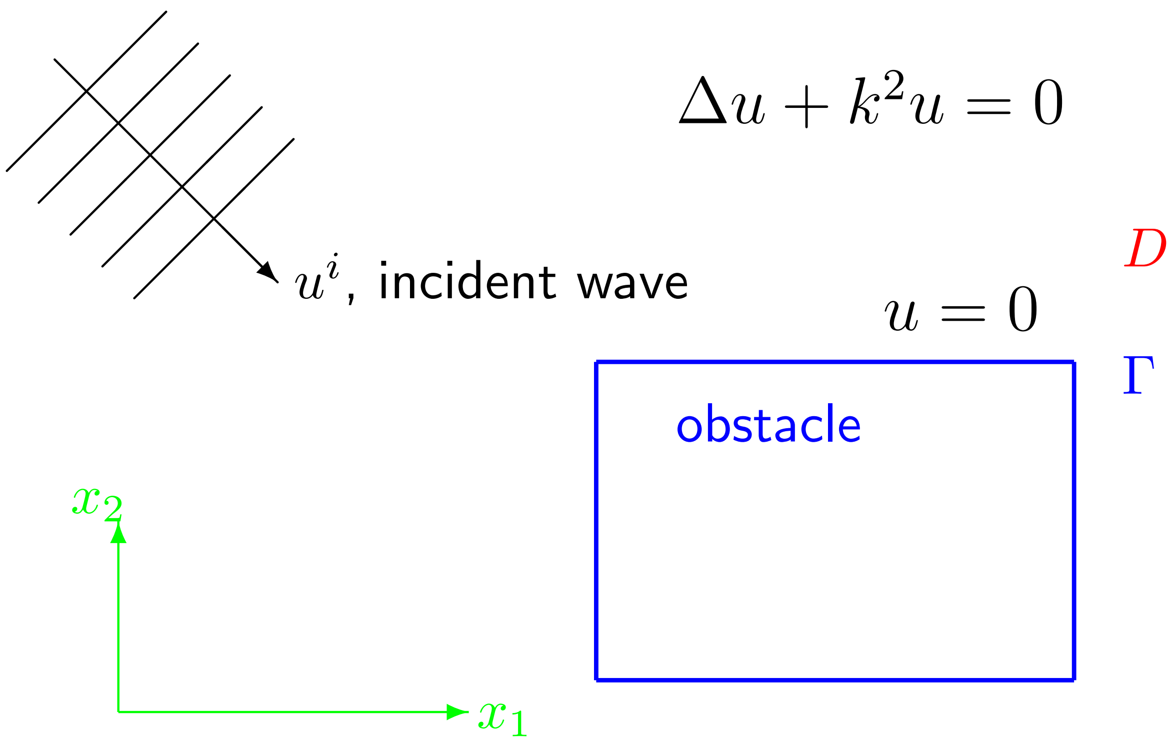
Funded by: Leverhulme Trust

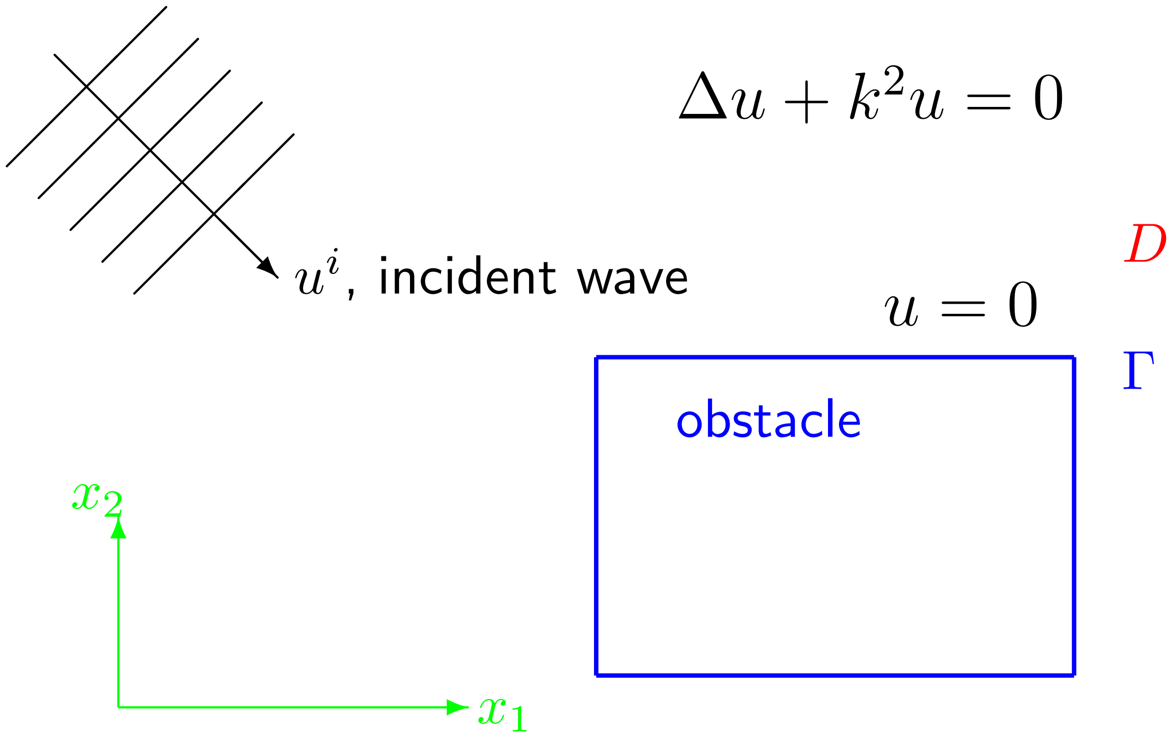
University of Maryland, September 2005

Contents

- The scattering problem
- The Burton & Miller integral equation
- Understanding solution behaviour
- Approximating the solution efficiently
  - Galerkin method
  - Collocation method
- Numerical Results

The Scattering Problem

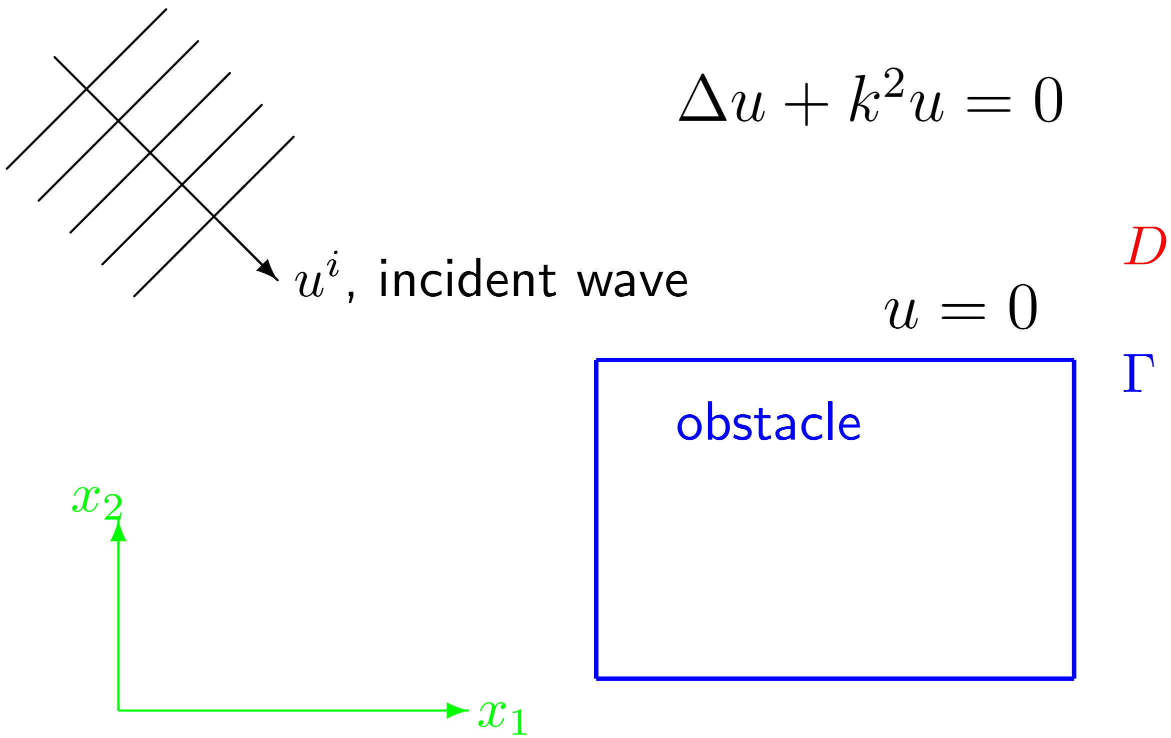




Green's representation theorem:

$$u(x) = u^i(x) - \int_\Gamma \Phi(x,y)\frac{\partial u}{\partial n}(y)ds(y), \quad x \in D,$$

where $\Phi(x,y) := \frac{\mathrm{i}}{4}H_0^{(1)}(k|x-y|)$.

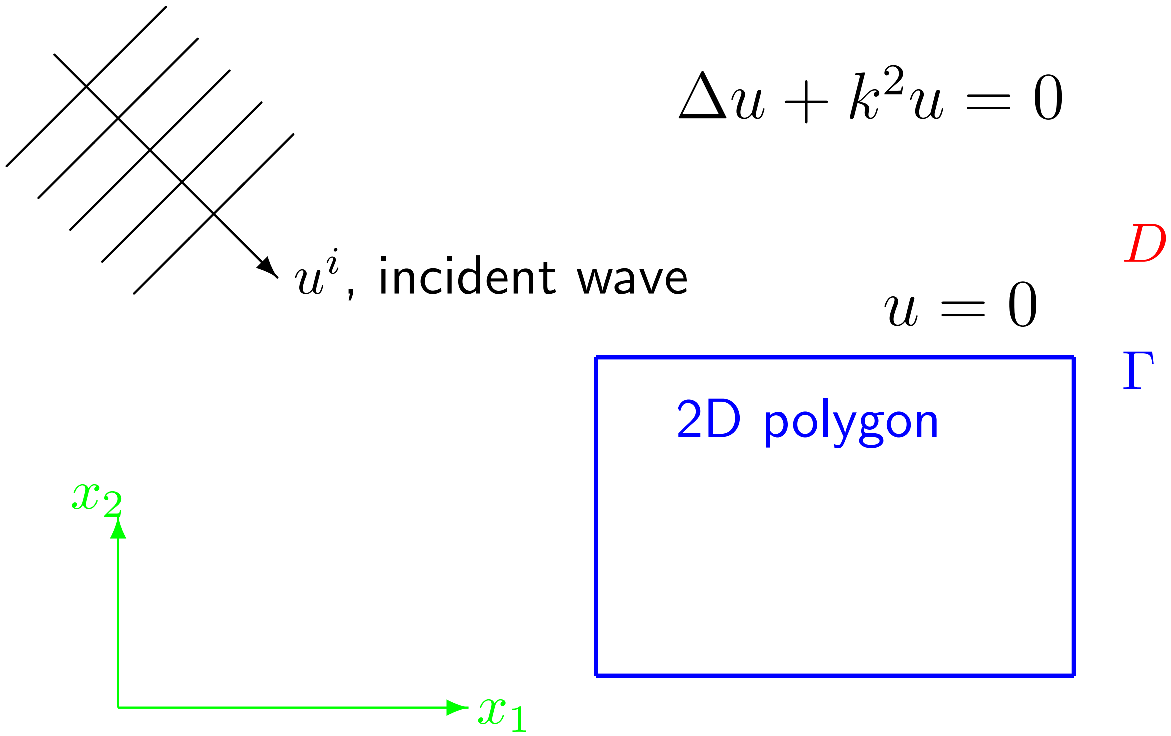


From Green's representation theorem (Burton & Miller 1971):

$$\frac{1}{2}\frac{\partial u}{\partial n}(x) + \int_\Gamma \left(\frac{\partial \Phi(x,y)}{\partial n(x)} + \mathrm{i}\eta\Phi(x,y) \right) \frac{\partial u}{\partial n}(y)ds(y) = f(x), \quad x \in \Gamma,$$

where

$$f(x) := \frac{\partial u^i}{\partial n}(x) + \mathrm{i}\eta u^i(x).$$



From Green's representation theorem:

$$\frac{1}{2}\frac{\partial u}{\partial n}(x) + \int_\Gamma \left(\frac{\partial \Phi(x,y)}{\partial n(x)} + \mathrm{i}\eta\Phi(x,y) \right) \frac{\partial u}{\partial n}(y) ds(y) = f(x), \quad x \in \Gamma.$$

Theorem (follows from Burton & Miller 1971, Selepov 1969) If $\eta \in \mathbb{R}$, $\eta \neq 0$, then this integral equation is uniquely solvable in $L^2(\Gamma)$.

$$\frac{1}{2}\frac{\partial u}{\partial n}(x) + \int_\Gamma \left(\frac{\partial \Phi(x,y)}{\partial n(x)} + \mathrm{i}\eta\Phi(x,y) \right) \frac{\partial u}{\partial n}(y) ds(y) = f(x), \quad x \in \Gamma.$$

Conventional BEM: Apply a Galerkin method, approximating $\partial u/\partial n$ by a piecewise polynomial of degree P, leading to a linear system to solve with N degrees of freedom. **Problem:** N of order of kL, where L is linear dimension, so cost is $O(N^2)$ to compute full matrix and apply iterative solver ... or close to $O(N)$ if a fast multipole method (e.g. Amini & Profit 2003, Darve 2004) is used.

This is **fantastic** but still infeasible as $kL \to \infty$.

$$\frac{1}{2}\frac{\partial u}{\partial n}(x) + \int_\Gamma \left(\frac{\partial \Phi(x,y)}{\partial n(x)} + \mathrm{i}\eta\Phi(x,y) \right) \frac{\partial u}{\partial n}(y) ds(y) = f(x), \quad x \in \Gamma.$$

Alternative: Reduce N by using new basis functions, e.g.

(i) approximate $\partial u/\partial n$ by taking a large number of plane waves and multiplying these by conventional piecewise polynomial basis functions (Perrey-Debain et al. 2003, 2004). **This is very successful (in 2D, 3D, for acoustic/elastic waves and Neumann/impedance b.c.s), reducing number of degrees of freedom per wavelength from e.g. 6-10 to close to 2.** However N still increases proportional to kL.

$$\frac{1}{2}\frac{\partial u}{\partial n}(x) + \int_\Gamma \left(\frac{\partial \Phi(x,y)}{\partial n(x)} + \mathrm{i}\eta\Phi(x,y) \right) \frac{\partial u}{\partial n}(y) ds(y) = f(x), \quad x \in \Gamma.$$

Alternative: Reduce N by using new basis functions, e.g.

(ii) for convex scatterers, remove some of the oscillation by factoring out the oscillation of the incident wave, i.e. writing

$$\frac{\partial u}{\partial n}(y) = \frac{\partial u^i}{\partial n}(y) \times F(y)$$

and approximating F by a conventional BEM (e.g. Abboud et al. 1994, Darrigrand 2002, Bruno et al 2004).

Alternative: Reduce N by using new basis functions, e.g.

(ii) for convex scatterers, remove some of the oscillation by factoring out the oscillation of the incident wave, i.e. writing

$$\frac{\partial u}{\partial n}(y) = \frac{\partial u^i}{\partial n}(y) \times F(y) \qquad (*)$$

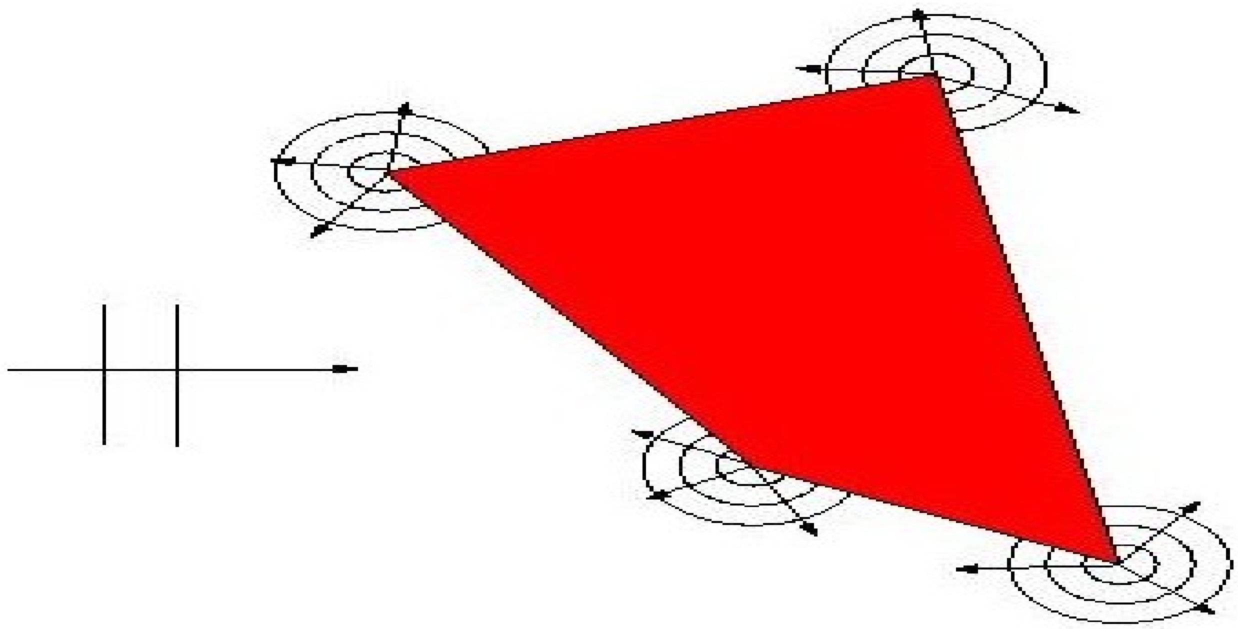
and approximating F by a conventional BEM.

For **smooth obstacles** this works well: equation $(*)$ holds with $F(y) \approx 2$ on the illuminated side (physical optics) and $F(y) \approx 0$ in the shadow zone.

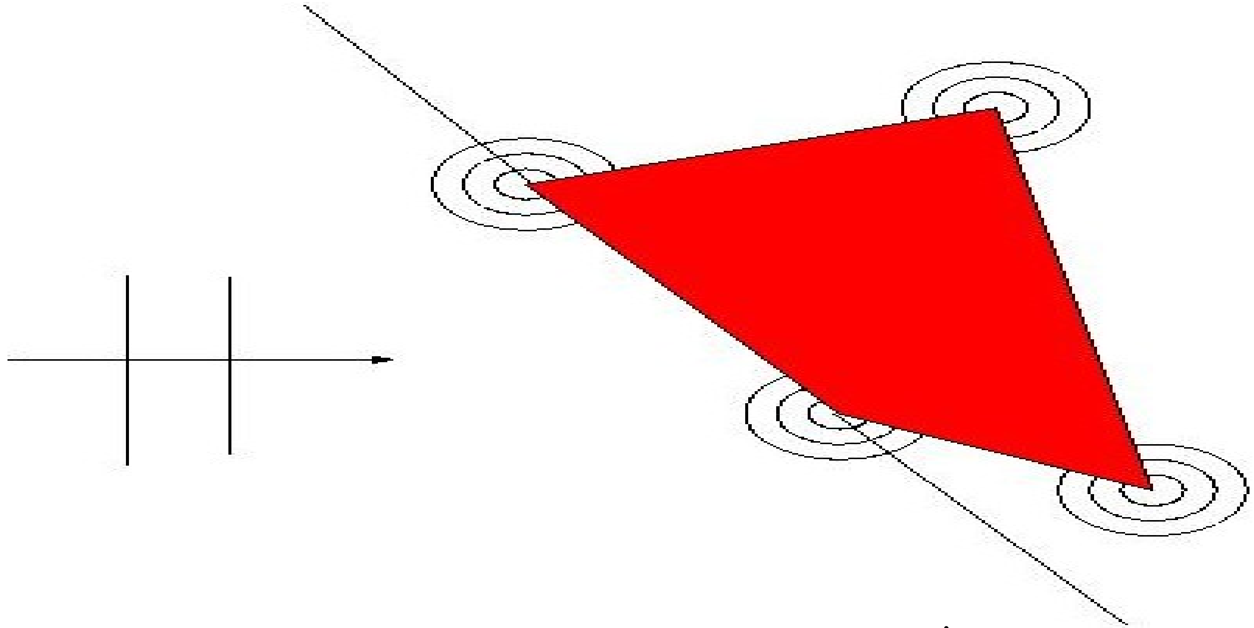
(ii) for convex scatterers, remove some of the oscillation by factoring out the oscillation of the incident wave, i.e. writing

$$\frac{\partial u}{\partial n}(y) = \frac{\partial u^i}{\partial n}(y) \times F(y) \qquad (*)$$

and approximating F by a conventional BEM. Not very effective for non-smooth scatterers.



Understanding solution behaviour



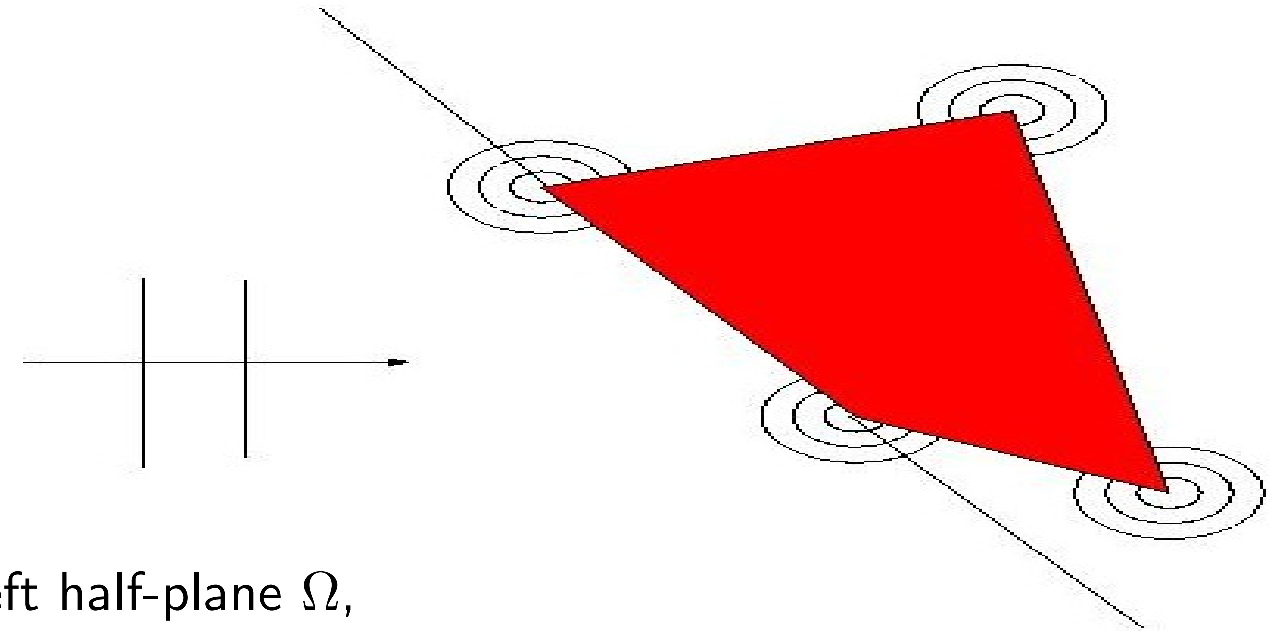
Let

$$G(x, y) := \Phi(x, y) - \Phi(x, y')$$

be the Dirichlet Green function for the left half-plane Ω. By Green's representation theorem,

$$u(x) = u^i(x) + u^r(x) + \int_{\partial\Omega\setminus\Gamma} \frac{\partial G(x, y)}{\partial n(y)} u(y) ds(y), \quad x \in \Omega.$$

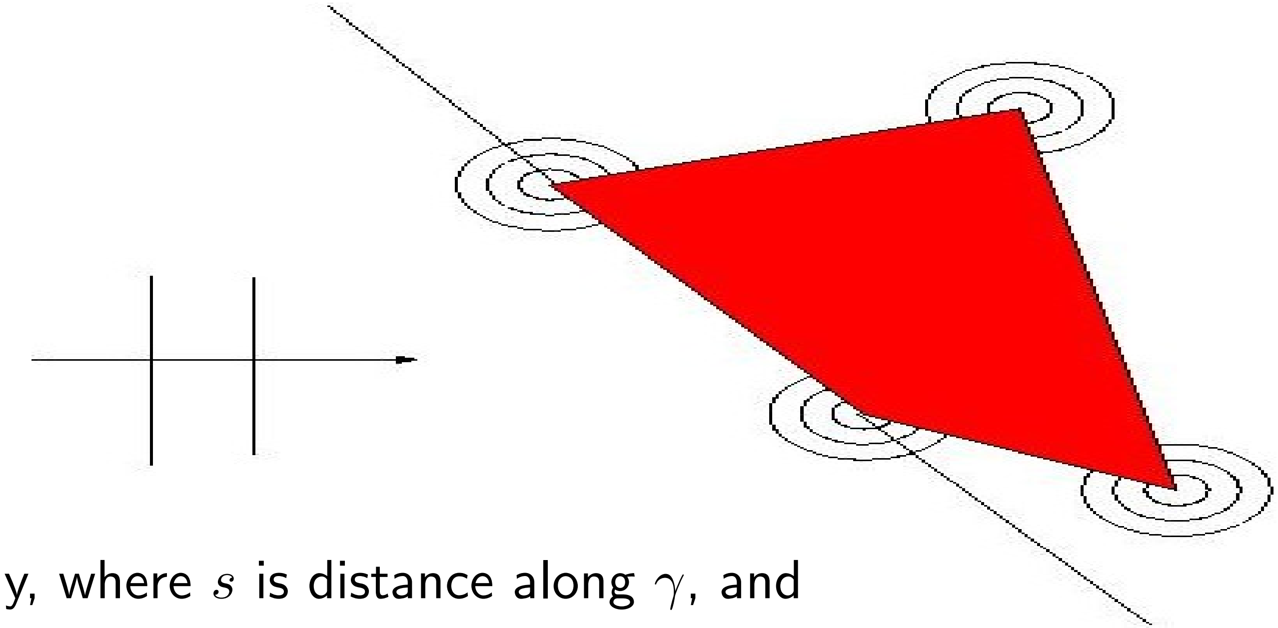
Understanding solution behaviour



In the left half-plane Ω,

$$u(x) = u^i(x) + u^r(x) + \int_{\partial\Omega\setminus\Gamma} \frac{\partial G(x,y)}{\partial n(y)} u(y) ds(y)$$

$$\Rightarrow \frac{\partial u}{\partial n}(x) = 2\frac{\partial u^i}{\partial n}(x) + 2\int_{\partial\Omega\setminus\Gamma} \frac{\partial^2 \Phi(x,y)}{\partial n(x)\partial n(y)} u(y) ds(y), \quad x \in \gamma = \partial\Omega \cap \Gamma.$$



Explicitly, where s is distance along γ, and
$\phi(s)$ and $\psi(s)$ are $k^{-1}\partial u/\partial n$ and u, at distance s along γ,

$$\phi(s) \quad = \quad P.O. + \frac{\mathrm{i}}{2}\left[\mathrm{e}^{\mathrm{i}ks}v_+(s) + \mathrm{e}^{-\mathrm{i}ks}v_-(s)\right]$$

where

$$v_+(s) := k\int_{-\infty}^{0} F\big(k(s-s_0)\big)\mathrm{e}^{-\mathrm{i}ks_0}\psi(s_0)ds_0.$$

and $F(z) := \mathrm{e}^{-\mathrm{i}z}H_1^{(1)}(z)/z$

$$\phi(s) \quad = \quad P.O. + \frac{\mathrm{i}}{2}\left[\mathrm{e}^{\mathrm{i}ks}v_+(s) + \mathrm{e}^{-\mathrm{i}ks}v_-(s)\right]$$

where

$$v_+(s) := k\int_{-\infty}^{0} F\big(k(s-s_0)\big)\mathrm{e}^{-\mathrm{i}ks_0}\psi(s_0)ds_0.$$

Now $F(z) := \mathrm{e}^{-\mathrm{i}z}H_1^{(1)}(z)/z$ which is non-oscillatory, in that

$$F^{(n)}(z) = O(z^{-3/2-n}) \text{ as } z \to \infty.$$

$$\phi(s) \quad = \quad P.O. + \frac{\mathrm{i}}{2}\left[\mathrm{e}^{\mathrm{i}ks} v_+(s) + \mathrm{e}^{-\mathrm{i}ks} v_-(s)\right]$$

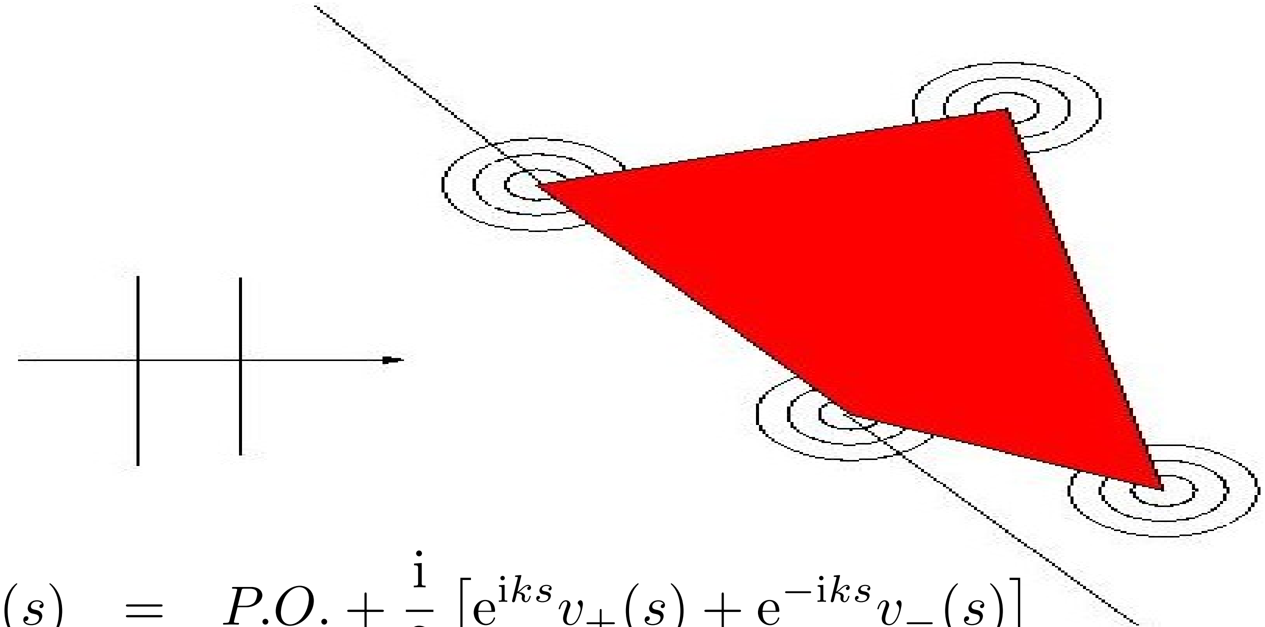
where

$$v_+(s) := k \int_{-\infty}^{0} F\big(k(s-s_0)\big)\mathrm{e}^{-\mathrm{i}ks_0}\psi(s_0)ds_0.$$

Now $F(z) := \mathrm{e}^{-\mathrm{i}z} H_1^{(1)}(z)/z$ which is non-oscillatory, in that

$$F^{(n)}(z) = O(z^{-3/2-n}) \text{ as } z \to \infty.$$

$$\Rightarrow v_+^{(n)}(s) = O(k^n (ks)^{-1/2-n}) \text{ as } ks \to \infty.$$



$$\phi(s) \quad = \quad P.O. + \frac{\mathrm{i}}{2}\left[\mathrm{e}^{\mathrm{i}ks}v_+(s) + \mathrm{e}^{-\mathrm{i}ks}v_-(s)\right]$$

where
$$k^{-n}|v_+^{(n)}(s)| = O\left((ks)^{-1/2-n}\right) \text{ as } ks \to \infty$$

and (by separation of variables local to the corner),

$$k^{-n}|v_+^{(n)}(s)| = O\left((ks)^{-\alpha-n}\right) \text{ as } ks \to 0,$$

where $\alpha < 1/2$ depends on the corner angle.

$$\phi(s) \quad = \quad P.O. + \frac{\mathrm{i}}{2}\left[\mathrm{e}^{\mathrm{i}ks} v_+(s) + \mathrm{e}^{-\mathrm{i}ks} v_-(s)\right]$$

where

$$k^{-n}|v_+^{(n)}(s)| = \begin{cases} O\left((ks)^{-1/2-n}\right) & \text{as } ks \to \infty \\ O\left((ks)^{-\alpha-n}\right) & \text{as } ks \to 0, \end{cases}$$

where $\alpha < 1/2$ depends on the corner angle.

Thus approximate

$$\phi(s) \quad \approx \quad P.O. + \frac{\mathrm{i}}{2}\left[\mathrm{e}^{\mathrm{i}ks} V_+(s) + \mathrm{e}^{-\mathrm{i}ks} V_-(s)\right],$$

where V_+ and V_- are piecewise polynomials on graded meshes.

Thus approximate

$$\phi(s) \quad \approx \quad P.O. + \frac{\mathrm{i}}{2}\left[\mathrm{e}^{\mathrm{i}ks}V_+(s) + \mathrm{e}^{-\mathrm{i}ks}V_-(s)\right],$$

where V_+ and V_- are piecewise polynomials on graded meshes.

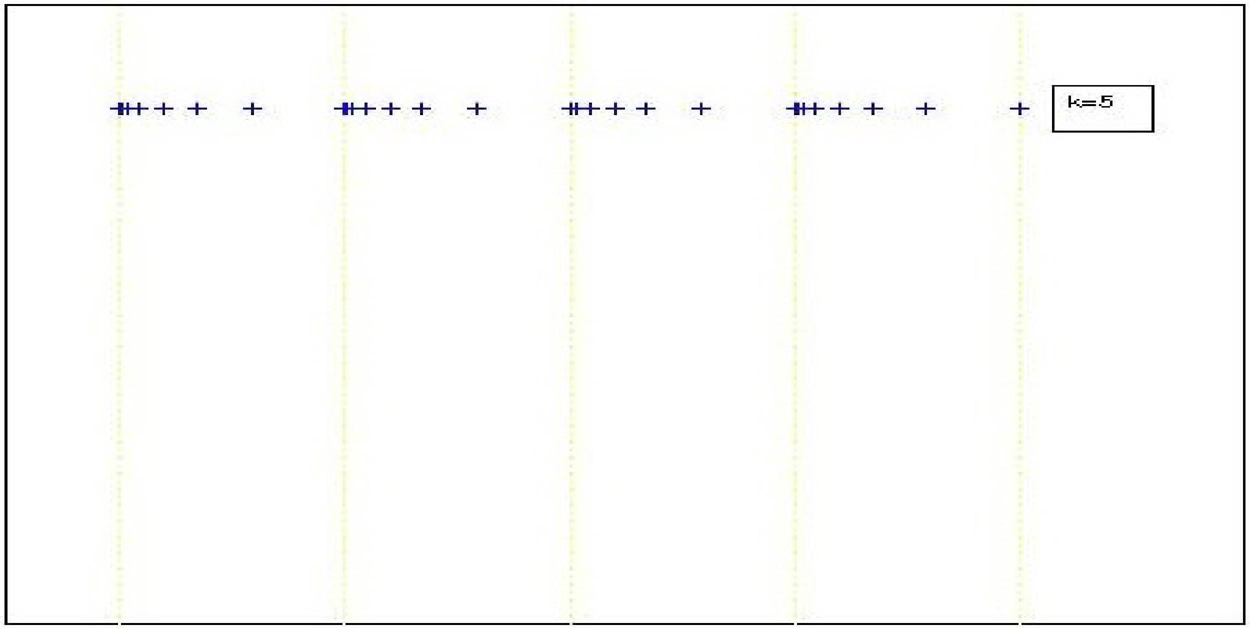


Figure 1: Scattering by a square

Thus approximate

$$\phi(s) \quad \approx \quad P.O. + \frac{\mathrm{i}}{2}\left[\mathrm{e}^{\mathrm{i}ks}V_+(s) + \mathrm{e}^{-\mathrm{i}ks}V_-(s)\right],$$

where V_+ and V_- are piecewise polynomials on graded meshes.

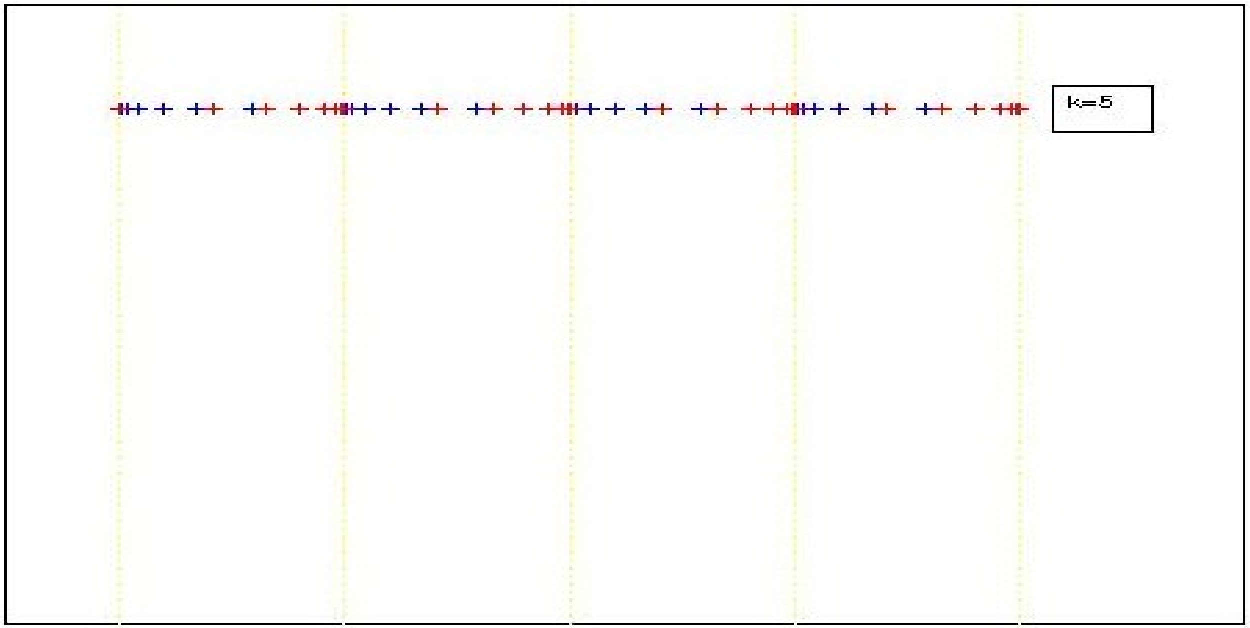


Figure 2: Scattering by a square

Thus approximate

$$\phi(s) \quad \approx \quad P.O. + \frac{\mathrm{i}}{2}\left[\mathrm{e}^{\mathrm{i}ks}V_+(s) + \mathrm{e}^{-\mathrm{i}ks}V_-(s)\right],$$

where V_+ and V_- are piecewise polynomials on graded meshes.

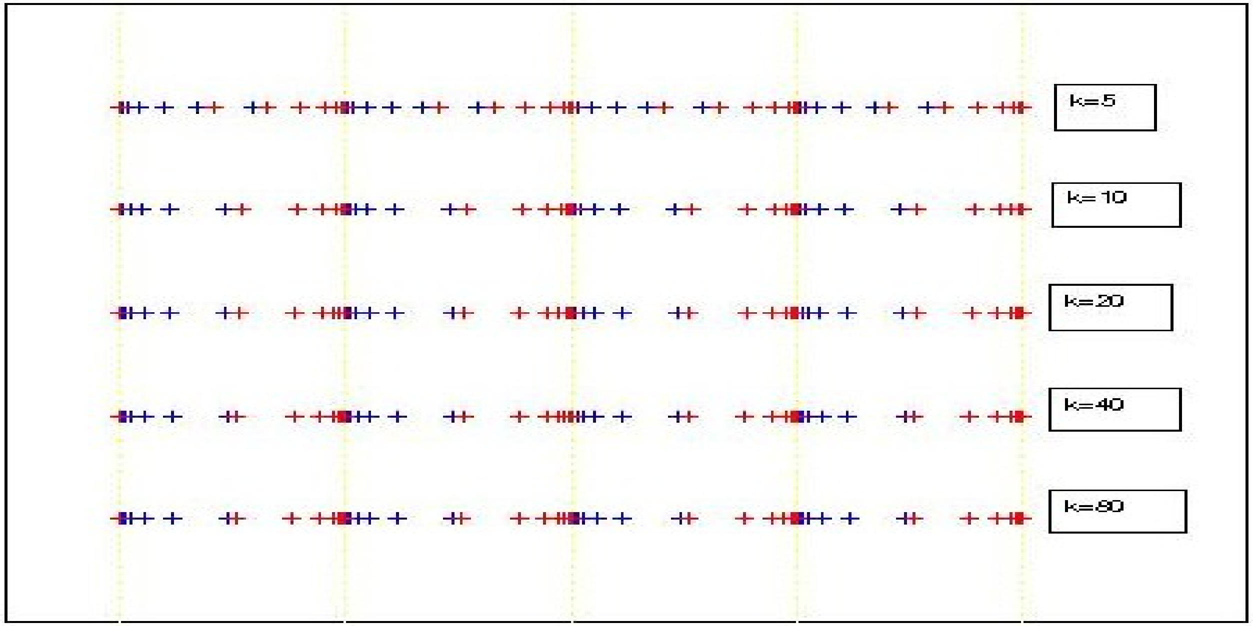


Figure 3: Scattering by a square

Approximation error

Theorem: If V_+ is the best L_2 approximation from the approximation space, then

$$k^{1/2}\|v_+ - V_+\|_2 \leq C_p \frac{n^{1/2}(1 + \log^{1/2}(kL))}{N^{p+1}},$$

where

- $N \propto$ degrees of freedom
- $p =$ polynomial degree
- $L = \max$ side length
- $n =$ number of sides of polygon

Boundary integral equation method

Integral equation in parametric form

$$\varphi(s) + \mathcal{K}\varphi(s) = F(s),$$

where

$$\varphi(s) := \frac{1}{k}\frac{\partial u}{\partial n}(x(s)) - P.O..$$

Theorem. The operator $(I + \mathcal{K}) : L_2(\Gamma) \mapsto L_2(\Gamma)$ is bijective with bounded inverse

$$\|(I + \mathcal{K})^{-1}\|_2 \leq C,$$

so that the integral equation has exactly one solution.

Boundary integral equation method

Integral equation in parametric form

$$\varphi(s) + \mathcal{K}\varphi(s) = F(s),$$

where

$$\varphi(s) := \frac{1}{k}\frac{\partial u}{\partial n}(x(s)) - P.O..$$

Difficulty 1 The operator $(I + \mathcal{K}) : L_2(\Gamma) \mapsto L_2(\Gamma)$ is bijective with bounded inverse

$$\|(I + \mathcal{K})^{-1}\|_2 \leq C(k),$$

where the dependence of $C(k)$ on k is not clear.

Approximation space: seek

$$\varphi_N(s) = \sum_{j=1}^{M} v_j \rho_j(s) \in V_N,$$

where

$$\rho_j(s) := \mathrm{e}^{\pm \mathrm{i} k s} \times \text{piecewise polynomial supported on graded mesh.}$$

Question: how do we compute v_j?

Galerkin method

To solve

$$\varphi(s) + \mathcal{K}\varphi(s) = F(s),$$

seek $\varphi_{N_G} \in V_N$ such that

$$(I + P_{N_G}\mathcal{K})\varphi_{N_G} = P_{N_G} F,$$

where P_{N_G} is the orthogonal projection onto the approximation space. Equivalently

$$(\varphi_{N_G}, \rho) + (\mathcal{K}\varphi_{N_G}, \rho) = (F, \rho), \quad \forall \rho \in V_N,$$

$$\Rightarrow \sum_{j=1}^{M} v_j [(\rho_j, \rho_m) + (\mathcal{K}\rho_j, \rho_m)] = (F, \rho_m).$$

If ρ_j, ρ_m supported on same side of polygon, integrals not oscillatory.

Galerkin method

Theorem. For $N \geq N^*$, the operator $(I + P_{N_G}\mathcal{K}) : L_2(\Gamma) \mapsto V_N$ is bijective with bounded inverse

$$\|(I + P_{N_G}\mathcal{K})^{-1}\|_2 \leq C_s.$$

Galerkin method

Difficulty 2. For $N \geq N^*(k)$, the operator $(I + P_{N_G}\mathcal{K}) : L_2(\Gamma) \mapsto V_N$ is bijective with bounded inverse

$$\|(I + P_{N_G}\mathcal{K})^{-1}\|_2 \leq C_s(k),$$

where the dependence of $N^*(k)$ and $C_s(k)$ on k is not clear.

Collocation method

To solve

$$\varphi(s) + \mathcal{K}\varphi(s) = F(s),$$

seek $\varphi_{N_C} \in V_N$ such that

$$(I + P_{N_C}\mathcal{K})\varphi_{N_C} = P_{N_C}F,$$

where P_{N_C} is the interpolatory projection onto the approximation space. Equivalently

$$\varphi_{N_C}(s_m) + \mathcal{K}\varphi_{N_C}(s_m) = F(s_m), \quad m = 1, \ldots, M,$$

$$\Rightarrow \sum_{j=1}^{M} v_j[\rho_j(s_m) + \mathcal{K}\rho_j(s_m)] = F(s_m).$$

If ρ_j supported on same side of polygon as s_m, integrals not oscillatory.

Collocation method

We have not shown that $(I + P_{N_C}\mathcal{K}) : L_2(\Gamma) \mapsto V_N$ is bijective with bounded inverse.

Galerkin vs. Collocation: error analysis

Theorem There exists a constant $C_p > 0$, independent of k, such that for $N \geq N^*$

$$
\begin{aligned}
k^{1/2}\|\varphi - \varphi_{N_G}\|_2 &\leq C_p C_s \sup_{x \in D} |u(x)| \frac{n^{1/2}(1 + \log^{1/2}(kL/n))}{N^{p+1}}, \\
k^{1/2}|u(x) - u_{N_G}(x)| &\leq C_p C_s \sup_{x \in D} |u(x)| \frac{n^{1/2}(1 + \log^{1/2}(kL/n))}{N^{p+1}}.
\end{aligned}
$$

- Stability and convergence not proven for collocation scheme.

Galerkin vs. Collocation: conditioning

Galerkin: mass matrix $M_G := [(\rho_j, \rho_m)]$ has cond$M \leq (1+\sigma)/(1-\sigma)$, where

$$\sigma \leq \max \left\{ \frac{\min(y_j^+, y_m^-) - \max(y_{j-1}^+, y_{m-1}^-)}{\sqrt{(y_j^+ - y_{j-1}^+)(y_m^- - y_{m-1}^-)}} \right\} < 1,$$

and if side lengths and angles are equal we can prove

$$\sigma < \left(\frac{1}{kL} \right)^{1/2N \log k} .$$

Collocation: difficulty with choice of collocation points, $M_C := [\rho_j(s_m)]$ may be ill conditioned.

Galerkin vs. Collocation: implementation

Galerkin: need to evaluate numerically many integrals of form

$$\int_{-b}^{-a}\int_{c}^{d}\left[H_0^{(1)}(k\sqrt{s^2+t^2})+\frac{\mathrm{i}tH_1^{(1)}(k\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\right]\mathrm{e}^{\mathrm{i}k(\sigma_j t-\sigma_m s)}\,\mathrm{d}t\,\mathrm{d}s.$$

Collocation: need to evaluate numerically many integrals of form

$$\int_{a}^{b}\left[H_0^{(1)}(k\sqrt{s_m^2+t^2})+\frac{\mathrm{i}tH_1^{(1)}(k\sqrt{s_m^2+t^2})}{\sqrt{s_m^2+t^2}}\right]\mathrm{e}^{\mathrm{i}k\sigma_j t}\,\mathrm{d}t.$$

- Collocation method easier to implement

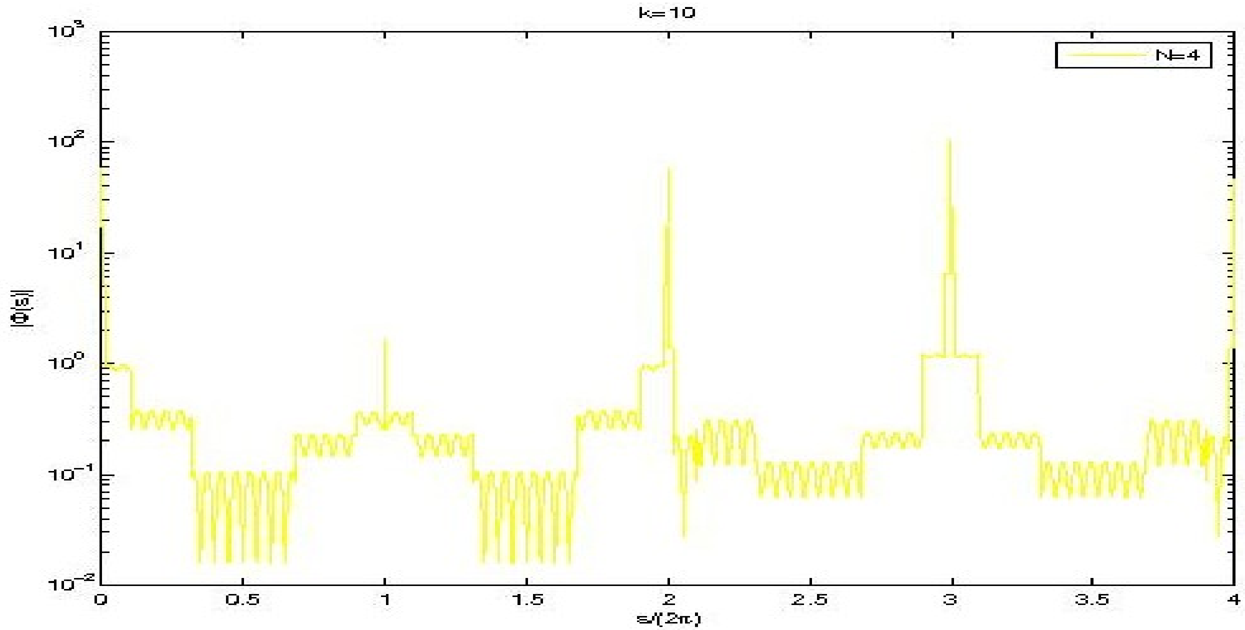
Numerical results

scattering by a square, $k = 5$

scattering by a square, $k = 10$

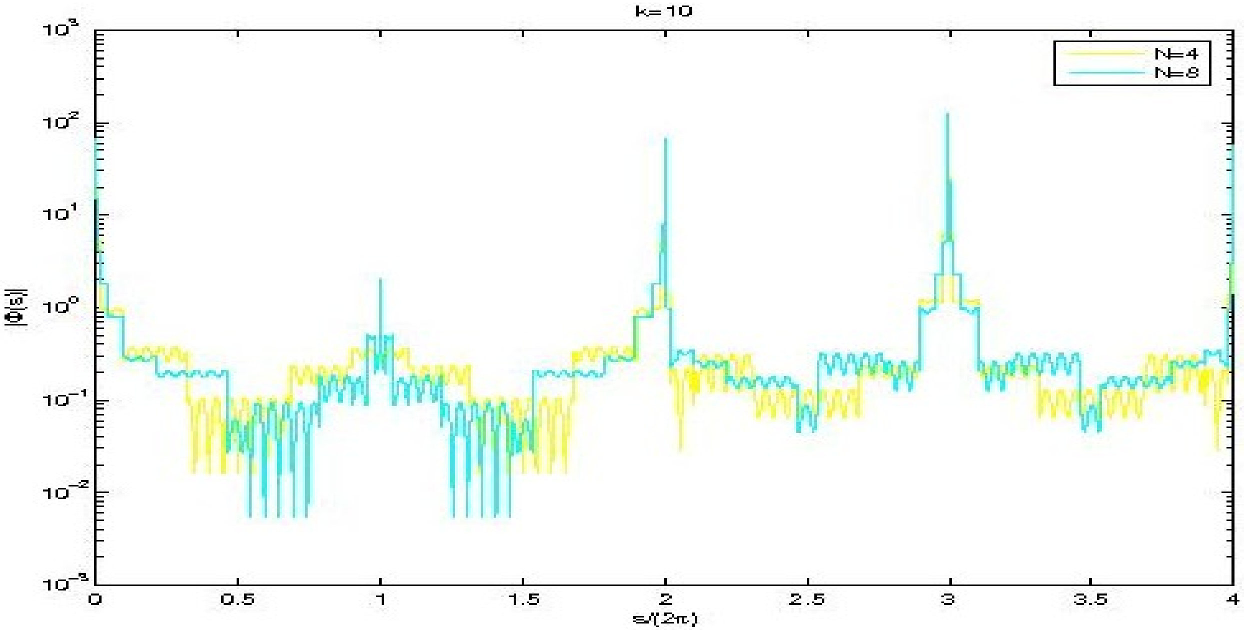
Numerical results (scattering by a square)

Solution minus P.O. approximation;



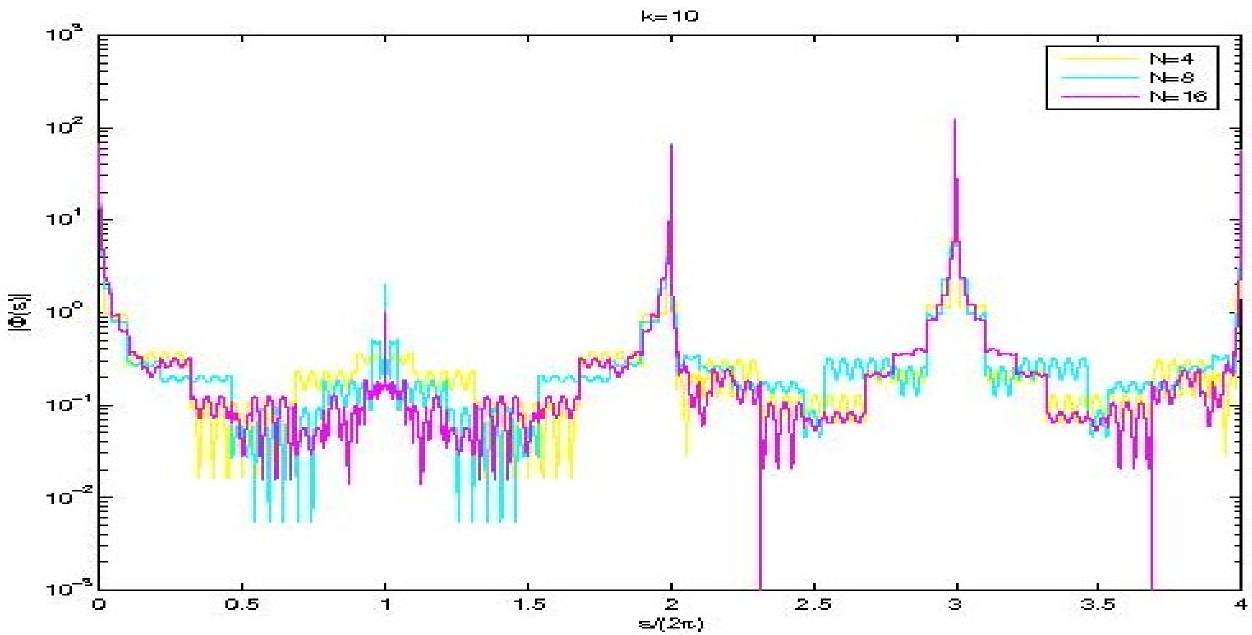
Numerical results (scattering by a square)

Solution minus P.O. approximation;



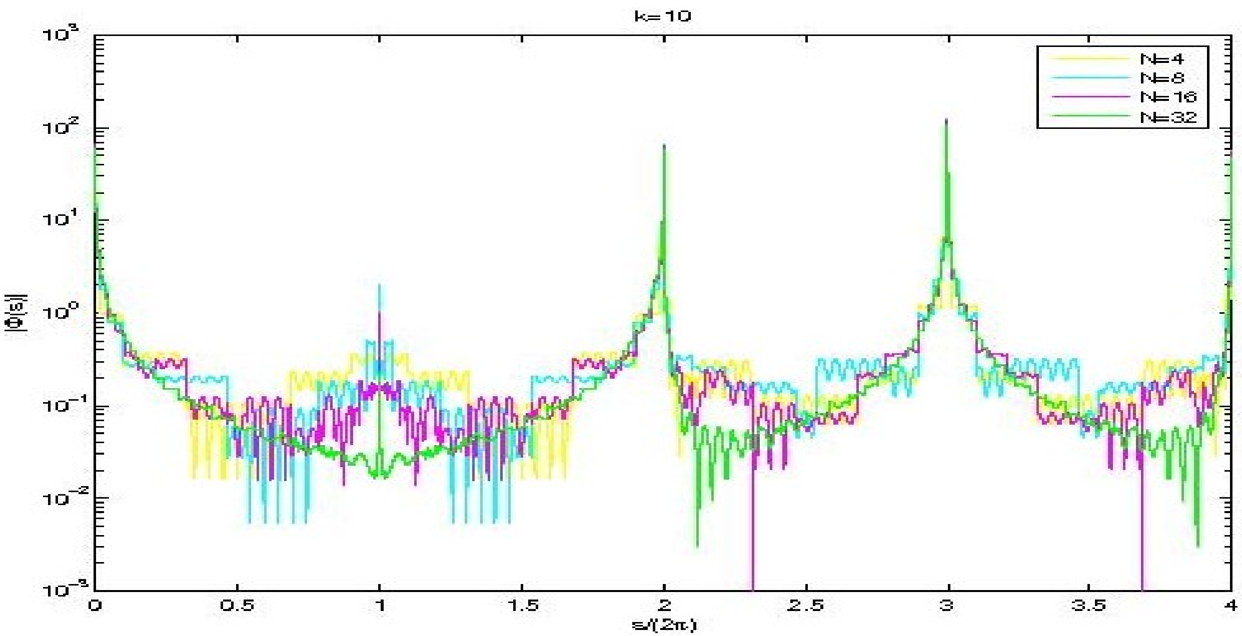
Numerical results (scattering by a square)

Solution minus P.O. approximation;



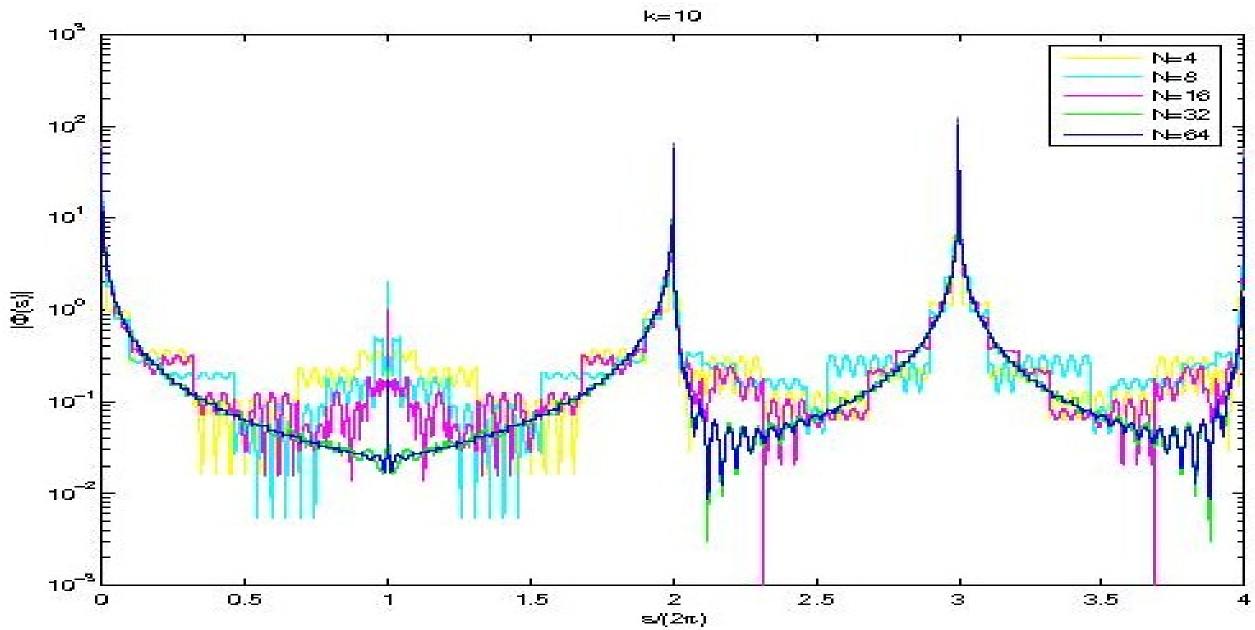
Numerical results (scattering by a square)

Solution minus P.O. approximation;



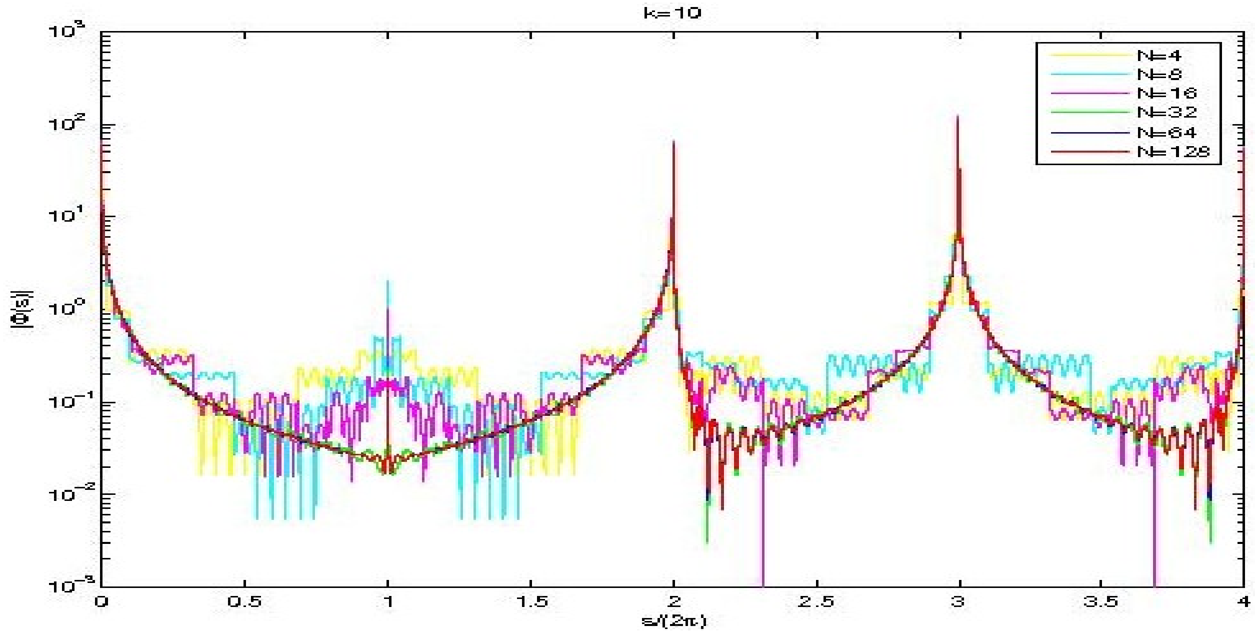
Numerical results (scattering by a square)

Solution minus P.O. approximation;



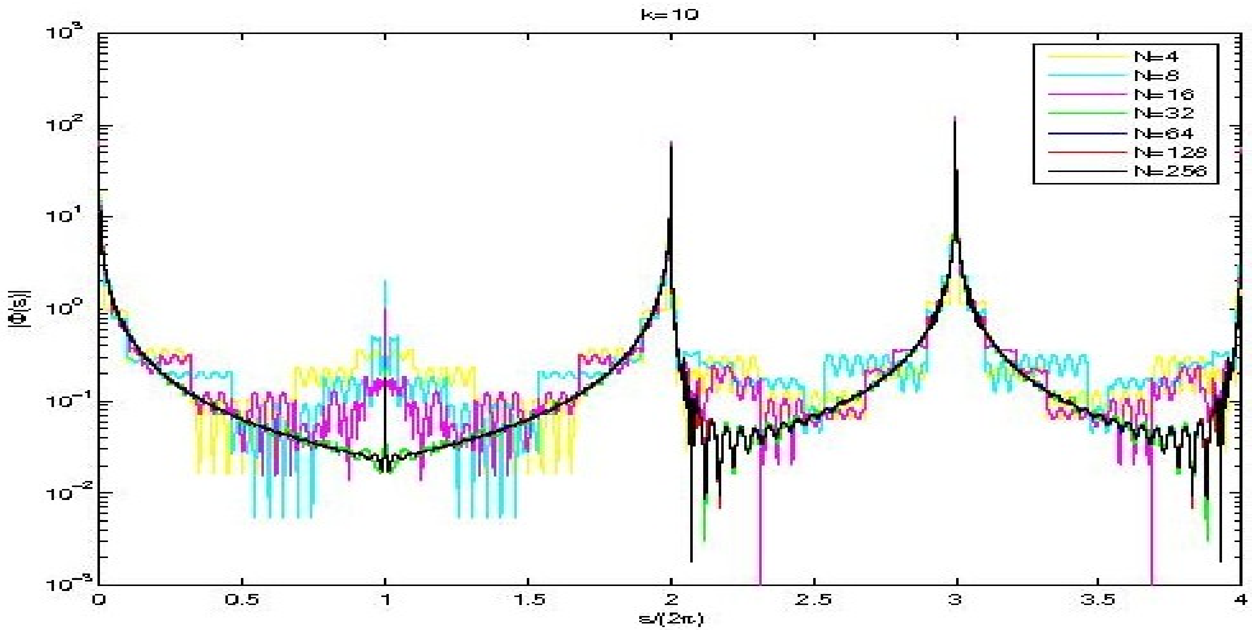
Numerical results (scattering by a square)

Solution minus P.O. approximation;



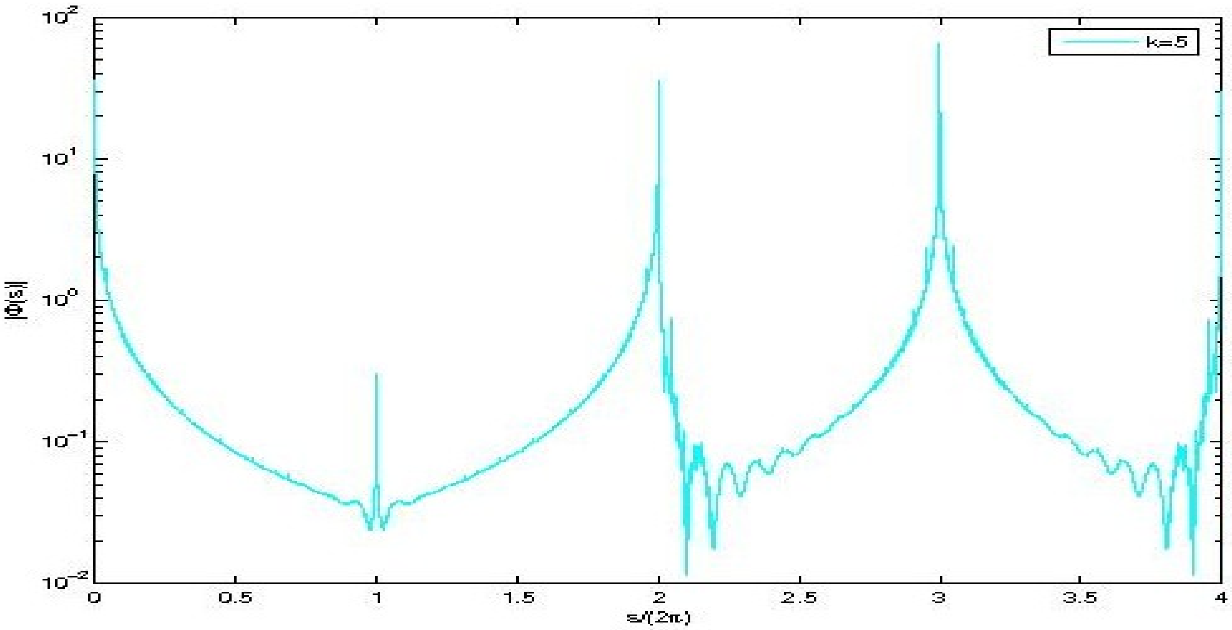
Numerical results (scattering by a square)

Solution minus P.O. approximation;



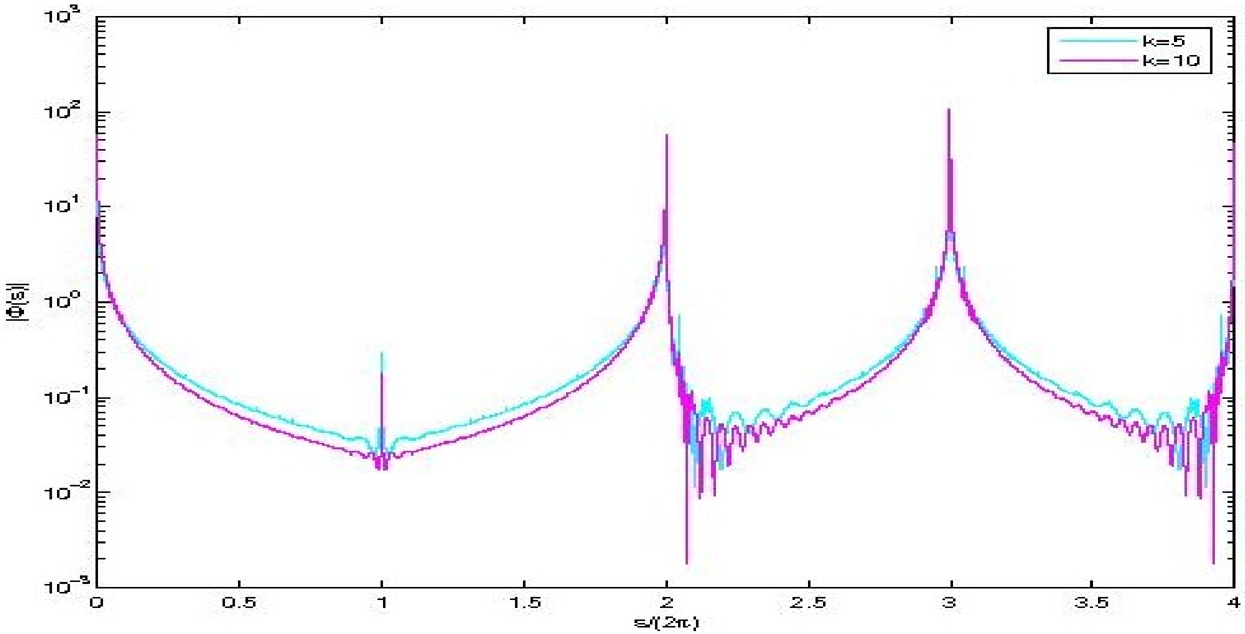
Numerical results (scattering by a square)

"Exact" solution minus P.O. approximation, $k = 5$;



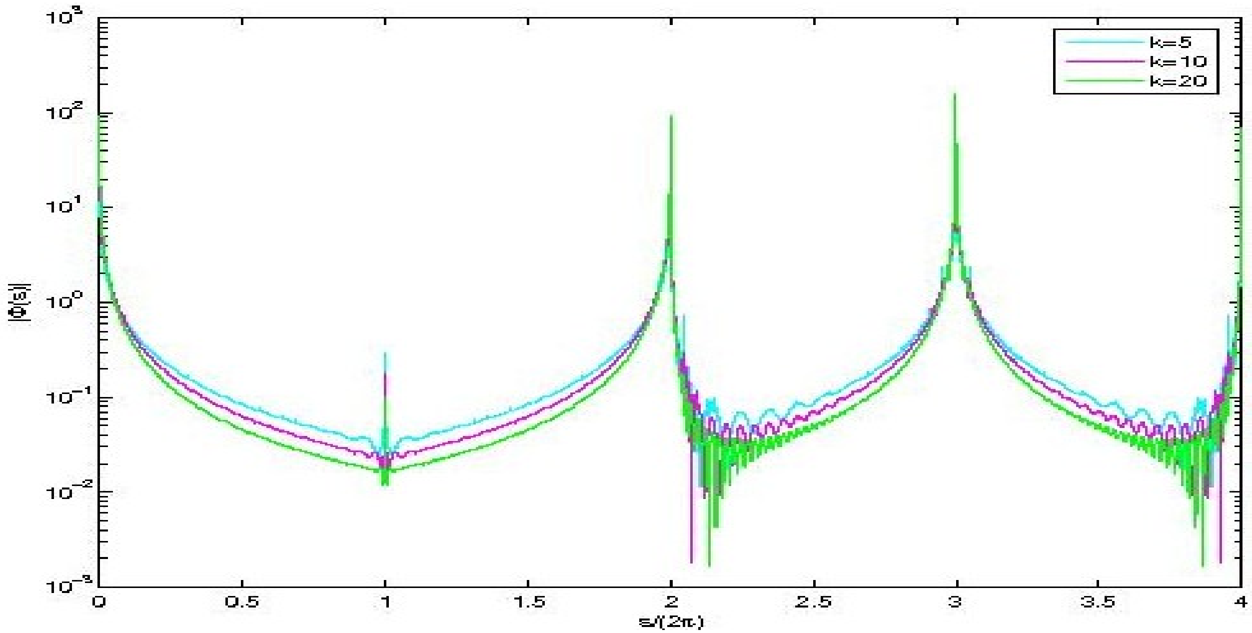
Numerical results (scattering by a square)

"Exact" solution minus P.O. approximation, $k = 10$;



Numerical results (scattering by a square)

"Exact" solution minus P.O. approximation, $k = 20$;



Numerical results (scattering by a square)

"Exact" solution minus P.O. approximation, $k = 40$;

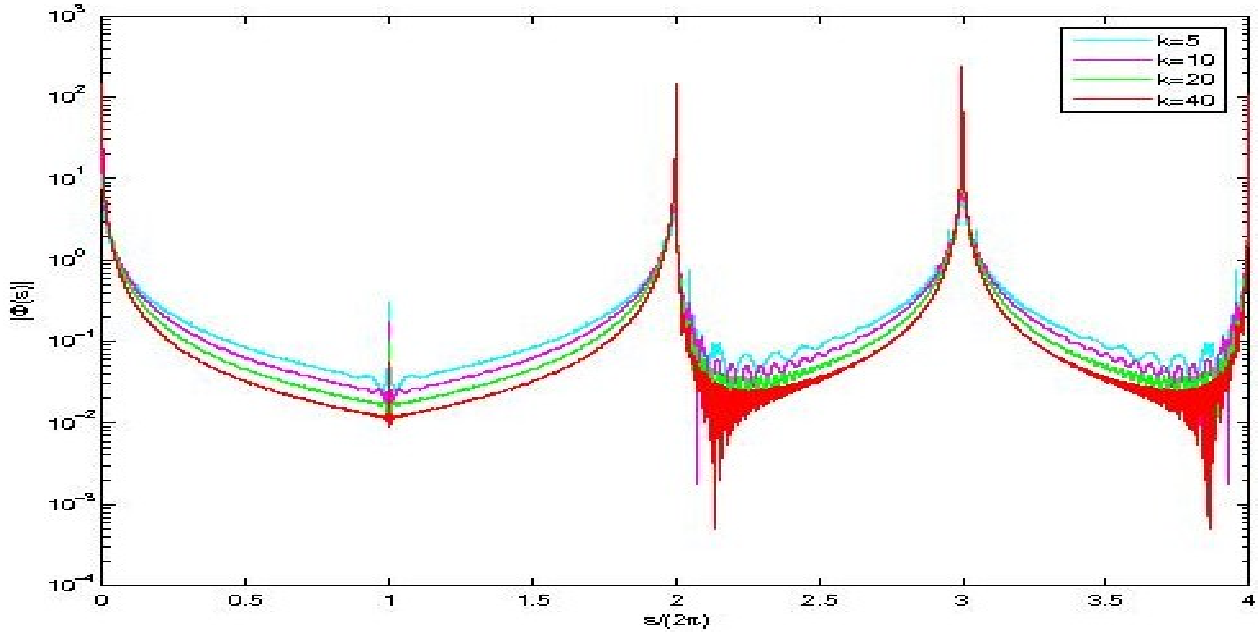


Table 1: Relative errors, $k = 10$

k	N	dof	$\frac{\|\varphi-\varphi_{N_G}\|_2}{\|\varphi\|_2}$	$\frac{\|\varphi-\varphi_{N_C}\|_2}{\|\varphi\|_2}$
10	2	24	$1.1691\times10^{+0}$	7.5453×10^{-1}
	4	48	4.3784×10^{-1}	4.7335×10^{-1}
	8	96	2.2320×10^{-1}	2.6980×10^{-1}
	16	192	1.2106×10^{-1}	1.2670×10^{-1}
	32	376	1.1633×10^{-1}	6.8440×10^{-2}
	64	752	2.8702×10^{-2}	3.3034×10^{-2}

Table 2: Relative errors, $k = 160$

k	N	dof	$\frac{\|\varphi-\varphi_{N_G}\|_2}{\|\varphi\|_2}$	$\frac{\|\varphi-\varphi_{N_C}\|_2}{\|\varphi\|_2}$
160	2	32	7.2765×10^{-1}	6.8901×10^{-1}
	4	56	4.2628×10^{-1}	4.4455×10^{-1}
	8	112	4.9060×10^{-1}	4.6445×10^{-1}
	16	224	1.2847×10^{-1}	2.3456×10^{-1}
	32	456	8.4578×10^{-2}	9.3327×10^{-2}
	64	904	3.4570×10^{-2}	4.8153×10^{-2}

k	M_N	$\|\varphi - \varphi_N\|_2$	$\|\varphi - \varphi_N\|_2 / \|\varphi\|_2$	COND
5	360	3.6171×10^{-1}	6.8909×10^{-2}	2.6×10^{1}
10	376	8.5073×10^{-1}	1.1633×10^{-1}	1.8×10^{2}
20	392	8.0941×10^{-1}	7.9909×10^{-2}	1.0×10^{3}
40	416	1.1252×10^{0}	8.0909×10^{-2}	2.4×10^{2}
80	432	1.6630×10^{0}	8.7071×10^{-2}	5.9×10^{2}
160	456	2.1936×10^{0}	8.4578×10^{-2}	5.2×10^{2}
320	472	3.5185×10^{0}	1.0211×10^{-1}	8.1×10^{2}

Table 3: Relative L_2 errors, various k, $N = 32$

k	N	$\lvert\frac{u_N - u_{256}}{u_{256}}(-\pi, 3\pi)\rvert$	$\lvert\frac{u_N - u_{256}}{u_{256}}(3\pi, 3\pi)\rvert$	$\lvert\frac{u_N - u_{256}}{u_{256}}(3\pi, -\pi)\rvert$
5	4	1.9588×10^{-2}	1.0071×10^{-3}	1.5885×10^{-2}
	8	4.2631×10^{-3}	2.8032×10^{-3}	2.3213×10^{-3}
	16	3.6178×10^{-4}	3.1438×10^{-4}	1.3514×10^{-3}
	32	6.6463×10^{-5}	2.9271×10^{-5}	1.7115×10^{-5}
	64	1.1634×10^{-5}	5.4525×10^{-6}	3.8267×10^{-6}

Table 4: Relative errors, for $u_N(x)$

k	N	$\lvert\frac{u_N - u_{256}}{u_{256}}(-\pi, 3\pi)\rvert$	$\lvert\frac{u_N - u_{256}}{u_{256}}(3\pi, 3\pi)\rvert$	$\lvert\frac{u_N - u_{256}}{u_{256}}(3\pi, -\pi)\rvert$
320	4	7.2339×10^{-6}	9.1702×10^{-6}	6.5155×10^{-5}
	8	1.3617×10^{-5}	4.7357×10^{-6}	3.6329×10^{-5}
	16	1.0694×10^{-5}	3.0122×10^{-6}	2.9284×10^{-5}
	32	1.0691×10^{-6}	5.3066×10^{-7}	2.8225×10^{-6}
	64	3.1606×10^{-7}	3.0148×10^{-7}	8.1702×10^{-7}

Table 5: Relative errors, for $u_N(x)$

What we actually are computing . . .

The difference between the exact solution and a leading order approximation;

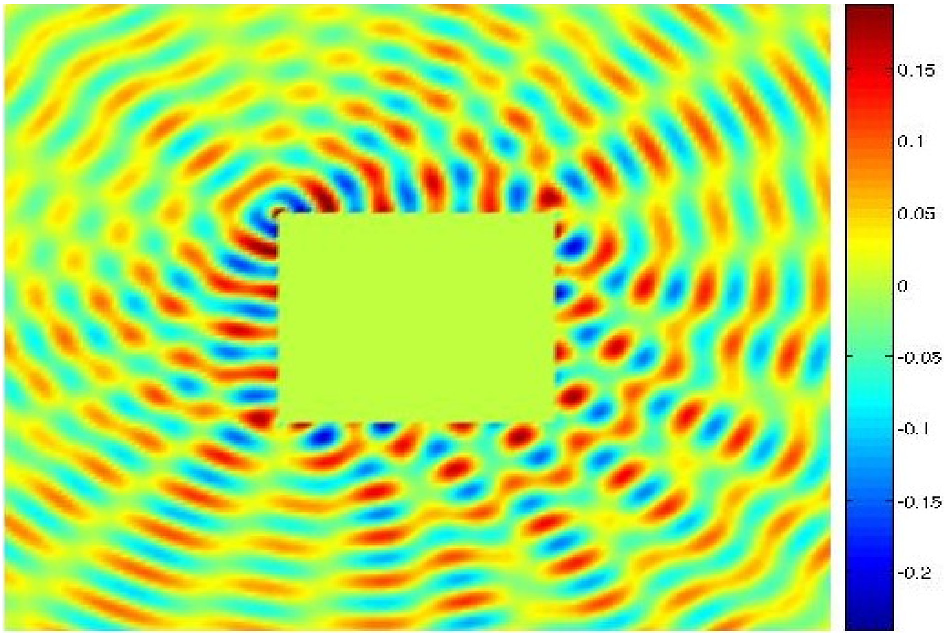


Figure 4: square, $k = 5$

What we actually are computing . . .

The difference between the exact solution and a leading order approximation;

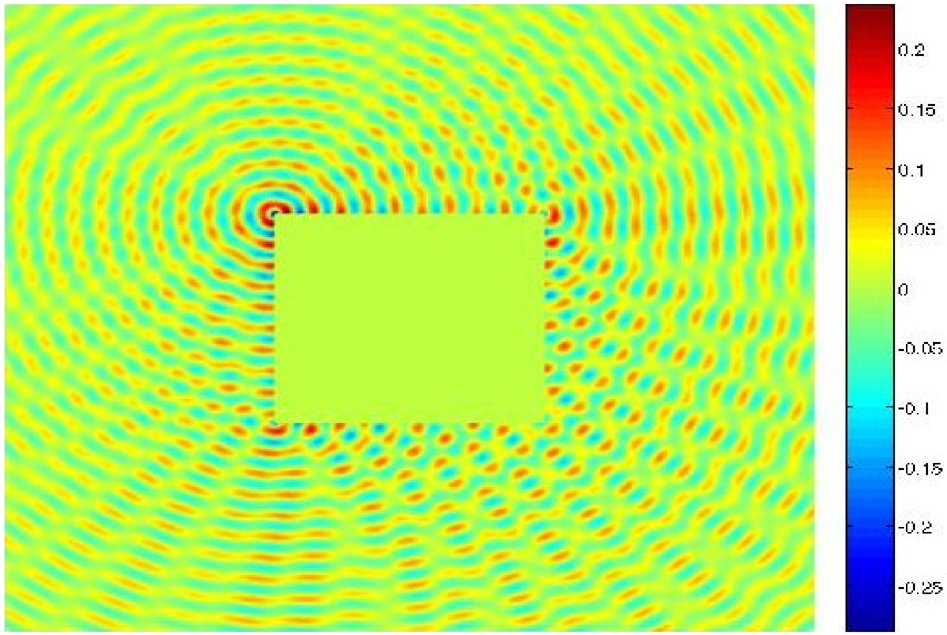


Figure 5: square, $k = 10$

What we actually are computing . . .

The difference between the exact solution and a leading order approximation;

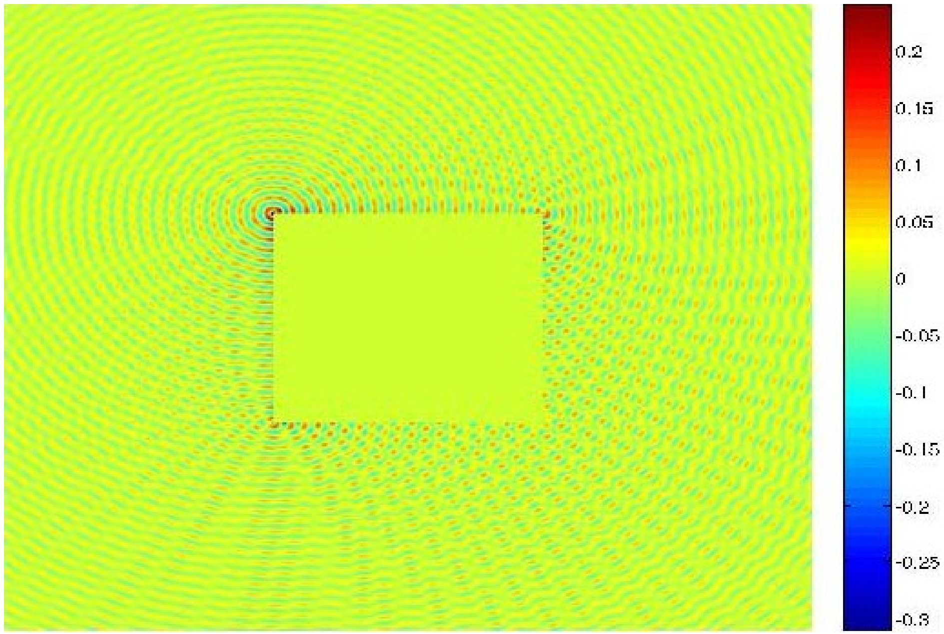


Figure 6: square, $k = 20$

What we actually are computing . . .

The difference between the exact solution and a leading order approximation;

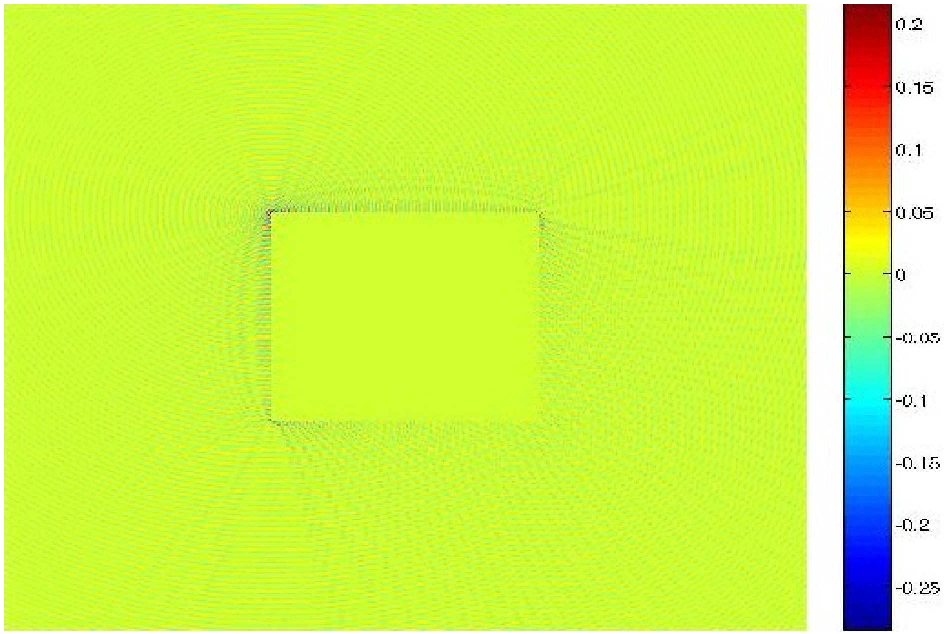


Figure 7: square, $k = 40$

Summary and Conclusions

- Using Green's representation theorem in a half-plane we can understand behaviour of the field on the boundary and its derivatives for scattering by a convex polygon (extends to convex polyhedron in 3D)
- For a convex polygon, design of an optimal graded mesh for piecewise polynomial approximation is then straightforward
- The number of degrees of freedom need only grow logarithmically with the wavenumber to maintain a fixed accuracy
- Ongoing considerations
  - Galerkin vs. Collocation - stability and convergence analysis
  - Better schemes for evaluating oscillatory integrals
  - hp ideas