Partially integrable dynamics of ensembles of nonidentical nonlinearly coupled oscillators

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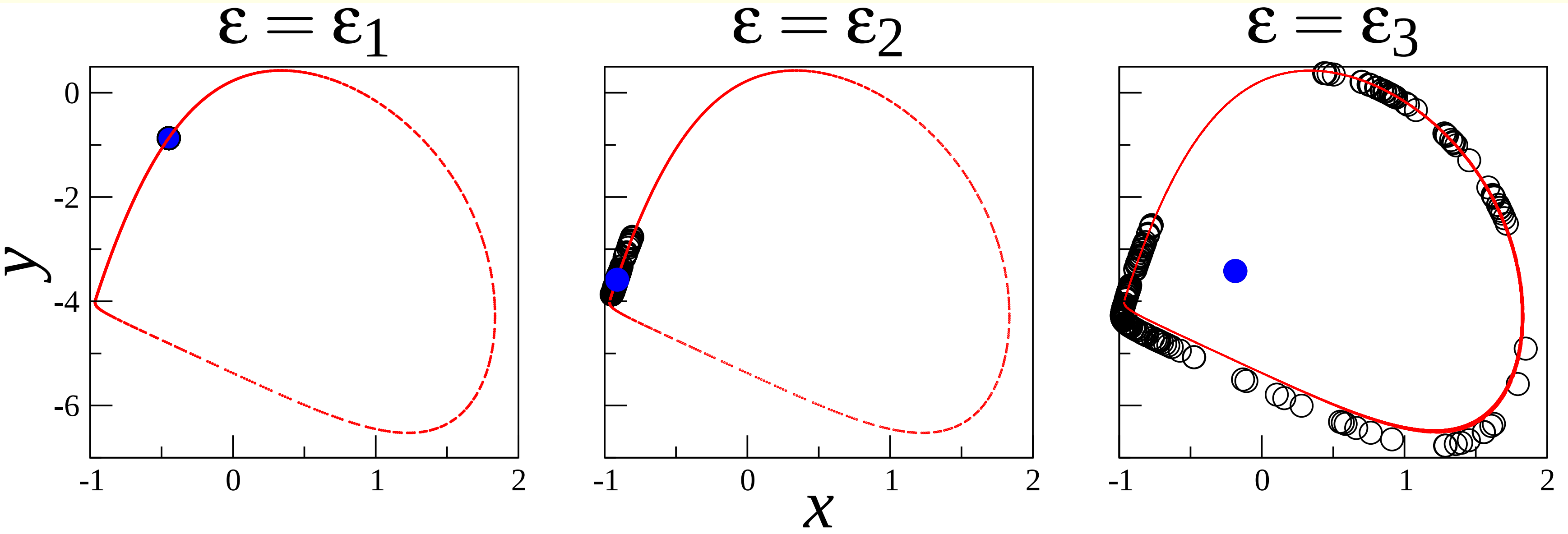
An example: coupled identical Hindmarsh-Rose neurons

$$
\begin{aligned}
\dot{x}_k &= y_k - x_k^3 + 3x_k^2 - z_k + 5.1 + \frac{\varepsilon}{N}\sum_{j=1}^{N} x_j \\
\dot{y}_k &= 1 - 5x_k^2 - y_k \\
\dot{z}_k &= 0.006 \cdot \left(4(x_k + 1.56) - z_k\right)
\end{aligned}
$$



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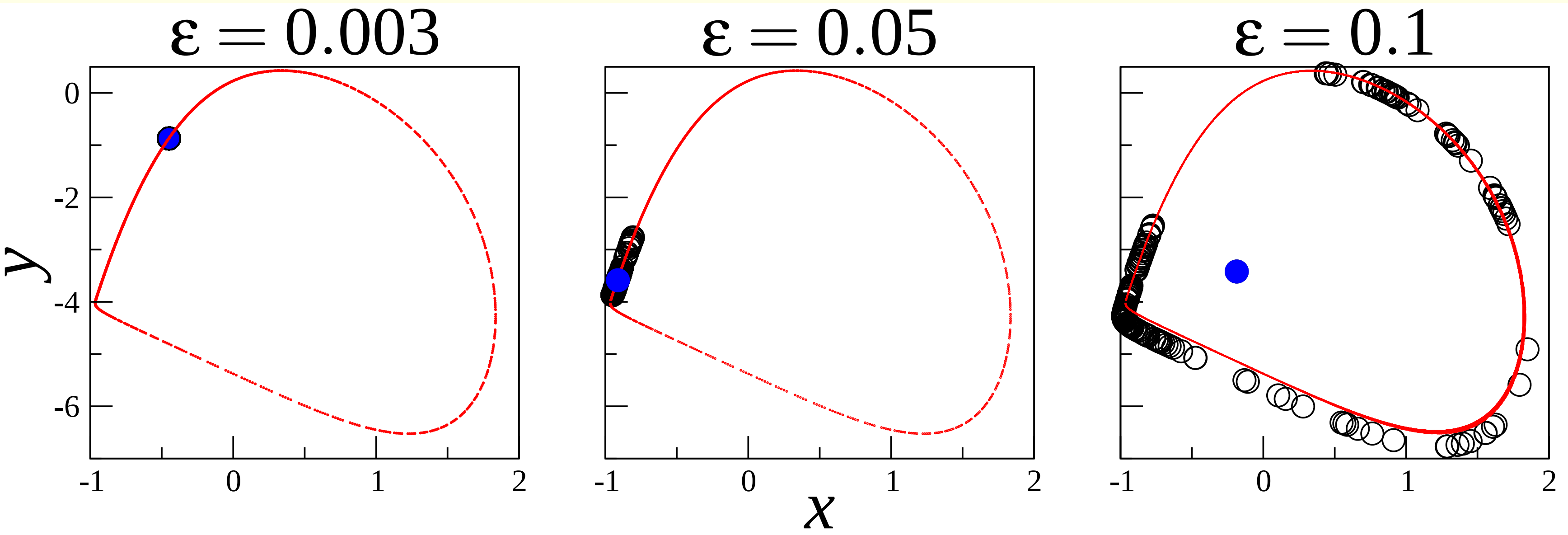

Ensembles of globally coupled oscillators

- Global coupling = all-to-all coupling = mean field coupling
- Models of many natural phenomena
  - in physics: arrays of Josephson junctions, multimode lasers,...
  - in neuroscience: pathological brain rhythms, binding problem,...
  - in social behavior: synchronously blinking fireflies, pedestrians on the Millennium Bridge, rhythmical applause in a large audience...
  - etc
- Main effect: synchronization $\Longrightarrow$
  - adjustment of phases of indvidual oscillators due to coupling
  - appearance of a **macroscopic mean field**

Typically (most studied): increase of coupling facilitates synchrony

We concentrate on the opposite case

Object and methods of our study

Large ensembles

with

global nonlinear

coupling

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Watanabe-Strogatz

(WS) theory

PRL 1993

Physica D 1994

Ott-Antonsen

(OA) ansatz

Chaos 2008, 2009

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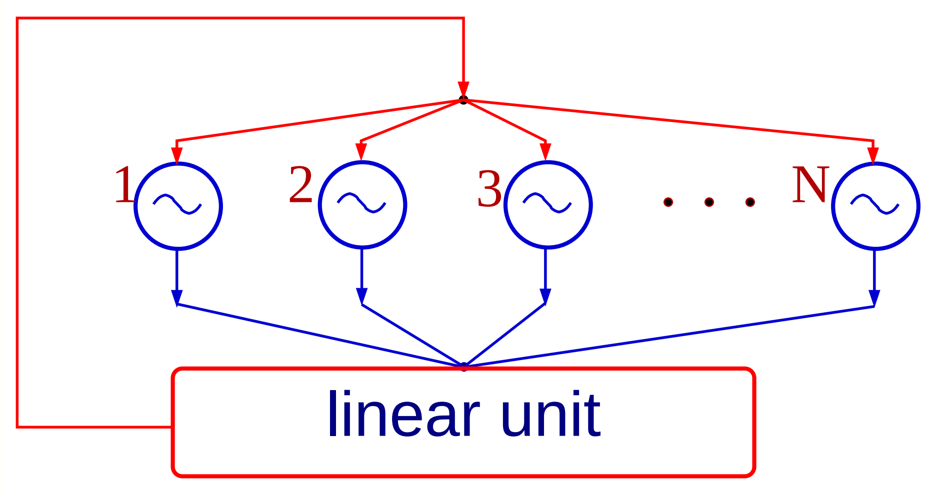
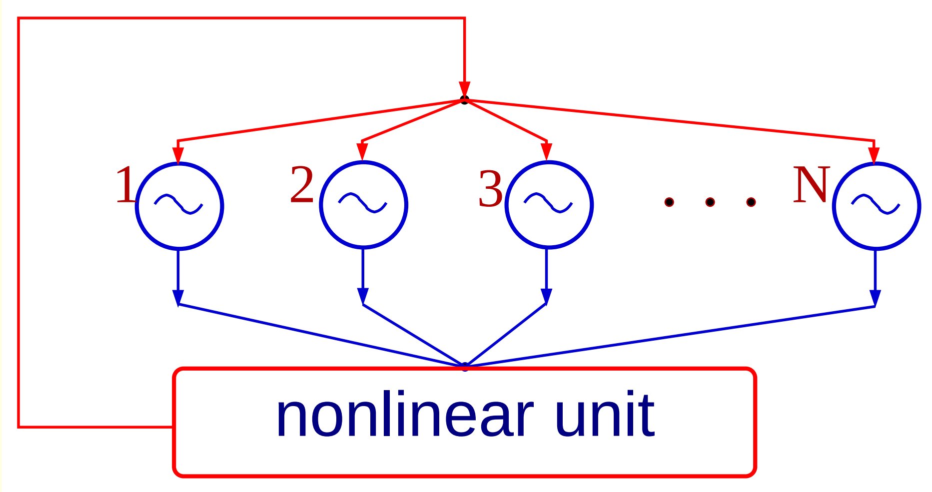
Linear vs nonlinear coupling





Linear coupling $\Longleftrightarrow$ form of interaction does not depend on the mean field amplitude r

Nonlinear coupling $\Longleftrightarrow$ form of interaction **depends** on r

i.e. interaction is attractive for small r and repulsive for large r

Stability of the synchronous cluster

- consider a general system of identical oscillators

$$\dot{\mathbf{x}}_k = \mathbf{F}(\mathbf{x}_k, \mathbf{u}, \mathbf{Y}; \varepsilon) \ , \qquad \dot{\mathbf{u}} = \mathbf{G}(\mathbf{u}, \mathbf{Y}; \varepsilon) \ ,$$

  where
  - $\mathbf{x}_k$ describe individual systems,
  - $\mathbf{Y}(\mathbf{x})$ are mean fields,
  - $\mathbf{u}$ describe the dynamics of the coupling,
  - ε is the coupling strength.
- generally, we expect that the stability of full synchrony can break with an increase of ε $\Longrightarrow$ we call such coupling nonlinear; in this case we expect complex dynamics

An example: nonlinearly coupled Landau-Stuart oscillators

1) coupled via a common nonlinear load

$$\frac{da_k}{dt} = (1+i)a_k - |a_k|^2 a_k + e^{i\xi} u$$

$$\frac{du}{dt} = (-\gamma + i)u + i\eta |u|^2 u + Y$$

Complex mean field $Y = N^{-1} \sum_k a_k = r e^{i\Theta}$

2) nonlinearly coupled via mean field

$$\frac{da_k}{dt} = (1+i)a_k - |a_k|^2 a_k + (\mu_1 + i\mu_2)Y - (\eta_1 + i\eta_2)|Y|^2 Y$$

In phase approximation both models yield the same phase model

Landau-Stuart model I:

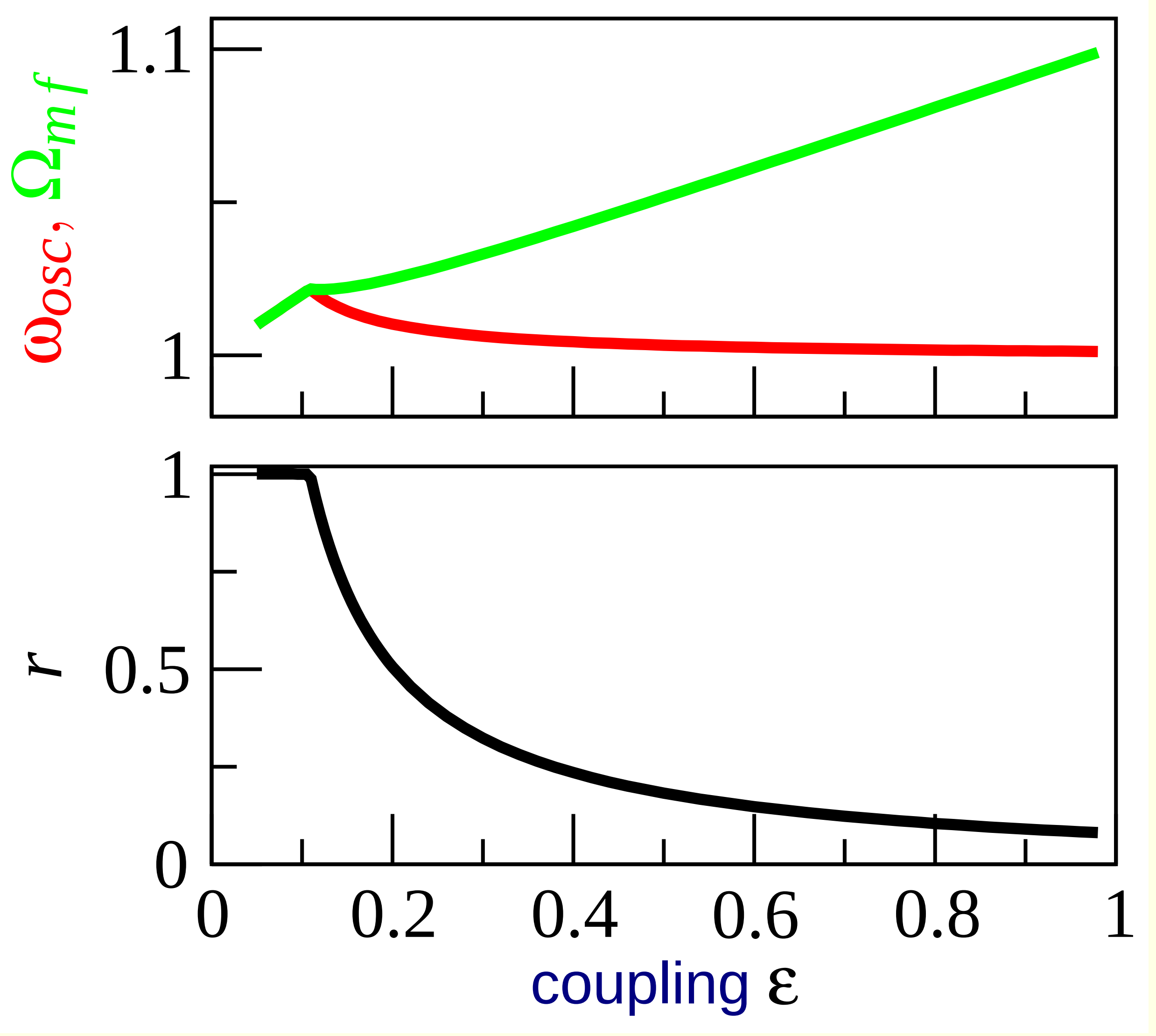
loss of synchrony with increase of coupling

Parameters:

$\gamma = 0.5$

$\eta = 10^3$

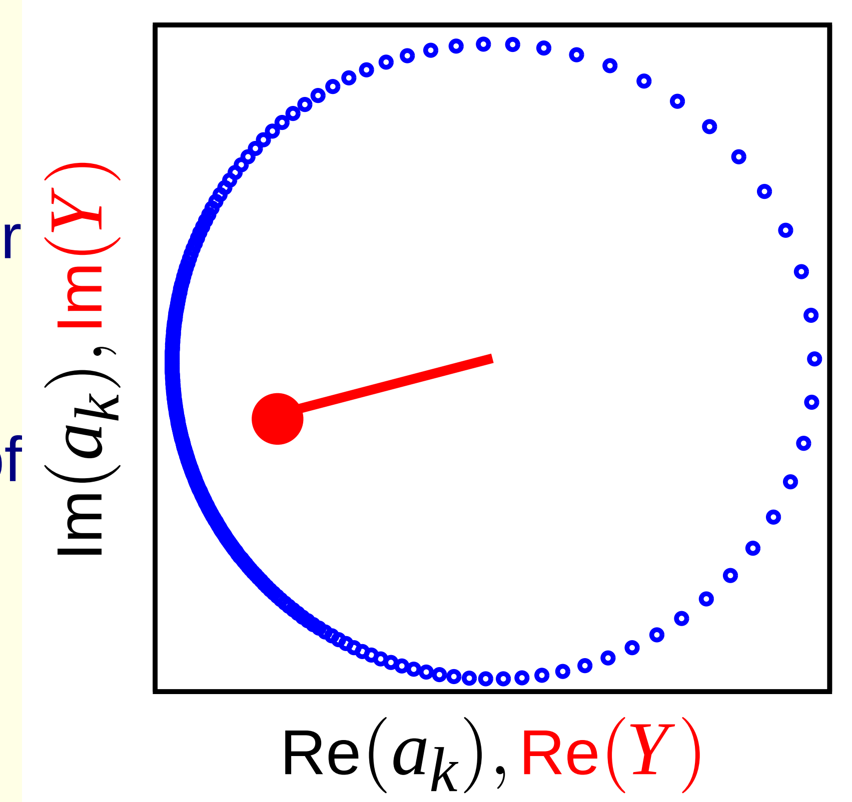
$\xi = 0.475\pi$



Landau-Stuart model I:

snapshot of the ensemble

- non-uniform distribution of oscillator phases, here for $\varepsilon - \varepsilon_q = 0.05$
- different velocities of oscillators and of the mean field



Nonlinear phase model

$$\dot{\phi}_k = \omega_k(r,\varepsilon) + A(r,\varepsilon)\varepsilon\,\mathrm{Im}\left[e^{i\beta(r,\varepsilon)} Y e^{-i\phi_k}\right]$$

Dependence of **coupling strength** $A\varepsilon$ and **phase shift** β

on the mean field amplitude $r = |Y|$

Landau-Stuart models in phase approximation yield this model

with $\omega =$ const, $A =$ const, and $\beta = \beta_0 + \beta_1 \varepsilon^2 r^2$

Rosenblum and Pikovsky, *PRL*, **98**, 054102 (2007)

Pikovsky and Rosenblum, *Physica D*, **238** (1), 27-37 (2009)

Nonlinear phase model II

$$\boxed{\dot{\phi}_k = \omega_k(r,\varepsilon) + A(r,\varepsilon)\varepsilon \,\mathrm{Im}\left[e^{i\beta(r,\varepsilon)} Y e^{-i\phi_k}\right]}$$

Particular cases:

- $\omega =$ const, $A = 1$, $\beta =$ const: the Kuramoto-Sakaguchi model
- $\omega =$ const, $\beta =$ const: Filatrela *et al.*, PRE 2007,
  Gianuzzi *et al.*, PRE 2007

Linear vs nonlinear coupling once again

Let us consider phase model

$$\dot{\phi}_k = \omega + \varepsilon r \sin(\Theta - \phi_k + \beta(r, \varepsilon))$$

and quantify stability of the synchronous solution

- For the Kuramoto-Sakaguchi model with $\beta =$ const:

  the eigenvalue is $\lambda = -\varepsilon \cos\beta$ $\Longrightarrow$ stability grows with ε

  $\Longrightarrow$ the coupling is **linear**

- For the nonlinear model $\lambda = -\varepsilon \cos(\beta(1, \varepsilon))$

  Example: $\beta = \beta_0 + \varepsilon^2 r^2$ $\Longrightarrow$ synchrony breaks for $\varepsilon = \sqrt{\frac{\pi}{2} - \beta_0}$

  $\Longrightarrow$ the coupling is **nonlinear**

Analytical tools

- Watanabe-Strogatz theory for identical oscillators

- Extending WS theory for the case of nonidentical oscillators

- Linking WS and OA theories

Watanabe-Strogatz theory (PRL 1993, Physica D 1994)

The model: sine-coupled **identical** phase oscillators

$$\dot{\phi}_k = \omega(t) + \mathrm{Im}\left[H(t)e^{-i\phi_k}\right] , \quad k = 1,\ldots,N > 3$$

Particular cases:

- $\omega =$ const, $H = \varepsilon e^{i\beta} Y$ (the Kuramoto-Sakaguchi model)
- $H = A(|Y|,\varepsilon)\varepsilon e^{i\beta(|Y|,\varepsilon)} Y$ (the generalized phase model)

Main result: for **any** functions $\omega(t)$, $H(t)$, N-dimensional system is completely described by 3 global variables plus $N-3$ constants of motion

- global amplitude ρ, $0 \le \rho \le 1$
- global phases Ψ, Φ
- constants of motion ψ_k

WS equations (in our notations)

$$\frac{d\rho}{dt} = \frac{1-\rho^2}{2}\mathrm{Re}\left(He^{-i\Phi}\right)$$

$$\frac{d\Phi}{dt} = \omega + \frac{1+\rho^2}{2\rho}\mathrm{Im}\left(He^{-i\Phi}\right)$$

$$\frac{d\Psi}{dt} = \frac{1-\rho^2}{2\rho}\mathrm{Im}\left(He^{-i\Phi}\right)$$

Remark: the global variables used differ from originally WS variables:

$$\tilde{\rho} = \frac{2\rho}{1+\rho^2} \qquad \tilde{\Psi} = \Psi + \pi \qquad \tilde{\Phi} = \Phi + \pi$$

WS equations (equivalent form)

We introduce

$$z = \rho e^{i\Phi}$$

$$\alpha = \Phi - \Psi$$

and re-write the equations as

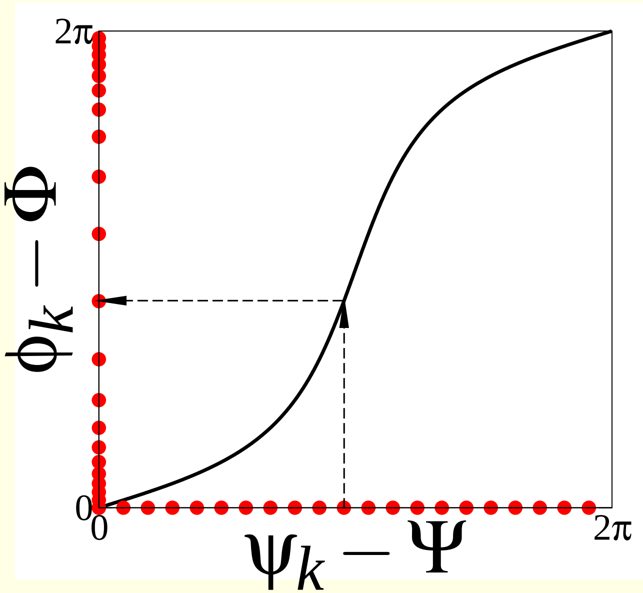
$$\frac{dz}{dt} = i\omega z + \frac{1}{2}H - \frac{z^2}{2}H^*$$

$$\frac{d\alpha}{dt} = \omega + \mathrm{Im}\left(z^* H\right)$$

Transformation to original variables

Time-dependent transformation:

$$\tan\left(\frac{\phi_k - \Phi}{2}\right) = \frac{1-\rho}{1+\rho}\tan\left(\frac{\psi_k - \Psi}{2}\right)$$



Hier ψ_k, $k = 1, \ldots, N$ are constants of motion, they obey 3 constraints:

$$\sum_{k=1}^{N} e^{i\psi_k} = 0\,, \qquad \sum \cos(2\psi_k) = 0$$

Meaning of the WS variables (qualitative)

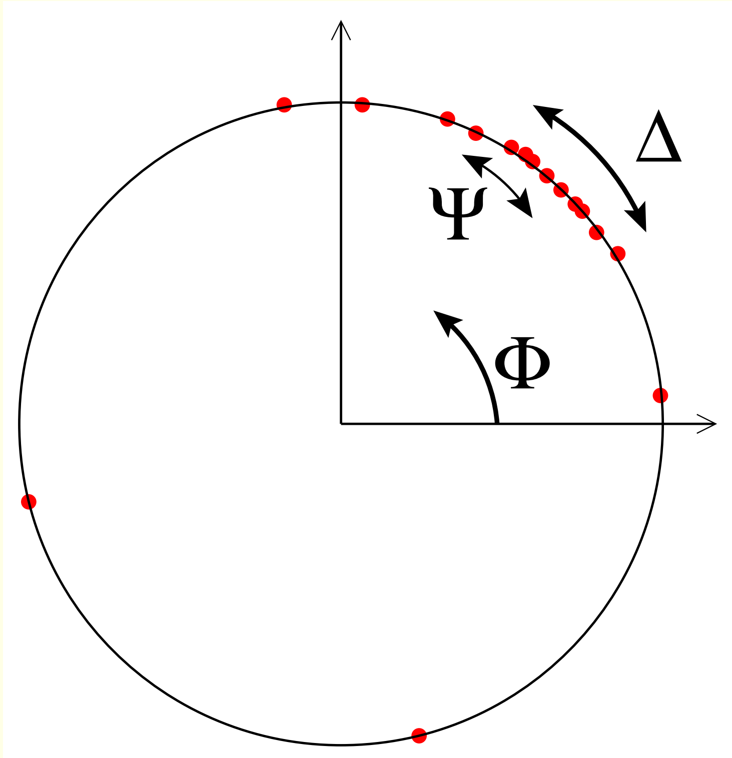
- Amplitude ρ is related to the width Δ of the bunch

  Easy to check: if $\rho = 0$ then $r = 0$

  $\rho = 1$ then $r = 1$

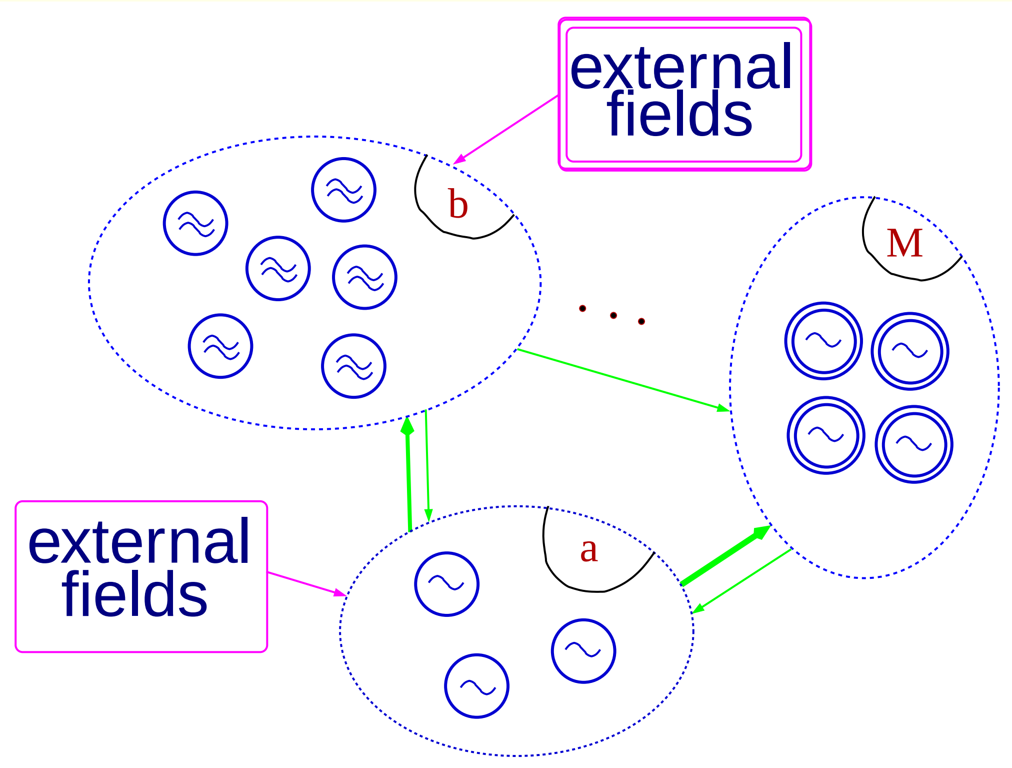
  Hence,

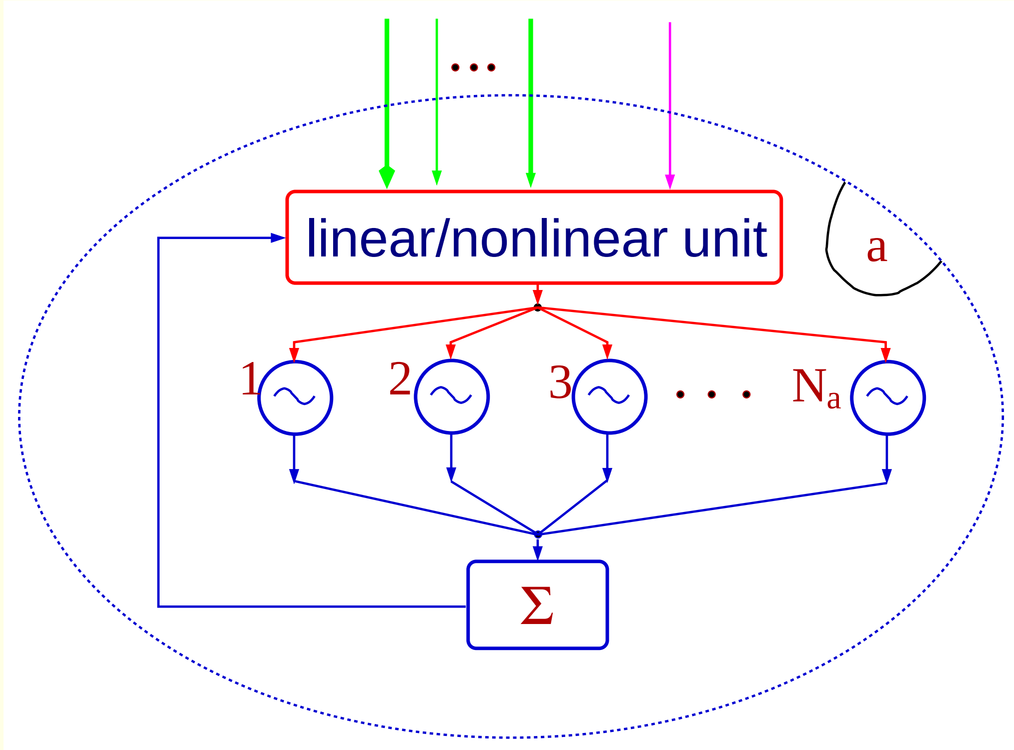
  amplitude $\rho \sim$ mean field amplitude r
- Phase $\Phi \sim$ mean field phase Θ
- Phase Ψ: shift of individual phases with respect to Φ
- Reminder: our variables differ from the original WS variables



How to extend the WS theory to treat non-identical oscillators: the main idea

- Group oscillators into M groups, $a = 1, \ldots, M$, so that each group contains identical units subject to common force





- Apply the WS theory to each group: the dynamics is then $3M$-dimensional

How to extend the WS theory to treat non-identical oscillators II

- Hierachically organized population: M subpopulations of size N_a, $\sum_a N_a = N$
- Microscopic equations of motion:

$$\frac{d\phi_k^{(a)}}{dt} = \omega_a + \mathrm{Im}\left(H_a e^{-i\phi_k^{(a)}}\right)$$

- Macroscopic equations of motion:

M coupled systems of 3 WS equations each

- Infinitely large population:
  - Thermodynamical limit I:

    $\Longrightarrow$ M finite, $N_a \to \infty$, several infinitely large subpopulations
  - Thermodynamical limit II:

    $\Longrightarrow$ $M \to \infty$, continuous frequency distribution

Local and global mean fields

We characterize every subpopulation by its own (local) Kuramoto mean field (order parameter):

$$Z_a = r_a e^{i\Theta_a} = N_a^{-1} \sum_{k=1}^{N_a} e^{i\phi_k^{(a)}} \quad \text{or} \quad Z_a = \int_0^{2\pi} w_a(\phi) e^{i\phi} d\phi \, ,$$

to be distinguished from the global mean field

$$Y = r e^{i\Theta} = N^{-1} \sum_{a=1}^{M} N_a Z_a \qquad \text{or} \quad Y = \int_{-\infty}^{\infty} n(\omega) Z(\omega) d\omega$$

where $n(\omega)$ is the frequency distribution

WS equations for hierarchical population

Finite number of subpopulations

We label subpopulations with indices $a = 1, \dots, M$, $b = 1, \dots, M$

$$\frac{dz_a}{dt} = i\omega_a z_a + \frac{1}{2}H_a - \frac{z_a^2}{2}H_a^*$$

$$\frac{d\alpha_a}{dt} = \omega_a + \mathrm{Im}(z_a^* H_a)$$

Here H_a is the effective force, common for all oscillators in a

E.g., for mean field coupling $H_a = \sum_b E_{ab} n_b Z_b$,

where E_{ab} quantifies coupling between subpopulations a and b

WS equations for hierarchical population

Infinite number of subpopulations

We consider a system with a continuous distribution of frequencies and identify subpopulation index a with the frequency ω

Performing the limit $M \to \infty$, we obtain:

$$\frac{\partial z(\omega,t)}{\partial t} = i\omega z + \frac{1}{2} H(\omega,t) - \frac{z^2}{2} H^*(\omega,t)$$

$$\frac{\partial \alpha(\omega,t)}{\partial t} = \omega + \text{Im}[z^* H(\omega,t)]$$

Direct WS reduction for a system with continuous distribution of parameters (following Watanabe and Strogatz, 1994)

- Continuity equation $\frac{\partial w}{\partial t} + \frac{\partial}{\partial \phi}(w\dot{\phi}) = 0$ for $w(\phi, \omega, t)$
- Variable substitution in the continuity equation:

$$t,\ \phi,\ \omega \quad \to \quad \tau = t,\ \psi = \psi(\omega, \phi, t),\ \nu = \omega$$

$$w(\omega, \phi, t) \quad \to \quad \sigma(\nu, \psi, \tau)$$

- New density function $\sigma(\nu, \psi, \tau)$ is stationary provided z and α obey the WS equations

Further simplification. Link to OA ansatz

Transformation to original variables II

We re-write the WS transformation

$$\tan\left(\frac{\phi_k - \Phi}{2}\right) = \frac{1-\rho}{1+\rho}\tan\left(\frac{\psi_k - \Psi}{2}\right)$$

in the exponential form

$$\boxed{e^{i\phi_k} = e^{i\Phi}\frac{\rho + e^{i(\psi_k - \Psi)}}{1 + \rho e^{i(\psi_k - \Psi)}}}$$

(Pikovsky and Rosenblum, *PRL* (2008))

(Möbius transformation, Marvel *et al.*, CHAOS 2009)

Meaning of the WS variables (quantitative)

Using the WS transformation $e^{i\phi_k} = e^{i\Phi}\frac{\rho + e^{i(\psi_k - \Psi)}}{1 + \rho e^{i(\psi_k - \Psi)}}$

we relate the WS variables to the complex Kuramoto mean field:

$$Z = re^{i\Theta} = N^{-1}\sum_{k=1}^{N} e^{i\phi_k} = \rho e^{i\Phi}\gamma(\rho, \Psi) = z\gamma(\rho, \Psi) \ ,$$

where

$$\gamma(\rho, \Psi) = N^{-1}\sum_{k=1}^{N}\frac{1 + \rho^{-1}e^{i(\psi_k - \Psi)}}{1 + \rho e^{i(\psi_k - \Psi)}}$$

Note: the relation is very simple, $\boxed{Z = z}$, if $\boxed{\gamma = 1}$

Remark: this is valid for each subpopulation, i.e. for γ_a or $\gamma(\omega)$

Meaning of the WS variables (quantitative) II

We write $\gamma(\rho, \Psi)$ as a series

$$\gamma = 1 + (1 - \rho^{-2}) \sum_{l=2}^{\infty} C_l (-\rho e^{-i\Psi})^l \, ,$$

where C_l are the amplitudes of the Fourier harmonics of the distribution of constants of motion ψ_k:

$$C_l = N^{-1} \sum_{k=1}^{N} e^{il\psi_k} \quad \text{or} \quad C_l = \int_{-\pi}^{\pi} \sigma(\psi) e^{il\psi} d\psi$$

$\Longrightarrow$ For uniform distribution of ψ and large N we have

$$\boxed{\gamma = 1 \quad \Longrightarrow \quad \rho = r, \Phi = \Theta \quad \text{or} \quad z = Z}$$

Reduction to one WS equation

For the case $\gamma_a = 1$, i.e. $z_a = Z_a$, WS equations are:

$$\frac{dZ_a}{dt} = i\omega_a Z_a + \frac{1}{2}H_a - \frac{Z_a^2}{2}H_a^*$$

$$\frac{d\alpha_a}{dt} = \omega_a + \text{Im}(Z_a^* H_a)$$

Most common case: mean field coupling, $H_a = \sum_{b=1}^{M} E_{ab} n_b Z_b$

H_a is independent of α $\Longrightarrow$ 2nd equation becomes irrelevant

Similarly, for a continuous distribution we are left with

$$\frac{\partial Z(\omega,t)}{\partial t} = i\omega Z + \frac{1}{2}H - \frac{Z^2}{2}H^*$$

$\boxed{\text{Ott–Antonsen equation (CHAOS 2008)}}$

Ott-Antonsen solution of the Kuramoto problem (CHAOS 2008)

- an infinitely large ensemble with the frequency distribution $n(\omega)$
- the density function $w(\omega,\phi,t)$ as a Fourier series

$$w(\omega,\phi,t)=\frac{n(\omega)}{2\pi}\left\{1+\left[\sum_{m=1}^{\infty} f_m(\omega,t)e^{-im\phi}+\text{c.c.}\right]\right\}$$

- density functions with $f_m(\omega,t)=[F(\omega,t)]^m$

  satisfy the continuity equation $\frac{\partial w}{\partial t}+\frac{\partial}{\partial\phi}(w\dot{\phi})=0$
- $\Longrightarrow$ equation for the temporal dynamics of the field
- we call the set of found solutions the OA reduced manifold
- these solutions are the **only** attractors if $n(\omega)$ is **smooth**

  (Ott and Antonsen, CHAOS 2009)

OA ansatz $\Longleftrightarrow$ uniform distribution of ψ in the WS theory

Application of the theory: Nonlinearly coupled ensemble

Lorentzian frequency distribution $n(\omega) = [\pi(\omega^2+1)]^{-1}$

Effective force: $H = A\varepsilon e^{i\beta} Y = A\varepsilon e^{i\beta} \int_{-\infty}^{\infty} n(\omega) Z(\omega) d\omega$

Full description:

$$\frac{\partial z(\omega,t)}{\partial t} = i\omega z + \frac{A\varepsilon e^{i\beta}}{2} Y - \frac{A\varepsilon e^{-i\beta}}{2} z^2 Y^*$$

$$\frac{\partial \alpha(\omega,t)}{\partial t} = \omega + \mathrm{Im}\left(A\varepsilon e^{i\beta} z^* Y\right)$$

Lorentzian distribution: reduced description

Reduced description (OA ansatz): $z = Z \quad \Longrightarrow \quad$ one WS equation

$$\frac{\partial Z(\omega,t)}{\partial t} = i\omega Z + \frac{\varepsilon A e^{i\beta}}{2} Y - \frac{\varepsilon A e^{-i\beta}}{2} Z^2 Y^*$$

plus the integral $Y = \int_{-\infty}^{\infty} n(\omega) Z(\omega) d\omega = Z(i)$ (see OA, Chaos 2008)

$\Longrightarrow$ WS-OA equation in the real form:

$$\frac{dr}{dt} = -r + \frac{\varepsilon A(r,\varepsilon)}{2} r(1-r^2)\cos\beta(r,\varepsilon)$$

$$\frac{d\Theta}{dt} = \Omega = \frac{\varepsilon A(r,\varepsilon)}{2}(1+r^2)\sin\beta(r,\varepsilon)\ ,$$

Condition $\dot{r} = 0$ yields an equation for r

If it is solved (numerically), then Ω can be found, too.

Example: $A = 1,\ \beta = \beta_0 + \varepsilon^2 r^2$ Main result: multistability

Diagram of regimes for $A = 1$ and $\beta = \beta_0 + \varepsilon^2 r^2$

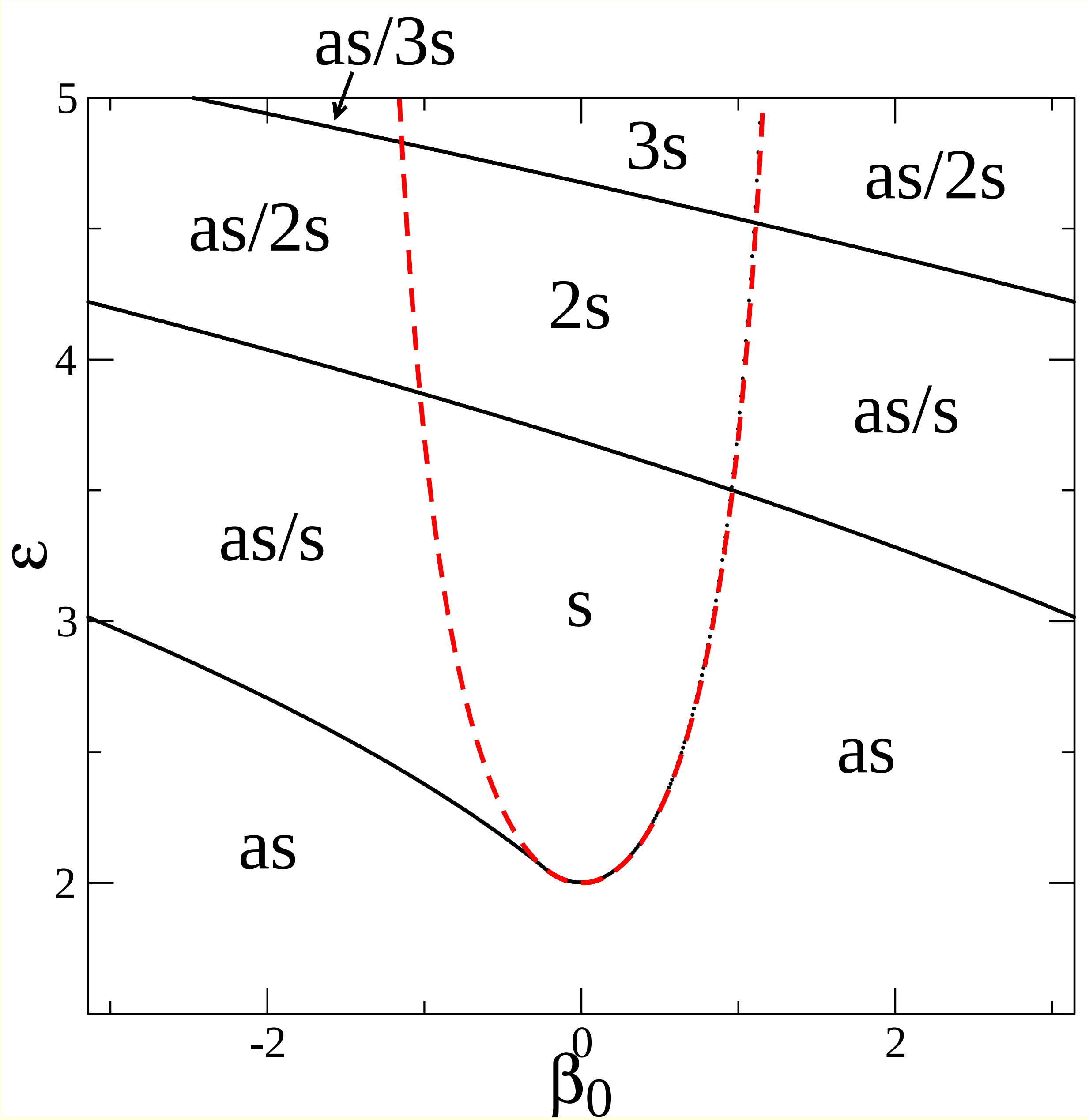
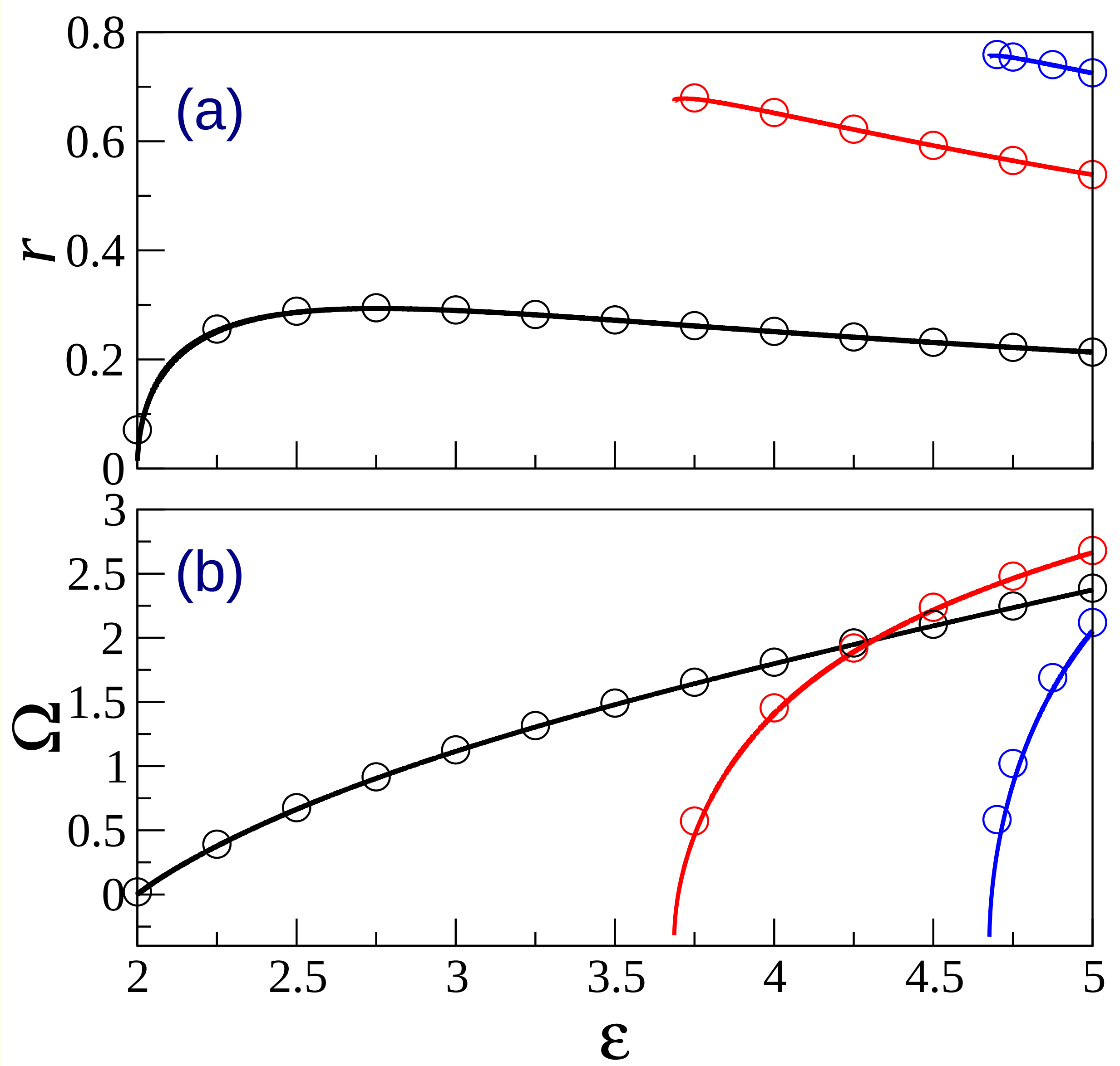


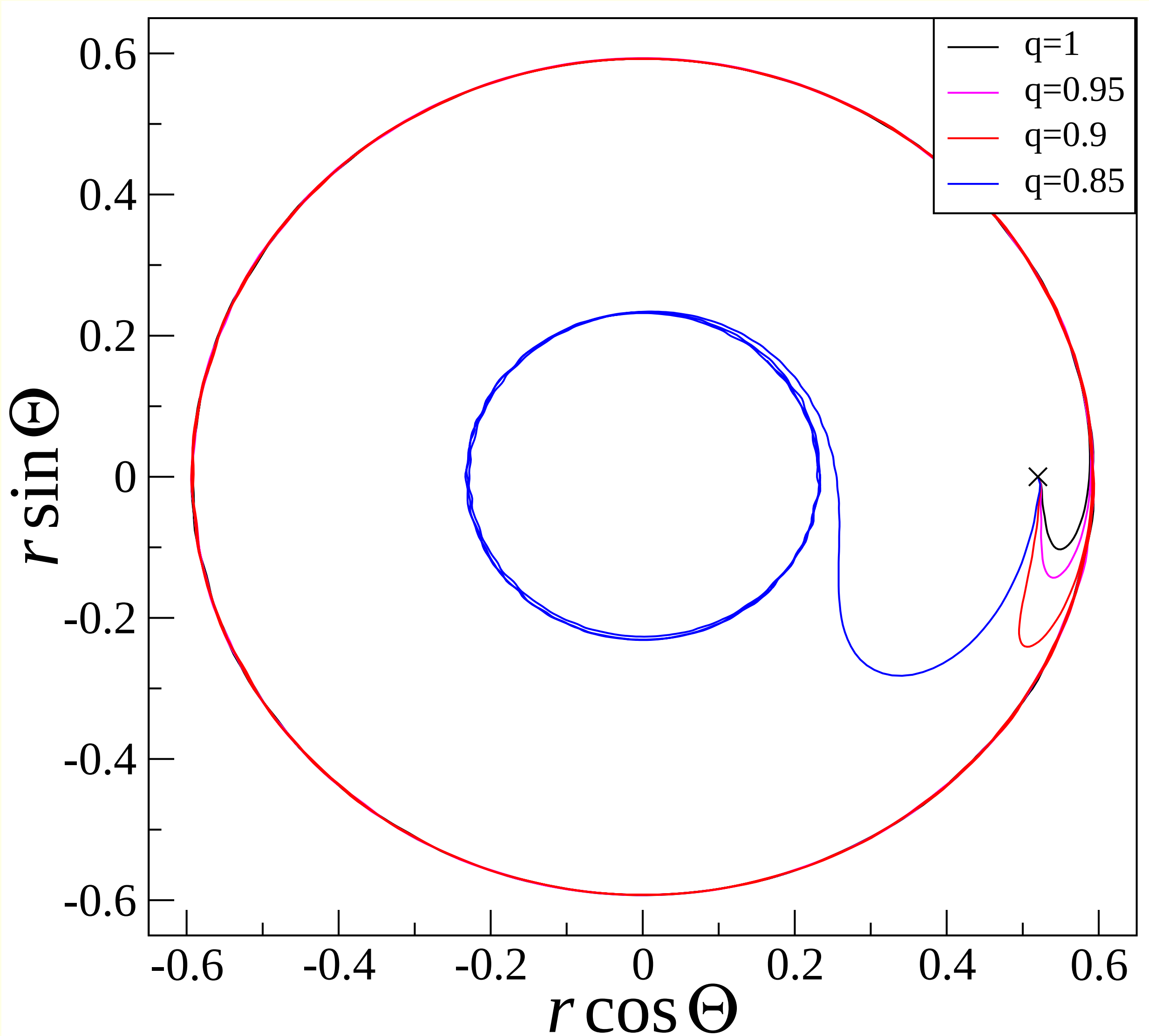
Illustration of multistability for $\beta_0 = 0$



Beyond the OA approximation

- OA ansatz yields the asymptotic solutions
- Basins of attraction depend on the WS constants of motion ψ
- For illustration, we simulate the ensemble for different dustributions of ψ
- The distributions are parameterized by $0 < q \leq 1$;
  $q = 1$ corresponds to the uniform distribution, and, hence,
  to the OA solution

Beyond the OA approximation: Numerics



Nonlinearly coupled ensemble with the uniform frequency distribution

Nonlinear model with $A = 1$, $\beta = \beta_0 + \varepsilon^2 r^2$, and $-\delta \le \omega \le \delta$

WS-equations in the OA approximation:

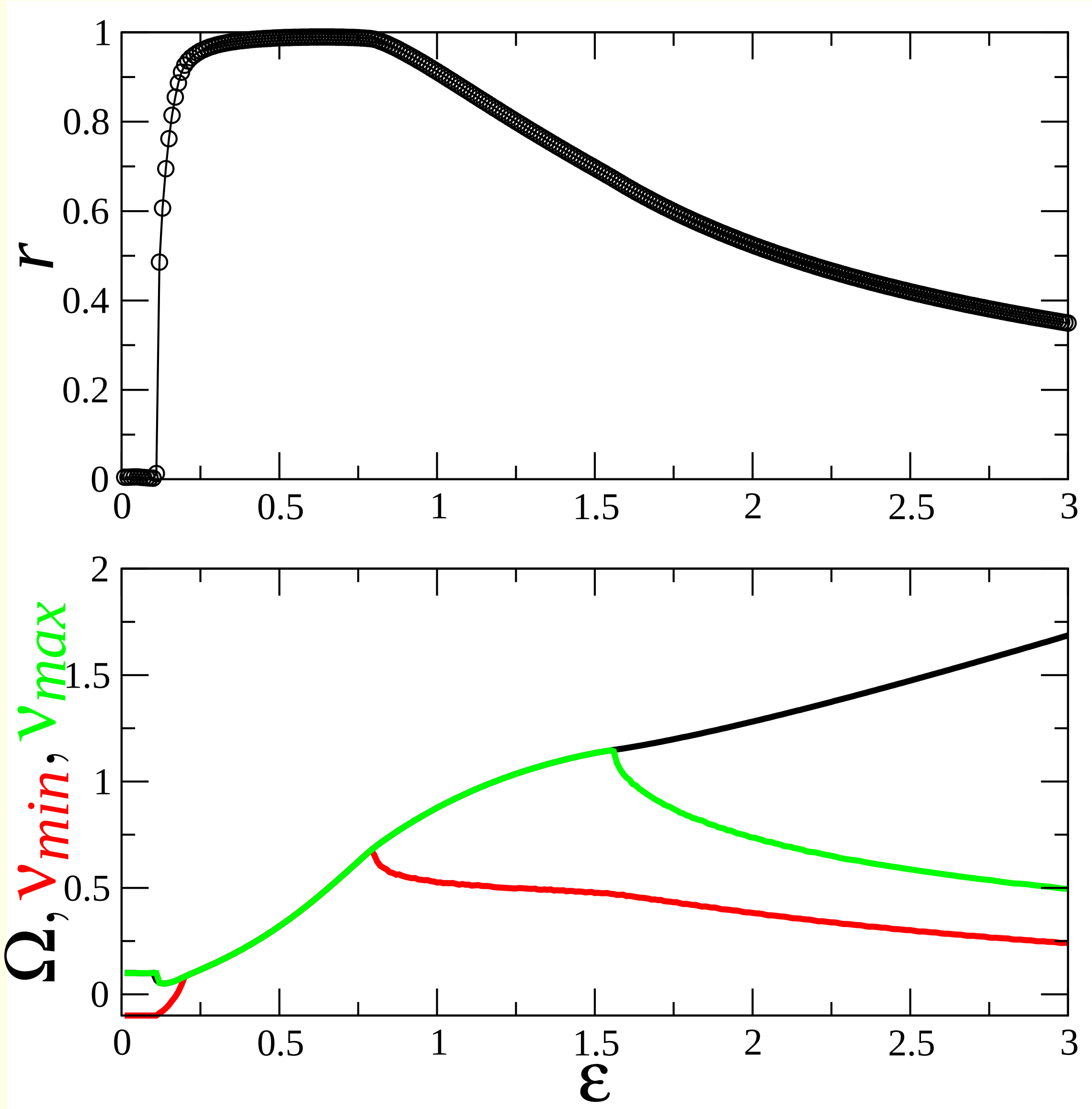
$$\frac{\partial \rho(\omega, t)}{\partial t} = \frac{1 - \rho^2}{2} \varepsilon r \cos(\Theta - \Phi + \beta)$$

$$\frac{\partial \Phi(\omega, t)}{\partial t} = \omega + \frac{1 + \rho^2}{2\rho} \varepsilon r \sin(\Theta - \Phi + \beta)$$

Equation for the mean field:

$$r e^{i\Theta} = (2\delta)^{-1} \int_{-\delta}^{\delta} Z(\omega, t) d\omega = (2\delta)^{-1} \int_{-\delta}^{\delta} \rho e^{i\Phi} d\omega$$

Uniform frequency distribution: results



$\delta = 0.1, \beta_0 = 0.15\pi$

Conclusions

- Watanabe–Strogatz theory extended to the case of nonidentical oscillators and linked to the Ott-Antonsen ansatz
- powerful technique which yields a low-dimensional description of the ensemble dynamics
- exact description of ensembles with global nonlinear coupling for Lorentzian and uniform frequency distribution

M. Rosenblum and A. Pikovsky, PRL **98**, 064101 (2007)

A. Pikovsky and M. Rosenblum, Physica D **238**, 27-37 (2009)

A. Pikovsky and M. Rosenblum, PRL **101**, 264103 (2008)

A. Pikovsky and M. Rosenblum, Physica D, submitted

Example: two interacting subpopulations with chimera

Model by Abrams, Mirollo, Strogatz, and Wiley (PRL 2008):

$\omega_1 = \omega_2 = 0$, $N_1 = N_2$

$\varepsilon_{11} = \varepsilon_{22} = \mu$, $\varepsilon_{12} = \varepsilon_{21} = \nu \neq \mu$, and $\beta_{ab} = \beta$.

Full description via WS equations

$$\dot{z}_{1,2} = \frac{1}{2}\left(H_{1,2} - z_{1,2}^2 H_{1,2}^*\right)$$

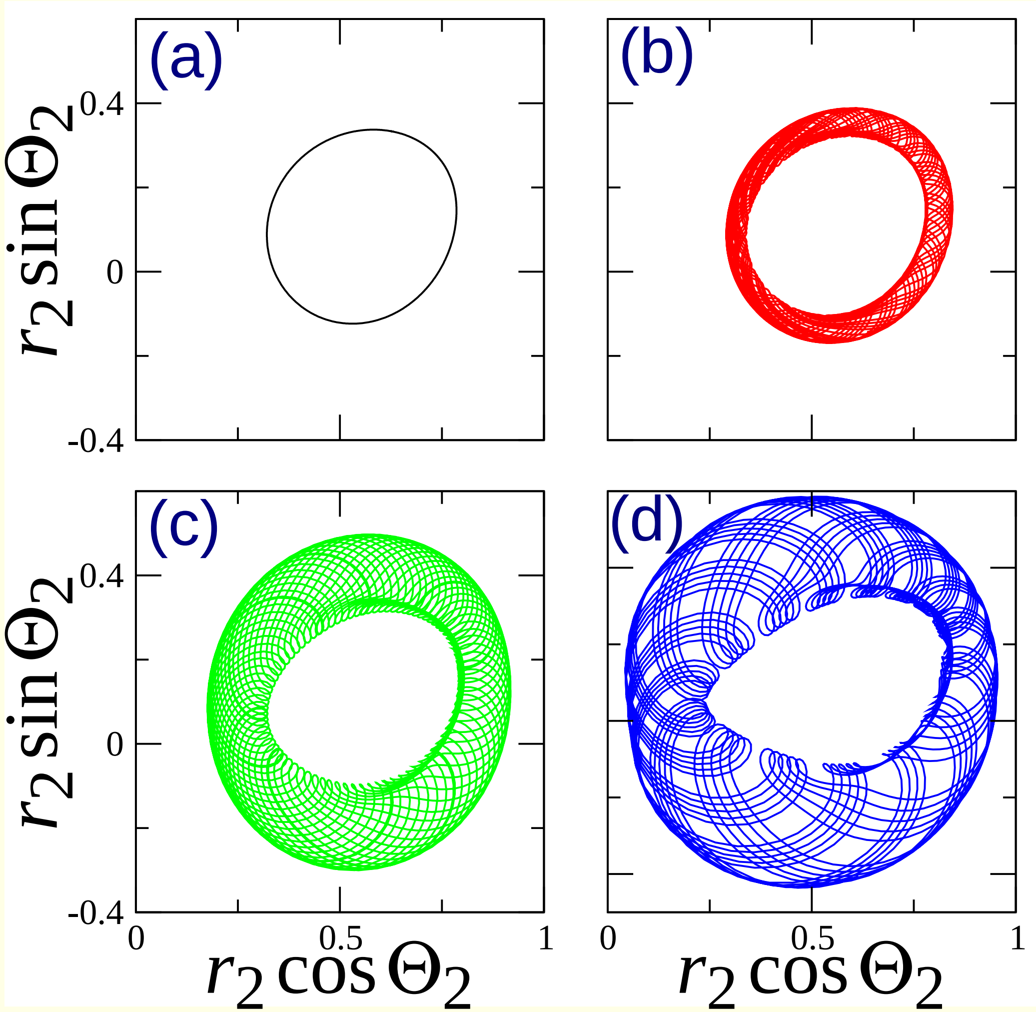
$$\dot{\alpha}_{1,2} = \mathrm{Im}(z_{1,2}^* H_{1,2})$$

$$H_{1,2} = (\mu Z_{1,2} + \nu Z_{2,1})e^{i\beta}, \qquad Z_{1,2} = z_{1,2}\gamma_{1,2}$$

Particular solution via OA ansatz, i.e. by setting $Z_{1,2} = z_{1,2}$

(Abrams et al.)

Results for the model of Abrams et al.



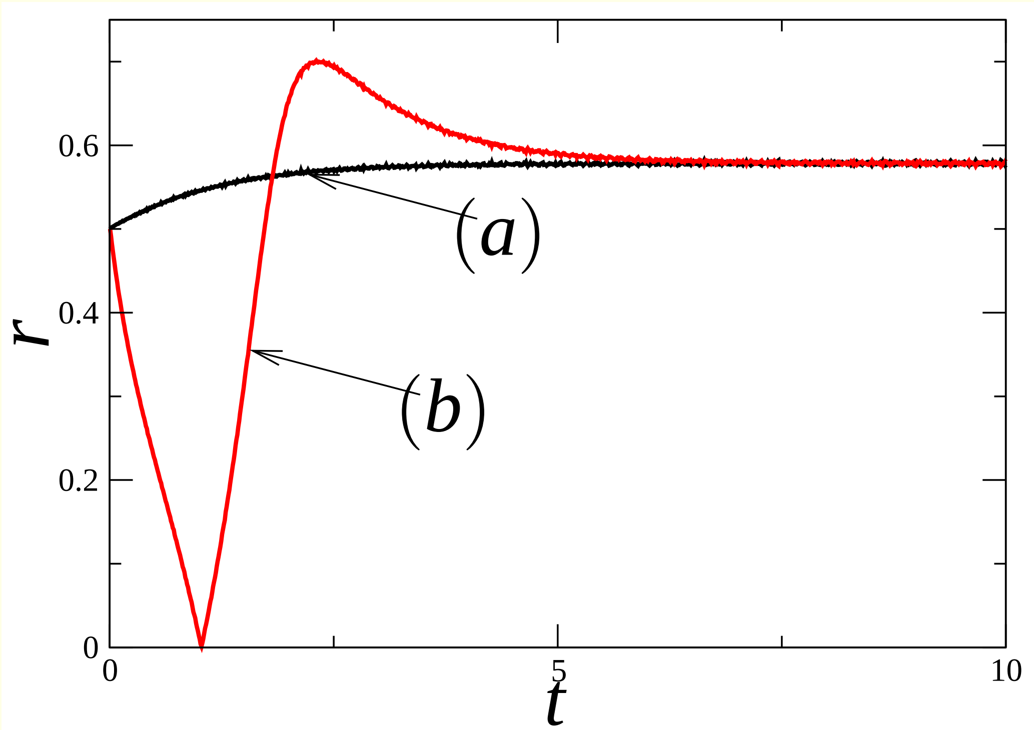
(a): uniform distribution of ψ (OA manifold)

(b-d): nonuniform distribution of ψ $\Longrightarrow$ qiasiperiodic chimeras

Example: the Kuramoto-Sakaguchi model III

Numerics: the same macroscopic initial conditions, $r(0) = 0.5$,

but different microscopic initial conditions, i.e. $\phi_k(0)$



Initial conditions (a): corresponds to the OA solution

Initial conditions (b): asymptotically tends to the OA solution

WS variables vs. generalized order parameters

Generalized Daido parameter of the order m

$$Z_m = N^{-1}\sum_{k=1}^{N} e^{im\phi_k} \qquad \text{or} \qquad Z_m = \int_0^{2\pi} w(\phi) e^{im\phi} d\phi \,.$$

Relation to WS variables:

$$Z_m = z^m \gamma_m(z, \alpha)$$

For uniform distribution of ψ and large N we have

$$\boxed{\gamma_m = 1 \quad \Longrightarrow \quad Z_m = z^m = Z^m}$$

OA ansatz in terms of generalized order parameters

Time derivative of the order parameter:

$$\dot{Z}_m = \int_0^{2\pi} \frac{\partial w(\omega,\phi,t)}{\partial t} e^{im\phi} d\phi = im \int_0^{2\pi} w(\omega,\phi,t)\dot{\phi} e^{im\phi} d\phi$$

Substitution of $\dot{\phi} = \omega + (He^{-i\phi} - H^* e^{i\phi})/2i$ yields

$$\dot{Z}_m = i\omega m Z_m + \frac{m}{2}(HZ_{m-1} - H^* Z_{m+1})$$

This system simplifies if $Z_m = Z^m$ $\Longrightarrow$ OA equation

$$\dot{Z}(\omega,t) = i\omega Z + \frac{1}{2}(H - H^* Z^2) \ .$$

OA ansatz $\Longleftrightarrow$ uniform distribution of ψ in the WS theory