Pressure estimate for Navier-Stokes equation in bounded domains

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- *Stability and convergence of efficient Navier-Stokes solvers via a commutator estimate*, to appear in Comm. Pure Appl. Math.
- *Error estimates for finite element Navier-Stokes solvers with explicit time stepping for pressure* (submitted)

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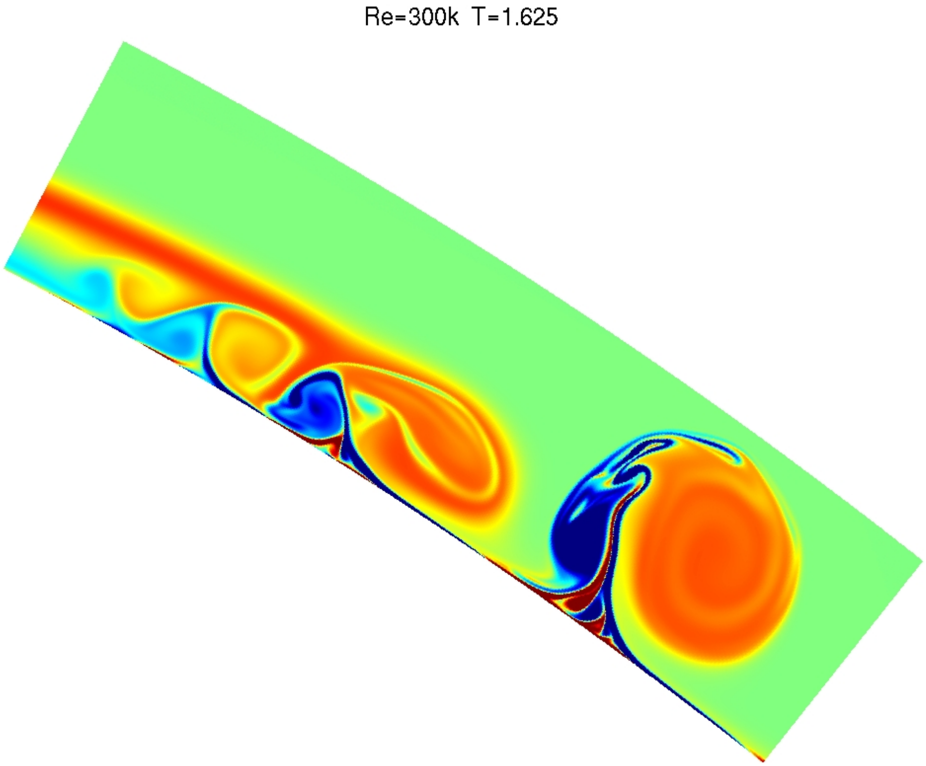

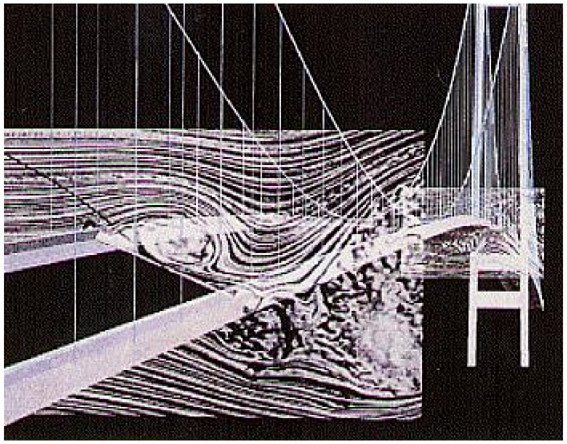

Phenomena modeled by Navier-Stokes dynamics:
Lift, drag, boundary-layer separation, vortex shedding, ...

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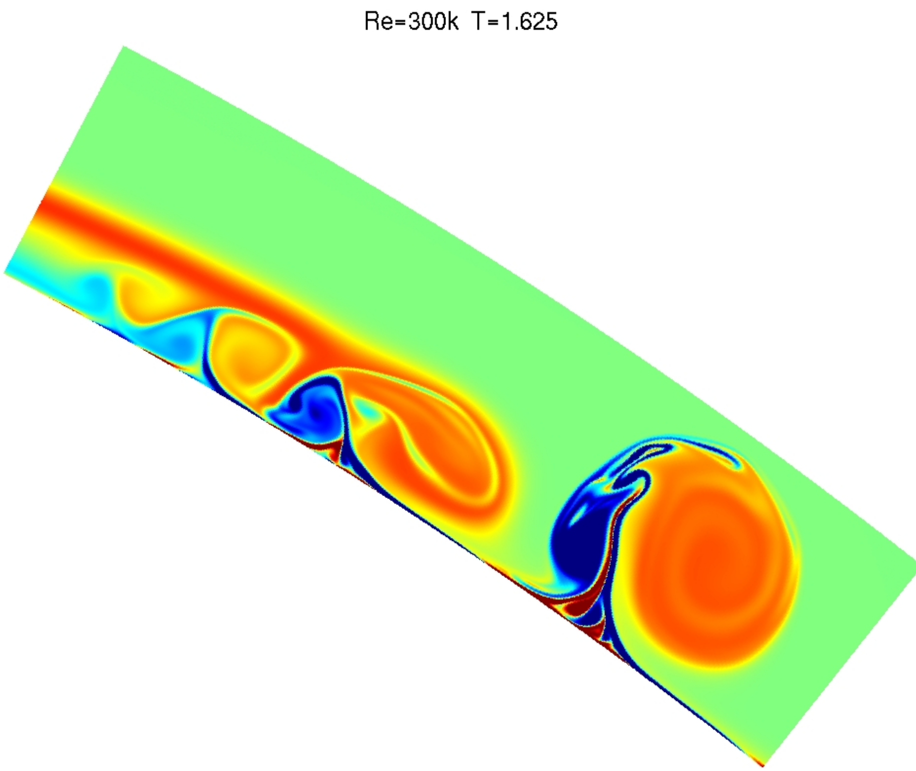

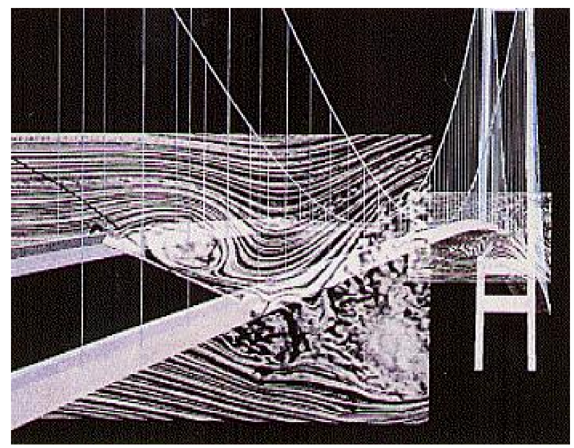

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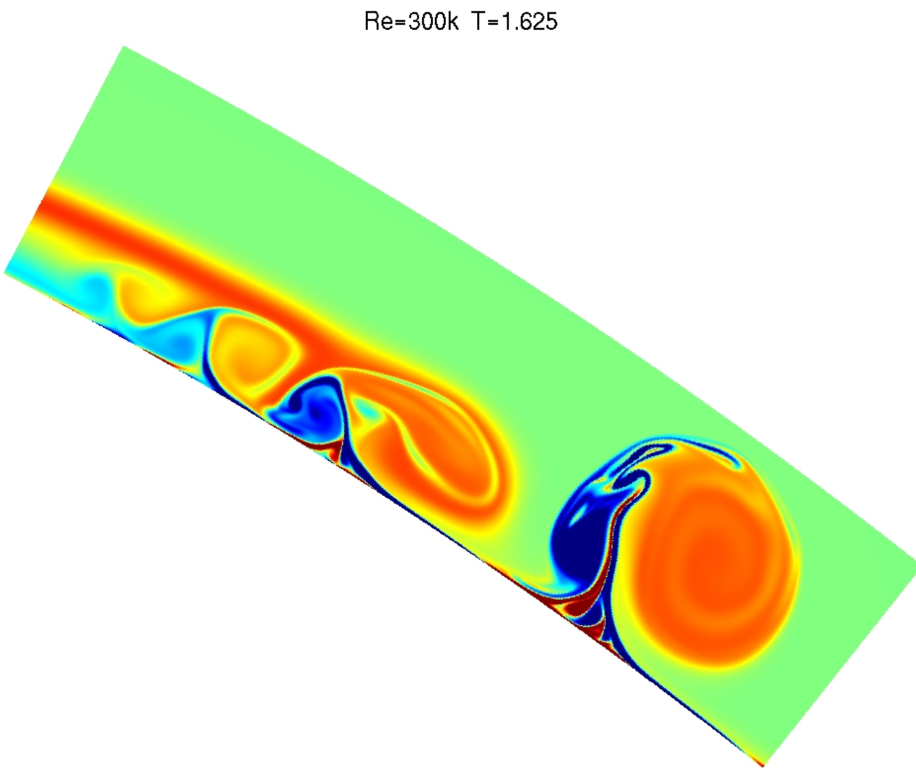

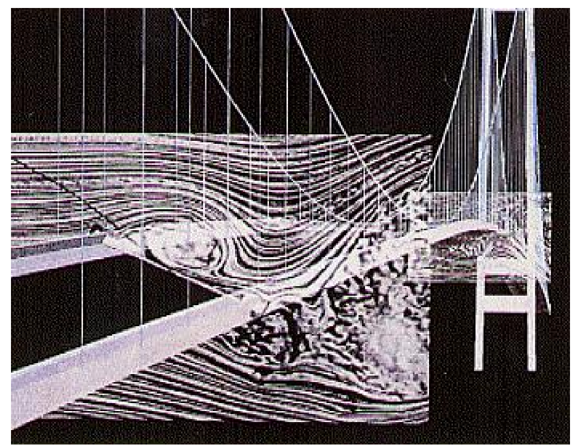

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Helmholtz projection onto divergence-free vector fields

$$L^2(\Omega, \mathbb{R}^N) = \mathcal{P}L^2(\Omega, \mathbb{R}^N) \oplus \nabla H^1(\Omega)$$

Given $\vec{v} \in L^2(\Omega, \mathbb{R}^N)$, there exists $q \in H^1(\Omega)$ so that

$$\vec{v} = \mathcal{P}\vec{v} - \nabla q$$

satisfies $\quad \langle \mathcal{P}\vec{v}, \nabla \phi \rangle = \langle \vec{v} + \nabla q, \nabla \phi \rangle = 0 \quad$ for all $\phi \in H^1(\Omega)$.

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Then

$$\nabla \cdot (\mathcal{P}\vec{v}) = 0 \quad \text{in } \Omega, \qquad \vec{n} \cdot \mathcal{P}\vec{v} = 0 \quad \text{on } \Gamma.$$

Note $\mathcal{P}\vec{v} \in H(\mathrm{div}; \Omega) = \{\vec{f} \in L^2(\Omega, \mathbb{R}^N) : \nabla \cdot \vec{f} \in L^2\}$,

so $\vec{n} \cdot \mathcal{P}\vec{v} \in H^{-1/2}(\Gamma)$ by a standard trace theorem.

Traditional unconstrained formulation of NSE

$$\vec{u}_t + \mathcal{P}(\vec{u} \cdot \nabla \vec{u} - \vec{f}) = \nu \mathcal{P} \Delta \vec{u}, \quad \vec{u}|_\Gamma = 0$$

- Formally $\quad \partial_t(\nabla \cdot \vec{u}) = 0$
- Perform analysis and computation in spaces of divergence-free fields (unconstrained Stokes operator $\mathcal{P}\Delta u$ is incompletely dissipative).
- Inf-Sup/LBB condition(Ladyzhenskaya-Babuška-Brezzi)

Alternative unconstrained formulation of NSE

$$\vec{u}_t + \mathcal{P}(\vec{u}\cdot\nabla\vec{u} - \vec{f}) = \nu\mathcal{P}\Delta\vec{u} + \nu\nabla(\nabla\cdot\vec{u}), \quad \vec{u}|_\Gamma = 0$$

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- For $\vec{u}\in H^2\cap H^1_0(\Omega,\mathbb{R}^N)$, Stokes pressure satisfies $\Delta p_{\mathsf{s}} = 0$ in Ω

  with BC: $\partial_n p_{\mathsf{s}} = \vec{n}\cdot(\Delta - \nabla\nabla\cdot)\vec{u} = -\vec{n}\cdot\nabla\times\nabla\times\vec{u}$ in $H^{-1/2}(\Gamma)$

  first used by Orszag (1986) for consistency in a projection step.

Pressure Poisson equation

- Unconstrained reformulation of Navier-Stokes equations:

$$u_t + \vec{u} \cdot \nabla \vec{u} + \nabla p = \nu \Delta \vec{u} + \vec{f}, \quad \vec{u}|_\Gamma = 0$$

Total pressure p is determined by the weak form

$$\langle \nabla p, \nabla \phi \rangle = \langle \vec{f} - \vec{u} \cdot \nabla \vec{u}, \nabla \phi \rangle + \nu \langle (\Delta - \nabla \nabla \cdot) \vec{u}, \nabla \phi \rangle \quad \forall \phi \in H^1(\Omega).$$

- In computation, we use $\Delta p^n = \nabla \cdot (\vec{f}^n - \vec{u}^n \cdot \nabla \vec{u}^n)$ in Ω

with BC: $\vec{n} \cdot \nabla p^n = \vec{n} \cdot \vec{f}^n - \nu \vec{n} \cdot (\nabla \times \nabla \times \vec{u}^n)$ on Γ

- ∇p_s arises from *tangential vorticity at the boundary*:

3D weak form: $$\int_\Omega \nabla p_s \cdot \nabla \phi = \int_\Gamma (\nabla \times \vec{u}) \cdot (\vec{n} \times \nabla \phi) \quad \forall \phi \in H^1(\Omega)$$

Space of Stokes pressures

$$\mathcal{S}_p = \{p \in H^1(\Omega)/\mathbb{R} : \Delta p = 0 \quad \text{in } \Omega,\ \vec{n} \cdot \nabla p \in \mathcal{S}_\Gamma\},$$
$$\mathcal{S}_\Gamma = \{f \in H^{-1/2}(\Gamma) : \int_G f = 0 \quad \forall \text{components } G \text{ of } \Gamma\}.$$

- $\exists$ a bounded right inverse $\nabla p_{\mathsf{s}} \mapsto \vec{u}$ from $\mathcal{S}_p \to H^2 \cap H^1_0(\Omega, \mathbb{R}^N)$
- In $\mathbb{R}^3$, $\nabla \mathcal{S}_p$ is the space of *simultaneous gradients and curls:*

$$\nabla \mathcal{S}_p = \nabla H^1(\Omega) \ \cap \ \nabla \times H^1(\Omega, \mathbb{R}^3)$$

The commutator term is strictly controlled by viscosity

If $\nabla \cdot \vec{u} = 0$, then $\|[\Delta, \mathcal{P}]\vec{u}\| = \|((I - \mathcal{P})\Delta\vec{u} - \nabla\nabla \cdot \vec{u}\| \leq \|\Delta\vec{u}\|$.
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Main Theorem Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$, bounded, $\partial\Omega \in C^3$. Then, $\forall \varepsilon > 0$, $\exists C \geq 0$, s.t. for all $\vec{u} \in H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$,

$$\int_\Omega |(\Delta\mathcal{P} - \mathcal{P}\Delta)\vec{u}|^2 \leq (\frac{1}{2} + \varepsilon) \int_\Omega |\Delta\vec{u}|^2 + C \int_\Omega |\nabla\vec{u}|^2$$

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Hence our unconstrained NSE is **fully dissipative**:

$$\vec{u}_t + \mathcal{P}(\vec{u} \cdot \nabla\vec{u} - \vec{f}) + \nu[\Delta, \mathcal{P}]\vec{u} = \nu\Delta\vec{u}, \quad \vec{u}|_\Gamma = 0$$

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NSE as perturbed heat equation!

Proof of the theorem – estimate on commutator $[\Delta, \mathcal{P}]$

Decompose $\vec{u} \in H^2 \cap H^1_0$ into parts parallel and normal to Γ:

Let $\Phi(x) = \mathrm{dist}(x, \Gamma)$, $\vec{n}(x) = -\nabla\Phi(x)$, ξ a cutoff $= 1$ near Γ.

$$\vec{u} = \vec{u}_{\|} + \vec{u}_{\perp}, \qquad \vec{u}_{\|} = \xi(I - \vec{n}\vec{n}^t)\vec{u}.$$

Boundary identities on Γ: $\nabla \cdot \vec{u}_{\|} = 0$, $\vec{n}\cdot\nabla\times\nabla\times\vec{u}_{\perp} = 0$.

Stokes pressure satisfies $\Delta p_{\mathsf{s}} = 0$, $\partial_n p_{\mathsf{s}} = -\vec{n}\cdot\nabla\times\nabla\times\vec{u}$

Hence $\nabla p_{\mathsf{s}} = (\Delta\mathcal{P} - \mathcal{P}\Delta)\vec{u} = (I - \mathcal{P})(\Delta - \nabla\nabla\cdot)(\vec{u}_{\|} + 0)$.

$$\langle \nabla p_{\mathsf{s}}, \nabla p_{\mathsf{s}} \rangle = \langle \Delta\vec{u}_{\|} - \nabla\nabla\cdot\vec{u}_{\|}, \nabla p_{\mathsf{s}} \rangle = \langle \Delta\vec{u}_{\|}, \nabla p_{\mathsf{s}} \rangle$$

$$\langle \nabla p_{\mathsf{s}} - \Delta\vec{u}_{\|}, \nabla p_{\mathsf{s}} \rangle = 0$$

$$\|\Delta\vec{u}_{\|}\|^2 = \|\nabla p_{\mathsf{s}}\|^2 + \|\nabla p_{\mathsf{s}} - \Delta\vec{u}_{\|}\|^2$$

D2N/N2D bounds on tubes $\Omega_s = \{x \in \Omega \mid \Phi(x) < s\}$

Lemma For $s_0 > 0$ small $\exists C_0$ such that whenever $\Delta p = 0$ in Ω_{s_0} and $0 < s < s_0$ then

$$\left| \int_{\Phi < s} |\vec{n} \cdot \nabla p|^2 - |(I - \vec{n}\vec{n}^t)\nabla p|^2 \right| \leq C_0 s \int_{\Phi < s_0} |\nabla p|^2$$

In the limit $s \to 0$, it reduce to

$$\left| \int_\Gamma |\vec{n} \cdot \nabla p|^2 - \int_\Gamma |(I - \vec{n}\vec{n}^t)\nabla p|^2 \right| \leq C_0 \int_\Omega |\nabla p|^2$$

In a half space: $\|\vec{n} \cdot \nabla p\|^2 = \|(I - \vec{n}\vec{n}^t)\nabla p\|^2$.

Known as Rellich identity in 2D circular disk.

Why factor $\frac{1}{2}$?

$$\text{Orthogonality:} \quad \left\langle \begin{pmatrix} a_{\|} \\ 0 \end{pmatrix} - \begin{pmatrix} b_{\|} \\ b_{\perp} \end{pmatrix}, \begin{pmatrix} b_{\|} \\ b_{\perp} \end{pmatrix} \right\rangle = 0$$

$$\text{Equal partition:} \quad b_{\perp} = b_{\|}$$

implies $a_{\|} - b_{\|} = b_{\|}$.

Hence

$$|b|^2 = (b_{\|})^2 + (b_{\perp})^2 = 2(b_{\|})^2 = \frac{1}{2}(a_{\|})^2$$

Half space:

$$u(x,y) = \sin(kx) y e^{-ky}, \quad v(x,y) = 0, \quad p(x,y) = \cos(kx) e^{-ky}$$

Equal partition: $\|p_x\|^2 = \|p_y\|^2 = \frac{\pi k}{2}$

Orthogonality: $\Delta u = -2k \sin(kx) e^{-ky} = 2\partial_x p$

Some details

$$\begin{aligned} \|\Delta\vec{u}\|^2 &= \|\Delta\vec{u}_{\|}\|^2 + 2\langle\Delta\vec{u}_{\|}, \Delta\vec{u}_{\perp}\rangle + \|\Delta\vec{u}_{\perp}\|^2 \\ &\geq (1-\varepsilon)\|\Delta\vec{u}_{\|}\|^2 - C\|\nabla\vec{u}\|^2 \end{aligned}$$

From orthogonality identity $\langle\nabla p - \Delta\vec{u}_{\|}, \nabla p\rangle = 0$

$$\|\Delta\vec{u}_{\|}\|^2 = \|\nabla p\|^2 + \|\nabla p - \Delta\vec{u}_{\|}\|^2$$

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$$\begin{aligned}\|\Delta\vec{u}_{\|}\|^2 &= \|\nabla p\|^2 + \|\nabla p - \Delta\vec{u}_{\|}\|^2 \\ &= \|\nabla p\|^2 + \|\nabla p\|^2_{\Phi>s} + \|\nabla p - \Delta\vec{u}_{\|}\|^2_{\Phi<s}\end{aligned}$$

$$\begin{aligned}\|\nabla p - \Delta\vec{u}_{\|}\|^2_{\Phi<s} &= \|(\nabla p - \Delta\vec{u}_{\|})_{\perp}\|^2_{\Phi<s} + \|(\nabla p - \Delta\vec{u}_{\|})_{\|}\|^2_{\Phi<s} \\ &\geq (1-\varepsilon)\|\nabla p_{\perp}\|^2_{\Phi<s} + \|(\nabla p - \Delta\vec{u}_{\|})_{\|}\|^2_{\Phi<s} - \text{junk}\end{aligned}$$

By orthogonality $\langle \nabla p - \Delta \vec{u}_{||}, \nabla p \rangle = 0$, with $\langle f, g \rangle_s := \int_{\Phi < s} f \cdot g$,

$$0 = \langle (\nabla p - \Delta \vec{u}_{||})_{||}, \nabla p_{||} \rangle_s + \langle (\nabla p - \Delta \vec{u}_{||})_{\perp}, \nabla p_{\perp} \rangle_s + \int_{\Phi > s} |\nabla p|^2$$

Hence

$$\begin{aligned} \|(\nabla p - \Delta \vec{u}_{||})_{||}\|_s^2 + \|\nabla p_{||}\|_s^2 &\geq -2\langle (\nabla p - \Delta \vec{u}_{||})_{||}, \nabla p_{||} \rangle_s \\ &\geq 2\langle (\nabla p - \Delta \vec{u}_{||})_{\perp}, \nabla p_{\perp} \rangle_s \end{aligned}$$

$$\|(\nabla p - \Delta \vec{u}_{||})_{||}\|_s^2 \geq \|\nabla p_{||}\|_s^2 + 2(\|\nabla p_{\perp}\|_s^2 - \|\nabla p_{||}\|_s^2) - \text{junk}$$

Done! $\quad (1 + \varepsilon)\|\vec{u}\|^2 \geq \|\nabla p\|^2 + \|\nabla p\|_{\Phi > s}^2 + \|\nabla p\|_{\Phi < s}^2 - \text{junk}$

Consequences of the main theorem

- Proof of unconditional stability and convergence for C^1 finite-element methods without regard to inf-sup compatibility conditions for velocity and pressure

- Numerical demonstration of stability and accuracy for efficient and easy-to-implement C^0 finite-element schemes regardless of velocity/pressure compatibility

- Additional benefit: Simple proof of existence and uniqueness for strong solutions in bounded domains

Time-differencing scheme with pressure explicit (related: Ti96,Pe01,GS03,JL04)

$$\frac{\vec{u}^{n+1}-\vec{u}^{n}}{\Delta t}-\nu\Delta\vec{u}^{n+1}=\vec{f}^{n}-\vec{u}^{n}\cdot\nabla\vec{u}^{n}-\nabla p^{n}, \quad \vec{u}^{n+1}|_{\Gamma}=0$$

$$\langle\nabla p^{n},\nabla\phi\rangle=\langle\vec{f}^{n}-\vec{u}^{n}\cdot\nabla\vec{u}^{n},\nabla\phi\rangle+\nu\langle\Delta\vec{u}^{n}-\nabla\nabla\cdot\vec{u}^{n},\nabla\phi\rangle \quad \forall\phi\in H^{1}(\Omega)$$

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Since $\langle\nabla p^{n},\mathcal{P}\Delta\vec{u}^{n}\rangle=0$, with $\nabla p_{\mathsf{s}}^{n}=(I-\mathcal{P})\Delta\vec{u}^{n}-\nabla\nabla\cdot\vec{u}^{n}$,

$$\langle\nabla p^{n},\nabla\phi\rangle=\langle\vec{f}^{n}-\vec{u}^{n}\cdot\nabla\vec{u}^{n},\nabla\phi\rangle+\nu\langle\nabla p_{\mathsf{s}}^{n},,\nabla\phi\rangle\quad\forall\phi\in H^{1}(\Omega).$$

Taking $\phi=p^{n}$ gives the pressure estimate

$$\|\nabla p^{n}\|\leq\|\vec{f}^{n}-\vec{u}^{n}\cdot\nabla\vec{u}^{n}\|+\nu\|\nabla p_{\mathsf{s}}^{n}\|$$

Stability analysis: dot with $-\Delta\vec{u}^{n+1}$

$$\frac{\vec{u}^{n+1}-\vec{u}^{n}}{\Delta t} - \nu\Delta\vec{u}^{n+1} = \vec{f}^{n} - \vec{u}^{n}\cdot\nabla\vec{u}^{n} - \nabla p^{n}$$

$$\begin{aligned}
&\frac{\|\nabla\vec{u}^{n+1}-\nabla\vec{u}^{n}\|^2 + \|\nabla\vec{u}^{n+1}\|^2 - \|\nabla\vec{u}^{n}\|^2}{2\Delta t} + \nu\|\Delta\vec{u}^{n+1}\|^2 \\
&\le \ \|\Delta\vec{u}^{n+1}\|(2\|\vec{f}^{n} - \vec{u}^{n}\cdot\nabla\vec{u}^{n}\| + \nu\|\nabla p_{\mathsf{s}}^{n}\|) \\
&\le (\frac{\varepsilon_1}{2}+\frac{\nu}{2})\|\Delta\vec{u}^{n+1}\|^2 + \frac{2}{\varepsilon_1}\|\vec{f}^{n} - \vec{u}^{n}\cdot\nabla\vec{u}^{n}\|^2 + \frac{\nu}{2}\|\nabla p_{\mathsf{s}}\|^2
\end{aligned}$$

This gives

$$\begin{aligned}
&\frac{\|\nabla\vec{u}^{n+1}\|^2 - \|\nabla\vec{u}^{n}\|^2}{\Delta t} + (\nu-\varepsilon_1)\|\Delta\vec{u}^{n+1}\|^2 \\
&\le \frac{8}{\varepsilon_1}(\|\vec{f}^{n}\|^2 + \|\vec{u}^{n}\cdot\nabla\vec{u}^{n}\|^2) + \nu\|\nabla p_{\mathsf{s}}^{n}\|^2
\end{aligned}$$

Handling the pressure

$$\frac{\|\nabla\vec{u}^{n+1}\|^2 - \|\nabla\vec{u}^n\|^2}{\Delta t} + (\nu - \varepsilon_1)\|\Delta\vec{u}^{n+1}\|^2$$
$$\leq \frac{8}{\varepsilon_1}(\|\vec{f}^n\|^2 + \|\vec{u}^n \cdot \nabla\vec{u}^n\|^2) + \nu\|\nabla p_{\mathsf{s}}^n\|^2$$

Use the theorem ($\beta = \frac{1}{2} + \varepsilon$): $\nu\|\nabla p_{\mathsf{s}}^n\|^2 \leq \nu\beta\|\Delta\vec{u}^n\|^2 + \nu C\|\nabla\vec{u}^n\|^2$

$$\frac{\|\nabla\vec{u}^{n+1}\|^2 - \|\nabla\vec{u}^n\|^2}{\Delta t} + (\nu - \varepsilon_1)(\|\Delta\vec{u}^{n+1}\|^2 - \|\Delta\vec{u}^n\|^2)$$
$$+ (\nu - \varepsilon_1 - \nu\beta)\|\Delta\vec{u}^n\|^2$$
$$\leq \frac{8}{\varepsilon_1}(\|\vec{f}^n\|^2 + \|\vec{u}^n \cdot \nabla\vec{u}^n\|^2) + \nu C\|\nabla\vec{u}^n\|^2$$

That's it for the pressure!

Handling the nonlinear term

Use Ladyzhenskaya's inequalities

$$\int_{\mathbb{R}^N} g^4 \le 2\left(\int_{\mathbb{R}^N} g^2\right)\left(\int_{\mathbb{R}^N} |\nabla g|^2\right) \qquad (N=2),$$

$$\int_{\mathbb{R}^N} g^4 \le 4\left(\int_{\mathbb{R}^N} g^2\right)^{1/2}\left(\int_{\mathbb{R}^N} |\nabla g|^2\right)^{3/2} \qquad (N=3),$$

to get for $N = 2, 3$ (details suppressed)

$$\int |\vec{u}^n \cdot \nabla \vec{u}^n|^2 \le C\|\nabla \vec{u}^n\|_{L^2}^3 \|\nabla \vec{u}^n\|_{H^1} \le \varepsilon_2 \|\Delta \vec{u}^n\|^2 + \frac{4C}{\varepsilon_2}\|\nabla \vec{u}^n\|^6$$

Take $\varepsilon_1,\ \varepsilon_2$ small so $\varepsilon := \nu(1-\beta) - \varepsilon_1 - 8\varepsilon_2/\varepsilon_1 > 0$. Gronwall:

Unconditional stability theorem for $N = 2, 3$

Theorem Take $\vec{f} \in L^2(0, T; L^2(\Omega, \mathbb{R}^N))$, $\vec{u}^0 \in H^2 \cap H^1_0(\Omega, \mathbb{R}^N)$. Then $\exists$ positive constants T^* and C^* depending only upon Ω, ν and

$$M_0 := \|\nabla \vec{u}^0\|^2 + \nu \Delta t \|\Delta \vec{u}^0\|^2 + \int_0^T \|\vec{f}\|^2,$$

so that whenever $n\Delta t \leq T^*$ we have

$$\sup_{0 \leq k \leq n} \|\nabla \vec{u}^k\|^2 + \sum_{k=0}^{n} \|\Delta \vec{u}^k\|^2 \Delta t \leq C^*,$$

$$\sum_{k=0}^{n-1} \left(\left\| \frac{\vec{u}^{k+1} - \vec{u}^k}{\Delta t} \right\|^2 + \|\vec{u}^k \cdot \nabla \vec{u}^k\|^2 \right) \Delta t \leq C^*.$$

Unconditional stability theorem for $N = 2, 3$

Theorem Take $\vec{f} \in L^2(0, T; L^2(\Omega, \mathbb{R}^N))$, $\vec{u}^0 \in H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$. Then $\exists$ positive constants T^* and C^* depending only upon Ω, ν and

$$M_0 := \|\nabla \vec{u}^0\|^2 + \nu \Delta t \|\Delta \vec{u}^0\|^2 + \int_0^T \|\vec{f}\|^2,$$

so that whenever $n\Delta t \leq T^*$ we have

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$$\sum_{k=0}^{n-1} \left(\left\| \frac{\vec{u}^{k+1} - \vec{u}^k}{\Delta t} \right\|^2 + \|\vec{u}^k \cdot \nabla \vec{u}^k\|^2 \right) \Delta t \leq C^*.$$

$\vec{u}_{\text{in}} \in H_0^1 \Rightarrow \exists$ strong solution $\vec{u} \in L^2(0, T^*; H^2) \cap H^1(0, T^*; L^2)$

Estimate on the divergence

Let $w^n = \nabla \cdot \vec{u}^n$. Then as long as $n\Delta t \le T_*$,

$$\sup_{0\le k\le n} \|w^k\|^2_{H^1(\Omega)'} + \sum_{k=1}^{n} \|w^k\|^2 \Delta t \le C(\|w^0\|^2_{H^1(\Omega)'} + \Delta t^{1/2})$$

C^0 finite element scheme (Johnston-L 04)

Finite element spaces: $X_h \subset H_0^1(\Omega, \mathbb{R}^N), \quad Y_h \subset H^1(\Omega)/\mathbb{R}$.

For all $\vec{v}_h \in X_h$ and $q_h \in Y_h$, require

$$\frac{1}{\Delta t}\left(\langle \vec{u}_h^{n+1}, \vec{v}_h\rangle - \langle \vec{u}_h^{n}, \vec{v}_h\rangle\right) + \nu\langle \nabla \vec{u}_h^{n+1}, \nabla \vec{v}_h\rangle$$
$$= -\langle \nabla p_h^n, \vec{v}_h\rangle + \langle \vec{f}^n - \vec{u}_h^n \cdot \nabla \vec{u}_h^n, \vec{v}_h\rangle.$$

$$\langle \nabla p_h^n, \nabla q_h\rangle = \langle \vec{f}^n - \vec{u}_h^n \cdot \nabla \vec{u}_h^n, \nabla q_h\rangle + \nu\langle \nabla \times \vec{u}_h^n, \vec{n} \times \nabla q_h\rangle_\Gamma,$$

Additional divergence damping can be achieved by adding $-\lambda\langle \nabla \cdot \vec{u}_h^n, \phi_h\rangle$ to the pressure equation.

C^1 finite element scheme

Finite element spaces: $X_h \subset H^2 \cap H_0^1(\Omega, \mathbb{R}^N), \quad Y_h \subset H^1(\Omega)/\mathbb{R}$.

For all $\vec{v}_h \in X_h$ and $q_h \in Y_h$, require

$$\frac{1}{\Delta t}\big(\langle \nabla \vec{u}_h^{n+1}, \nabla \vec{v}_h\rangle - \langle \nabla \vec{u}_h^{n}, \nabla \vec{v}_h\rangle\big) + \nu\langle \Delta \vec{u}_h^{n+1}, \Delta \vec{v}_h\rangle$$

$$= \langle \nabla p_h^n, \Delta \vec{v}_h\rangle - \langle \vec{f}^n - \vec{u}_h^n \cdot \nabla \vec{u}_h^n, \Delta \vec{v}_h\rangle.$$

$$\langle \nabla p_h^n, \nabla q_h\rangle = \langle \vec{f}^n - \vec{u}_h^n \cdot \nabla \vec{u}_h^n, \nabla q_h\rangle + \nu\langle \nabla \times \vec{u}_h^n, \vec{n} \times \nabla q_h\rangle_\Gamma,$$

Error estimates of C^1 FE scheme

Theorem Assume Ω is a bounded domain in $\mathbb{R}^N$ (N=2,3) with C^3 boundary. Let $M_0, > 0$, and let $T_* > 0$ be given by the stability theorem. Let $m \geq 2$, $m' \geq 1$ be integers, and assume

(i) The spaces $X_{0,h} \subset H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$ and $Y_h \subset H^1(\Omega)$ have the property that whenever $0 < h < 1$, $\vec{v} \in H^{m+1} \cap H_0^1(\Omega, \mathbb{R}^N)$ and $q \in H^{m'}(\Omega)$,

$$\begin{aligned} \inf_{\vec{v}_h \in X_{0,h}} \|\Delta(\vec{v} - \vec{v}_h)\| &\leq C_0 h^{k-1} \|\vec{v}\|_{H^{k+1}} \quad \text{for } 2 \leq k \leq m, \\ \inf_{q_h \in Y_h} \|\nabla(q - q_h)\| &\leq C_0 h^{m'-1} \|q\|_{H^{m'}}, \end{aligned}$$

where $C_0 > 0$ is independent of $\vec{v}$, q and h.

(ii) $\vec{f} \in C^1([0,T], L^2(\Omega, \mathbb{R}^N))$, $T > 0$, and a given solution of NSE satisfies

$$(\vec{u}, p) \in C^1([0,T]; H^{m+1} \cap H_0^1(\Omega, \mathbb{R}^N)) \times C^1([0,T]; H^{m'}(\Omega)/\mathbb{R}).$$

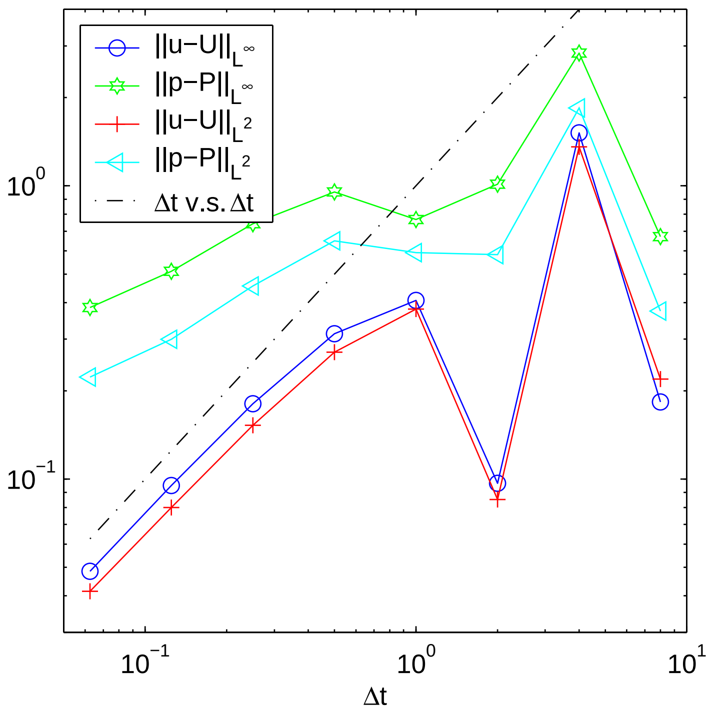
Then there exists $C_1 > 0$ with the following property. Whenever $\vec{u}_h^0 \in X_h$, $0 < h < 1$, $0 < n\Delta t \le \min(T, T_*)$, and

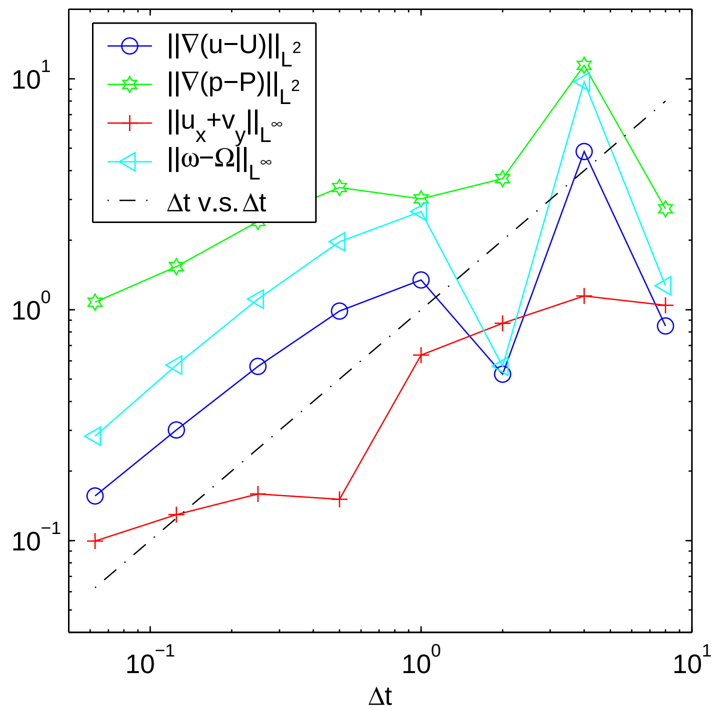
$$\|\nabla \vec{u}_h^0\|^2 + \Delta t \|\Delta \vec{u}_h^0\|^2 + \sum_{k=0}^{n} \|\vec{f}(t_k)\|^2 \Delta t \le M_0,$$

then $\vec{e}^n = \vec{u}(t_n) - \vec{u}_h^n$, $r^n = p(t_n) - p_h^n$ of C^1 finite element scheme satisfy

$$\sup_{0 \le k \le n} \|\nabla \vec{e}^k\|^2 + \sum_{k=0}^{n} \left(\|\Delta \vec{e}^k\|^2 + \|\nabla r^k\|^2 \right) \Delta t$$
$$\le C_1(\Delta t^2 + h^{2m-2} + h^{2m'-2} + \|\nabla \vec{e}^0\|^2 + \|\Delta \vec{e}^0\|^2 \Delta t).$$

Stability check for smooth solution

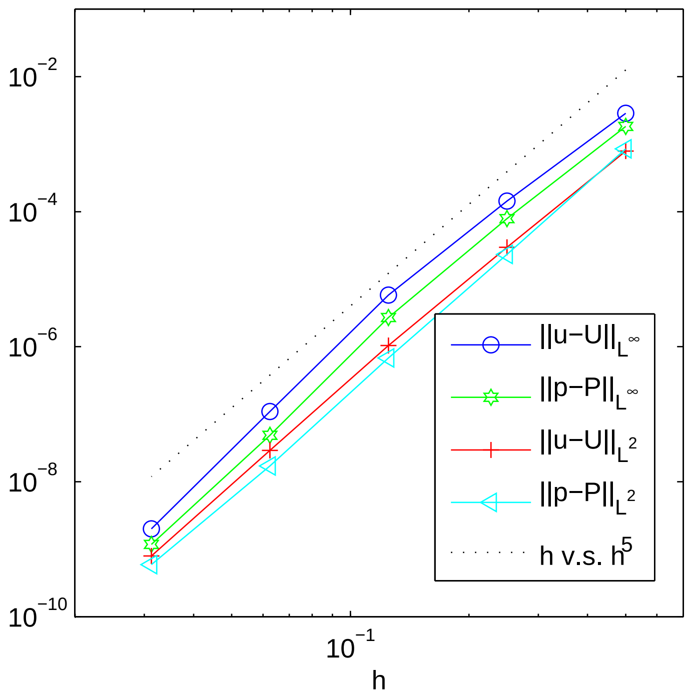


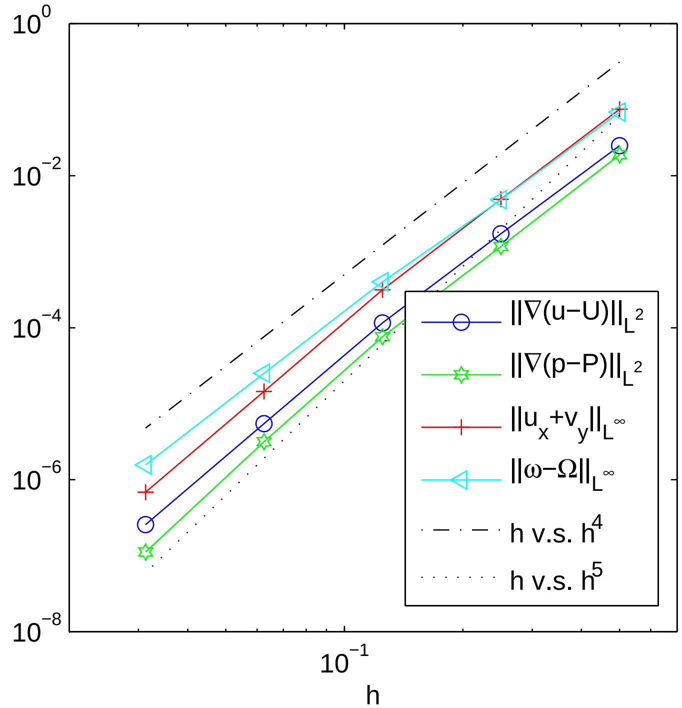


$$\Omega = [-1, 1]^2, \text{ Re=0.5, } t = 1000$$

$$\text{P1/P1 finite elements, } \Delta x = \frac{1}{16}, \text{ max } \Delta t = 8$$

Spatial accuracy check for smooth solution

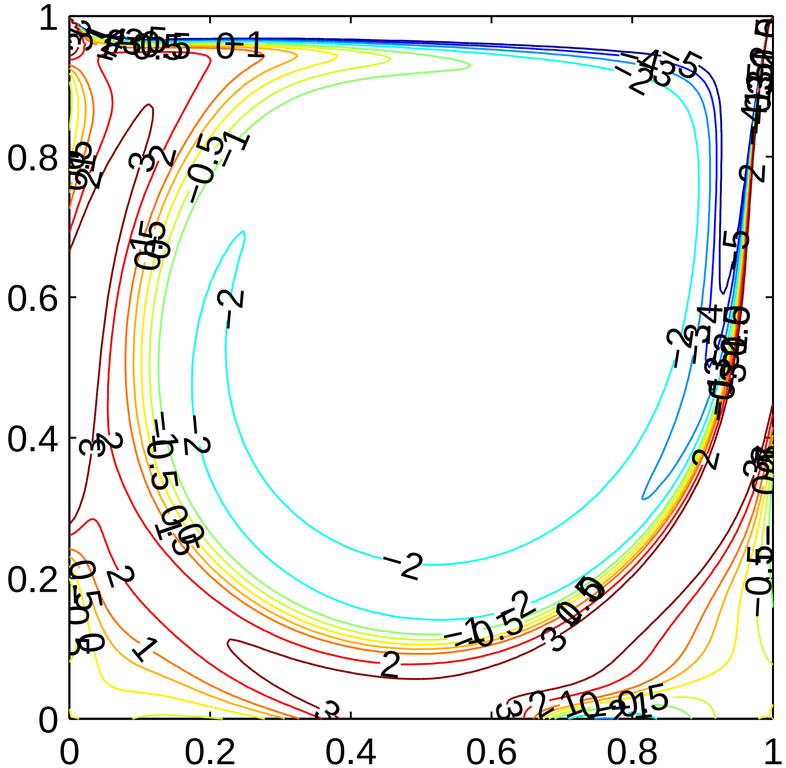


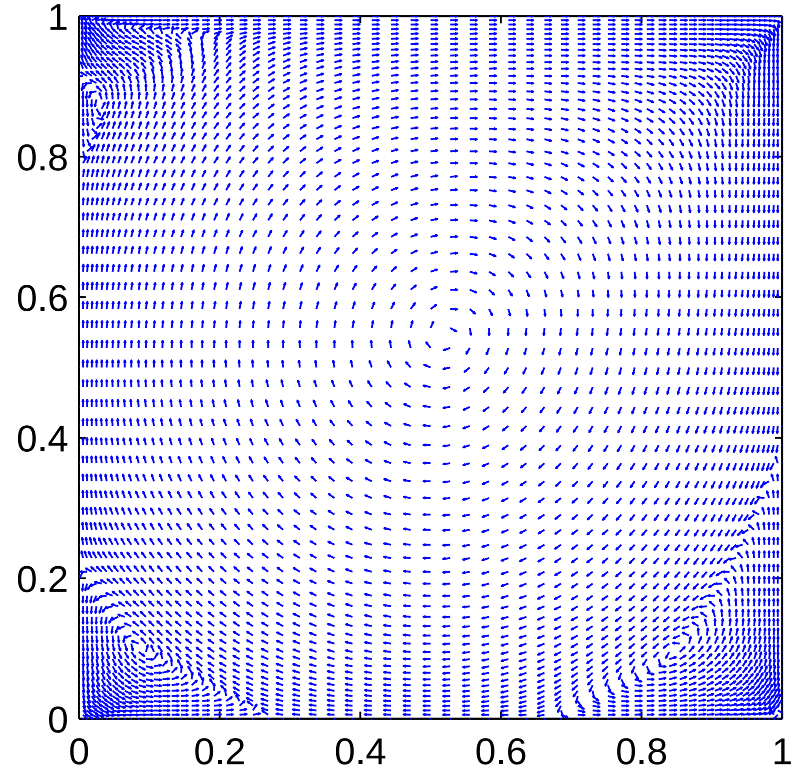


$$\Omega = [-1, 1]^2,\ \nu = .001,\ t = 2,$$

P4/P4 finite elements, RK4 time stepping, $\Delta t = .003$,

Driven cavity flow

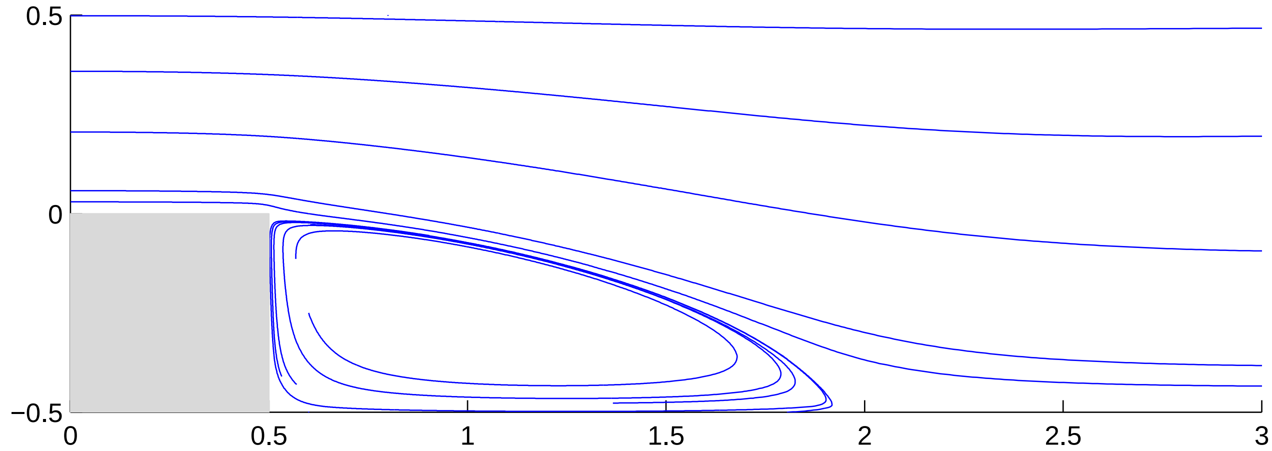




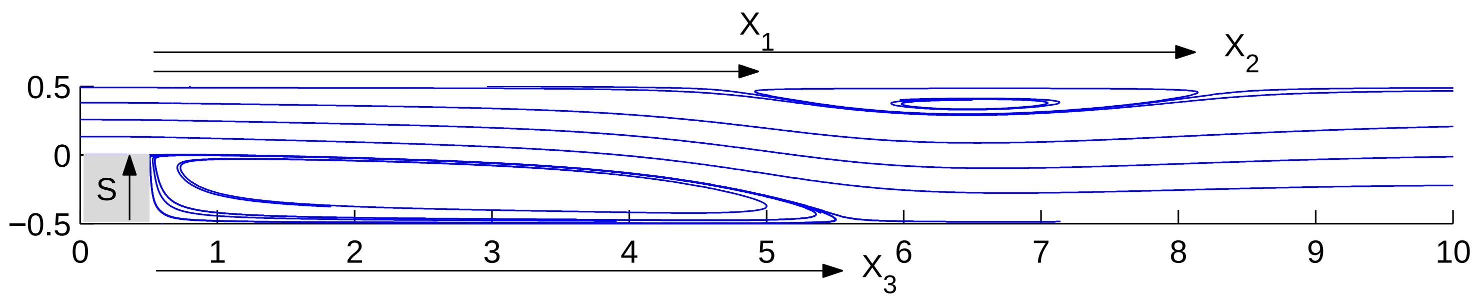
Re=2000. $2 \times 32 \times 32$ P4 elements. $\lambda = 15$.

CN/AB time stepping

Backward facing step flow



Re=100. 528 P4 elements. $\lambda = 20.$ CN/AB time stepping



Re=600. 984 P4 elements. $\lambda = 20.$ CN/AB time stepping

Non-homogeneous side conditions

$$
\begin{aligned}
\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} + \nabla p = \nu \Delta \vec{u} + \vec{f} \qquad & (t > 0, x \in \Omega), \\
\nabla \cdot \vec{u} = h \qquad & (t \geq 0, x \in \Omega), \\
\vec{u} = \vec{g} \qquad & (t \geq 0, x \in \Gamma), \\
\vec{u} = \vec{u}_{\mathrm{in}} \qquad & (t = 0, x \in \Omega).
\end{aligned}
$$

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\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} + \nabla p = \nu \Delta \vec{u} + \vec{f} \qquad & (t > 0, x \in \Omega), \\
\nabla \cdot \vec{u} = h \qquad & (t \geq 0, x \in \Omega), \\
\vec{u} = \vec{g} \qquad & (t \geq 0, x \in \Gamma), \\
\vec{u} = \vec{u}_{\text{in}} \qquad & (t = 0, x \in \Omega).
\end{aligned}
$$

Unconstrained formulation involves an inhomogeneous pressure p_{gh}:

$$
\partial_t \vec{u} + \mathcal{P}(\vec{u} \cdot \nabla \vec{u} - \vec{f} - \nu \Delta \vec{u}) + \nabla p_{gh} = \nu \nabla \nabla \cdot \vec{u}
$$

$$
\langle \nabla p_{gh}, \nabla \phi \rangle = -\langle \vec{n} \cdot \partial_t \vec{g}, \phi \rangle_\Gamma + \langle \partial_t h, \phi \rangle + \langle \nu \nabla h, \nabla \phi \rangle \quad \forall \phi \in H^1(\Omega).
$$

$\nabla \cdot \vec{u} - h$ satisfies a heat equation with Neumann BCs as before.

Summary: NSE is a perturbed heat equation!

$$\vec{u}_t + \mathcal{P}(\vec{u} \cdot \nabla \vec{u} - \vec{f}) \;\; = \;\; \nu \mathcal{P} \Delta \vec{u}$$

Summary: NSE is a perturbed heat equation!

$$\vec{u}_t + \mathcal{P}(\vec{u} \cdot \nabla \vec{u} - \vec{f}) = \nu \mathcal{P} \Delta \vec{u} + \nu \nabla (\nabla \cdot \vec{u})$$

Summary: NSE is a perturbed heat equation!

$$\begin{aligned} \vec{u}_t + \mathcal{P}(\vec{u} \cdot \nabla \vec{u} - \vec{f}) &= \nu \mathcal{P} \Delta \vec{u} + \nu \nabla (\nabla \cdot \vec{u}) \\ &= \nu \mathcal{P} \Delta \vec{u} + \nu \Delta (I - P) \vec{u} \end{aligned}$$

Summary: NSE is a perturbed heat equation!

$$
\begin{aligned}
\vec{u}_t + \mathcal{P}(\vec{u} \cdot \nabla \vec{u} - \vec{f}) &= \nu \mathcal{P} \Delta \vec{u} + \nu \nabla (\nabla \cdot \vec{u}) \\
&= \nu \mathcal{P} \Delta \vec{u} + \nu \Delta (I - P) \vec{u} \\
&= \nu \Delta \vec{u} - \nu [\Delta, \mathcal{P}] \vec{u}
\end{aligned}
$$

Summary: NSE is a perturbed heat equation!

$$
\begin{aligned}
\vec{u}_t + \mathcal{P}(\vec{u}\cdot\nabla\vec{u} - \vec{f}) &= \nu\mathcal{P}\Delta\vec{u} + \nu\nabla(\nabla\cdot\vec{u}) \\
&= \nu\mathcal{P}\Delta\vec{u} + \nu\Delta(I - P)\vec{u} \\
&= \nu\Delta\vec{u} - \nu[\Delta,\mathcal{P}]\vec{u} \\
&= \nu\Delta\vec{u} - \nu\nabla p_{\mathsf{s}}
\end{aligned}
$$

- $\nabla\cdot\vec{u}$ satisfies a heat equation with Neumann BCs
- The Stokes pressure term is strictly controlled by viscosity
- Stability, existence, uniqueness theory is greatly simplified
- Promise of enhanced flexibility in design of numerical schemes

Summary: NSE is a perturbed heat equation!

$$\begin{aligned}
\vec{u}_t + \mathcal{P}(\cancel{\vec{u}\cdot\nabla\vec{u}} - \vec{f}) &= \nu\mathcal{P}\Delta\vec{u} + \nu\nabla(\nabla\cdot\vec{u}) \\
&= \nu\mathcal{P}\Delta\vec{u} + \nu\Delta(I - P)\vec{u}) \\
&= \nu\Delta\vec{u} - \nu[\Delta, \mathcal{P}]\vec{u} \\
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\end{aligned}$$

- $\nabla\cdot\vec{u}$ satisfies a heat equation with Neumann BCs
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Existence and uniqueness theorem

Assume

$$\vec{u}_{\rm in} \in H_{uin} := H^1(\Omega, \mathbb{R}^N),$$
$$\vec{f} \in H_f := L^2(0,T; L^2(\Omega, \mathbb{R}^N)),$$
$$\vec{g} \in H_g := H^{3/4}(0,T; L^2(\Gamma, \mathbb{R}^N)) \cap L^2(0,T; H^{3/2}(\Gamma, \mathbb{R}^N))$$
$$\cap \{\vec{g} \mid \partial_t(\vec{n} \cdot \vec{g}) \in L^2(0,T; H^{-1/2}(\Gamma))\},$$
$$h \in H_h := L^2(0,T; H^1(\Omega)) \cap H^1(0,T; (H^1)'(\Omega)).$$

and the compatibility conditions

$$\vec{g} = \vec{u}_{\rm in} \quad (t = 0, x \in \Gamma), \qquad \langle \vec{n} \cdot \partial_t \vec{g}, 1 \rangle_\Gamma = \langle \partial_t h, 1 \rangle_\Omega.$$

Then $\exists T^* > 0$ so that a unique strong solution exists, with

$$\vec{u} \in L^2([0,T^*], H^2) \cap H^1([0,T^*], L^2) \hookrightarrow C([0,T^*], H^1).$$