Center for Scientific Computation and Mathematical Modeling
Modeling and Computations of Shallow-Water Coastal Flows
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A Lagrangian Particle/Panel Method for Incompressible Fluid Flow

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outline

1. incompressible fluid flow
2. vortex sheet model in 2D
3. vortex ring simulations in 3D
4. barotropic vorticity equation on a sphere

1. incompressible fluid flow : Eulerian form

$u(x,t)$: velocity , $\nabla \cdot u = 0$

$p(x,t)$: pressure

Navier-Stokes equation

$$u_t + (u \cdot \nabla)u = -\nabla p + \frac{1}{Re}\Delta u \ , \ Re = \frac{UL}{\nu}$$

Euler equation

$$\nu \to 0 \ , \ Re \to \infty$$

1. incompressible fluid flow : Lagrangian form

$$\nabla \cdot u = 0$$

$\nabla \times u = \omega$: vorticity

Biot-Savart law

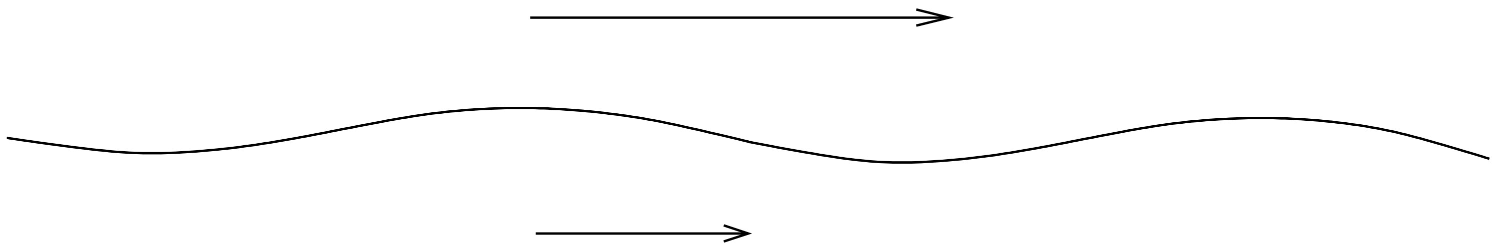
$$u(x,t) = \int_V K(x-\tilde{x}) \times \omega(\tilde{x},t) d\tilde{x} \ , \ \ K(x) = -\frac{x}{4\pi|x|^3}$$

flow map

$\alpha \ \rightarrow \ x(\alpha,t)$, α : Lagrangian parameter (name tag)

$$\frac{\partial x}{\partial t} = \int_V K(x-\tilde{x}) \times \nabla_\alpha \tilde{x} \cdot \tilde{\omega}_0 \, d\tilde{\alpha} \ \ , \ \ \tilde{x} = x(\tilde{\alpha},t)$$

2. vortex sheet model in 2D

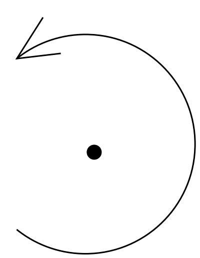


Birkhoff-Rott equation

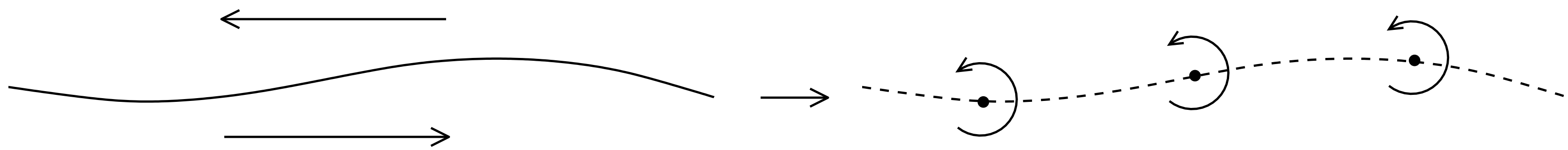
$z(\alpha, t)$: vortex sheet flow map , complex curve

$$\frac{\partial \overline{z}}{\partial t}(\alpha, t) = \text{pv}\int_a^b K(z(\alpha, t) - z(\widetilde{\alpha}, t))d\widetilde{\alpha}$$

$K(z) = \dfrac{1}{2\pi i z}$: Cauchy kernel



vortex blob method



regularized ODEs

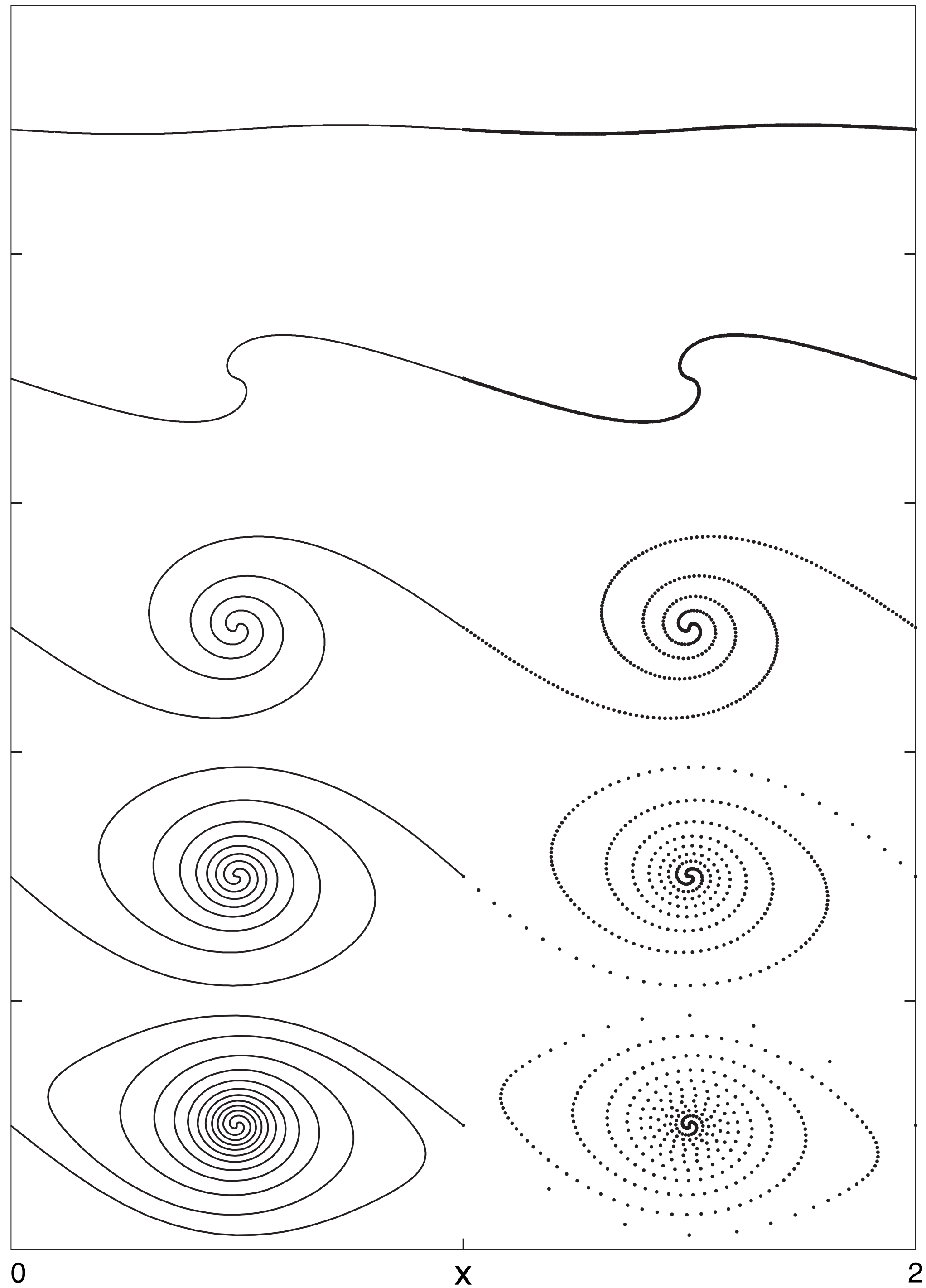
$$\overline{\frac{dz_j}{dt}} = \sum_{k=1}^{N} K_\delta(z_j - z_k)\,\Gamma_k$$

$$K_\delta(z) \;=\; \frac{1}{2\pi i z} \cdot \frac{|z|^2}{|z|^2 + \delta^2}$$

↑ smoothing parameter

Chorin & Bernard (1973) , Anderson (1985) , K (1986)

example : Kelvin-Helmholtz instability



example : wing tip vortex, NASA experiment

Prandtl : vortex sheet model of an airplane wake

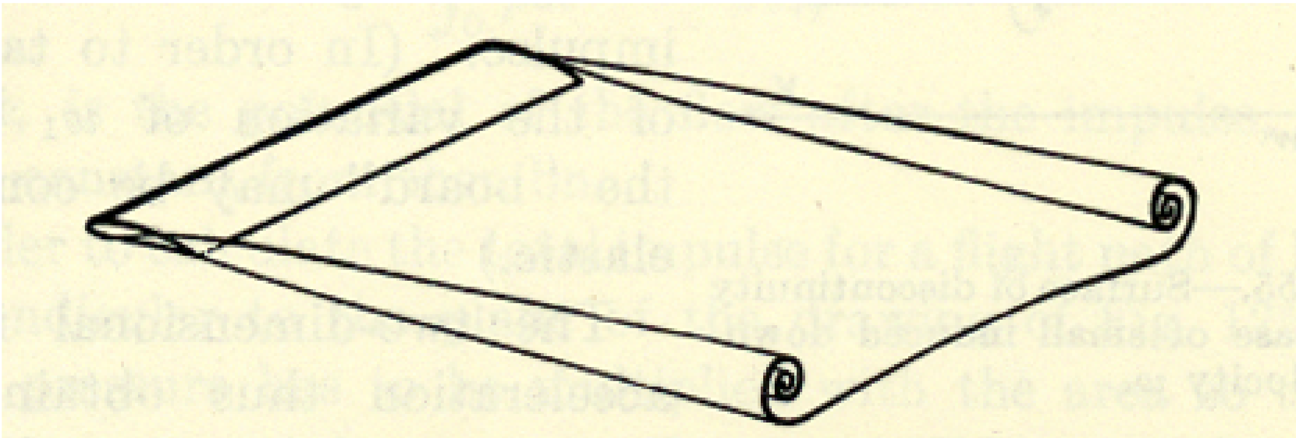
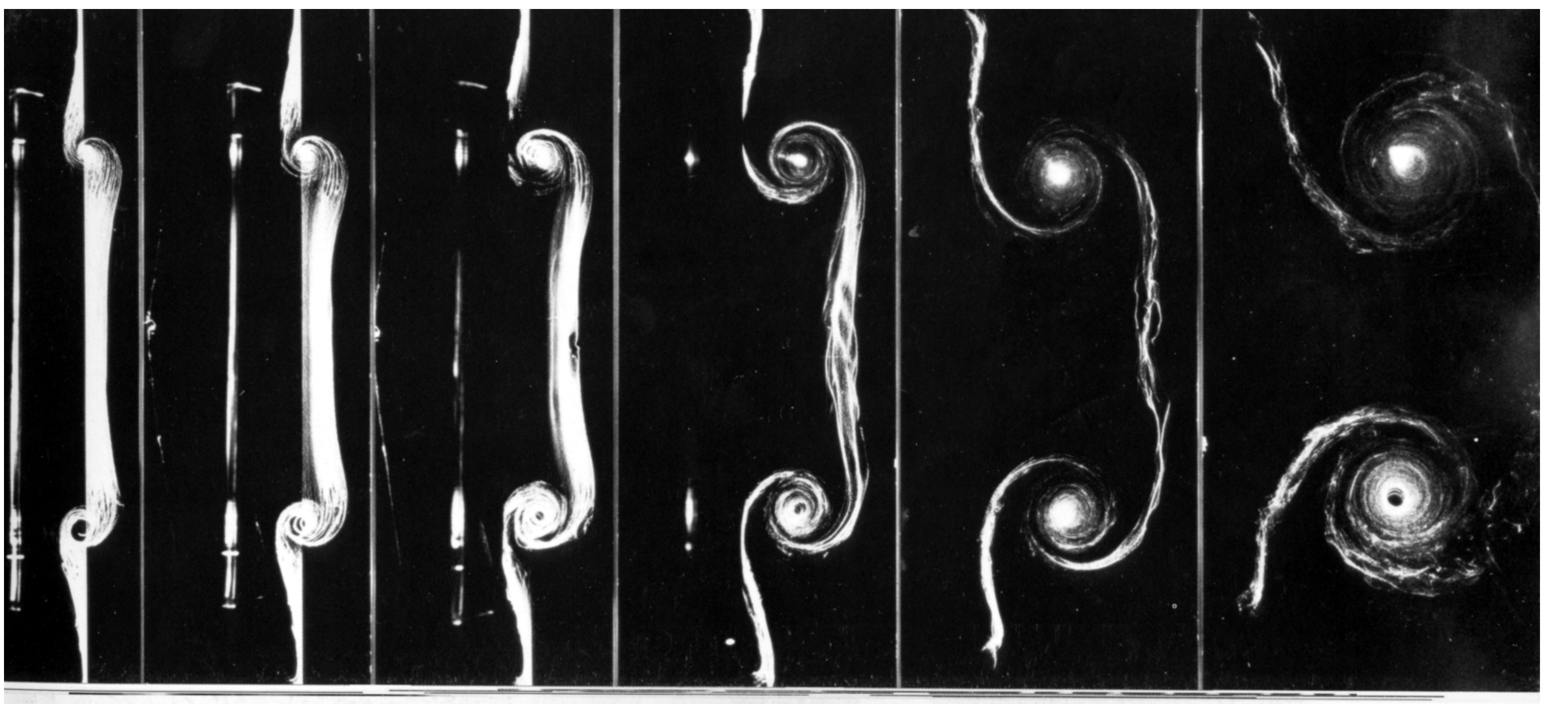


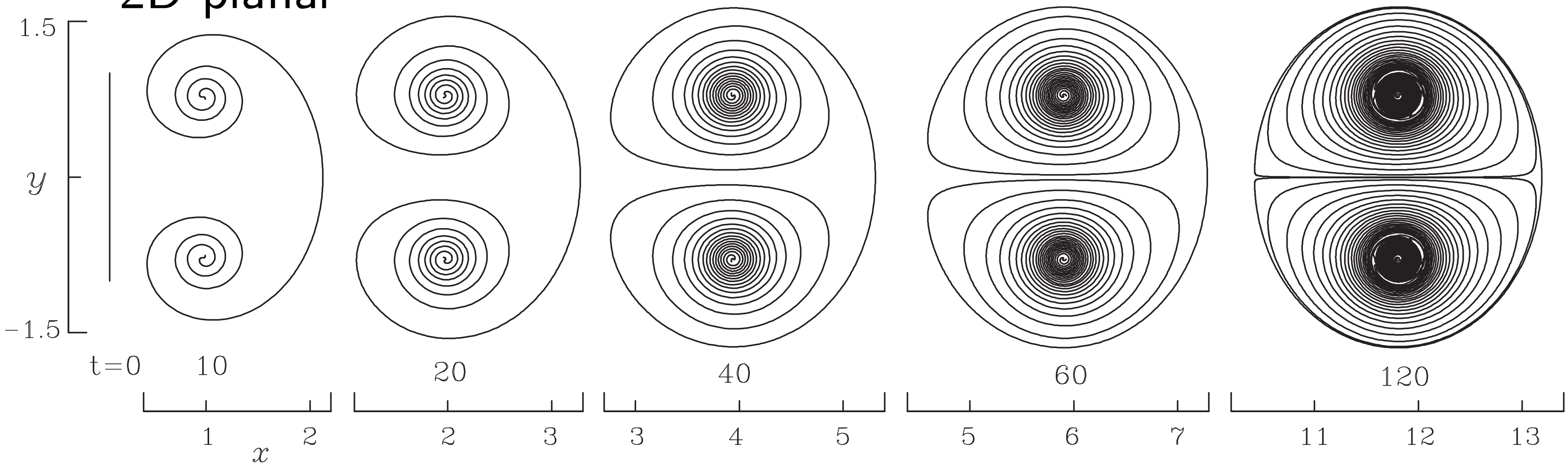
FIG. 154.—Surface of discontinuity behind a wing.

cross-section

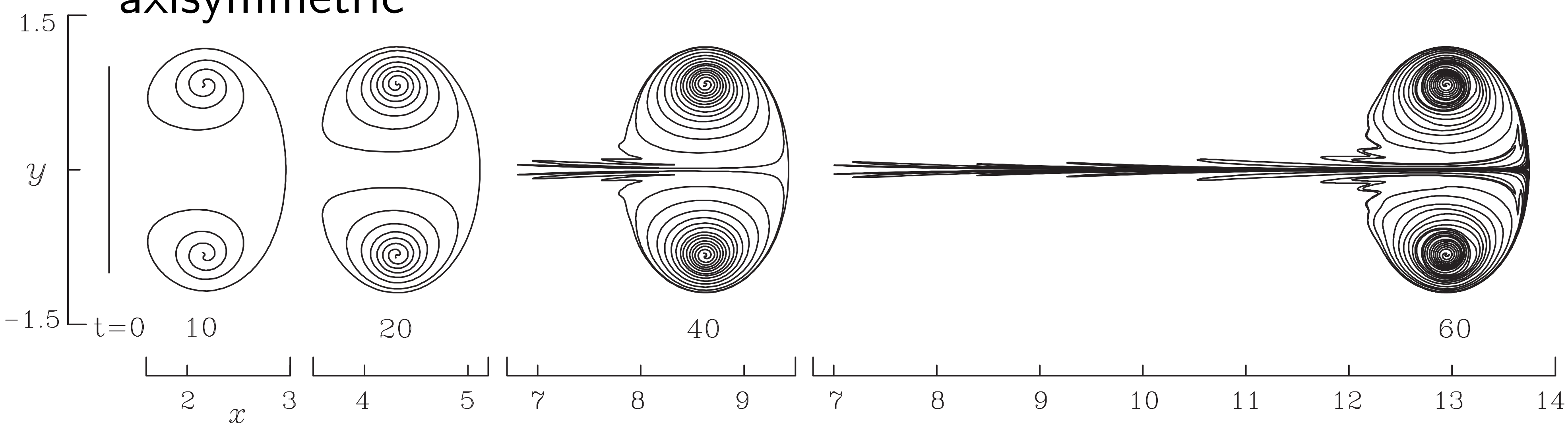


simulations : K-Nitsche (2002) JFM 454

2D planar

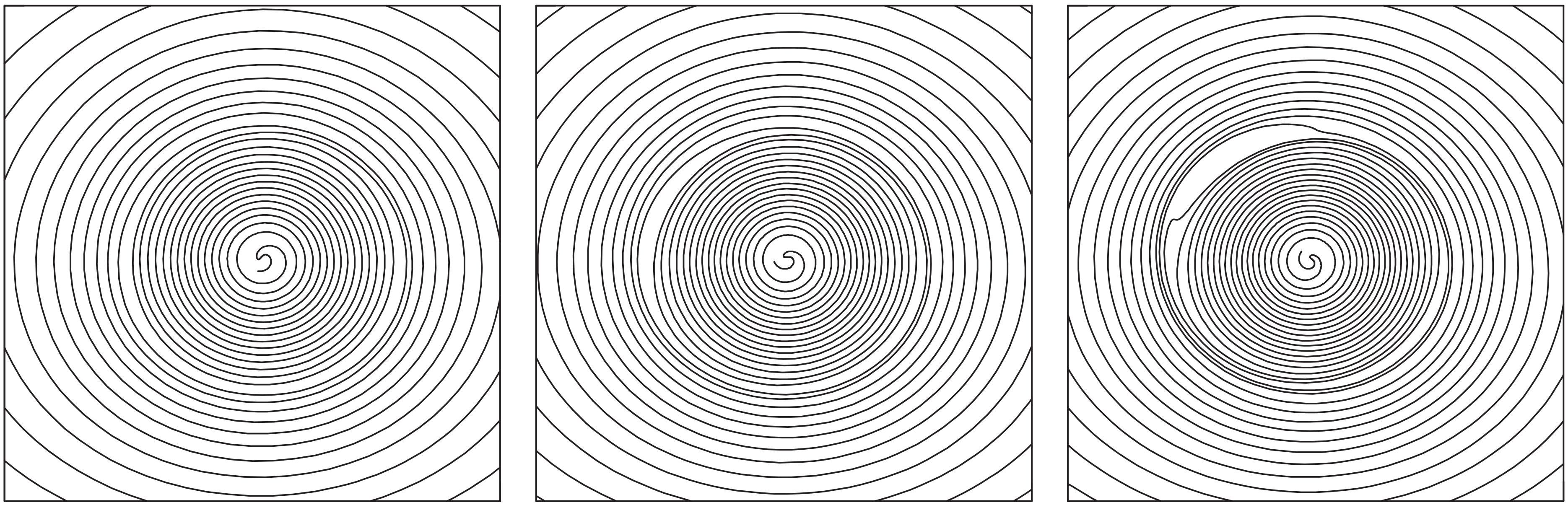


axisymmetric

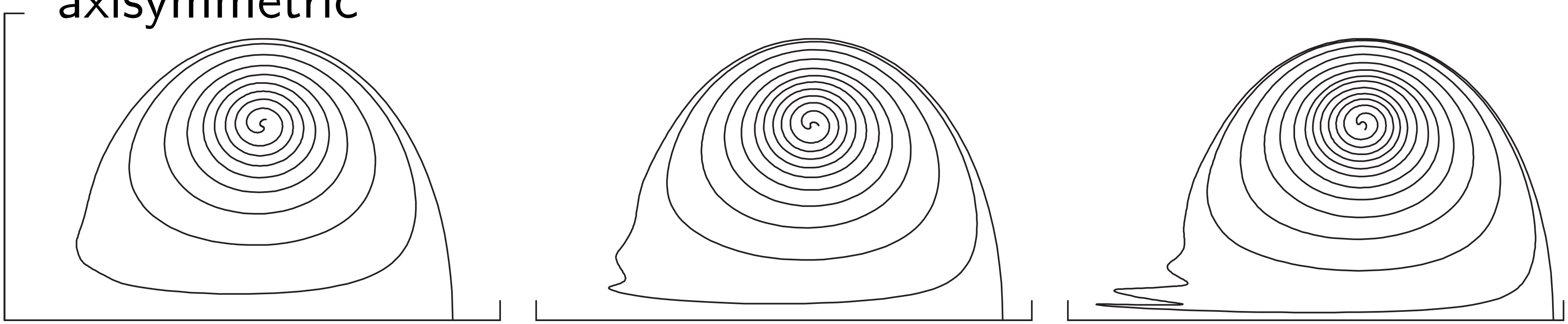


onset of irregular features

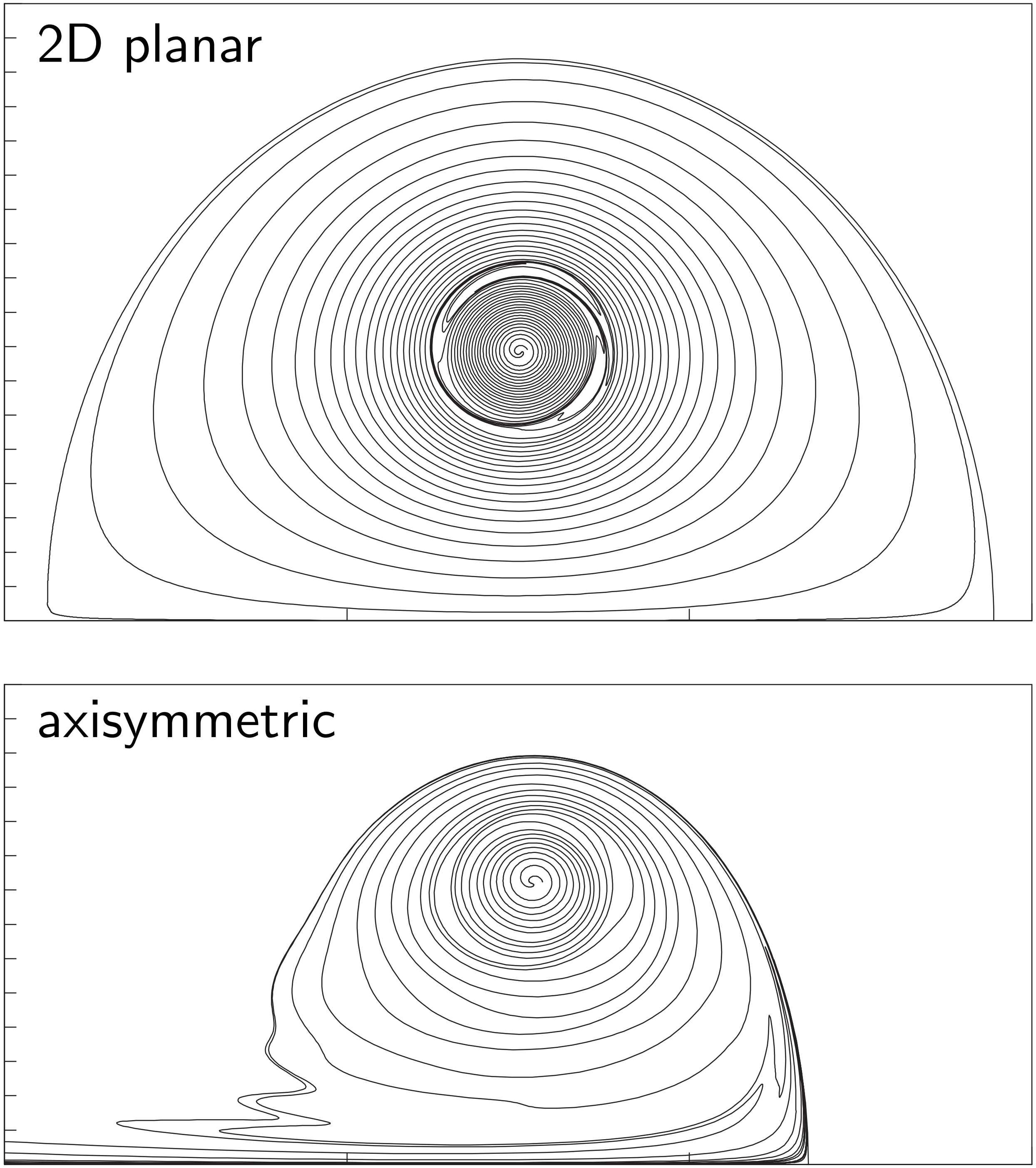
2D planar



axisymmetric



closeup at final time



explanation

2D incompressible flow $=$ Hamiltonian system

$$\psi(x,y,t) : \text{stream function} \quad \rightarrow \quad \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial \psi}{\partial y} \\ \dfrac{dy}{dt} = -\dfrac{\partial \psi}{\partial x} \end{cases}$$

model : oscillating vortex pair

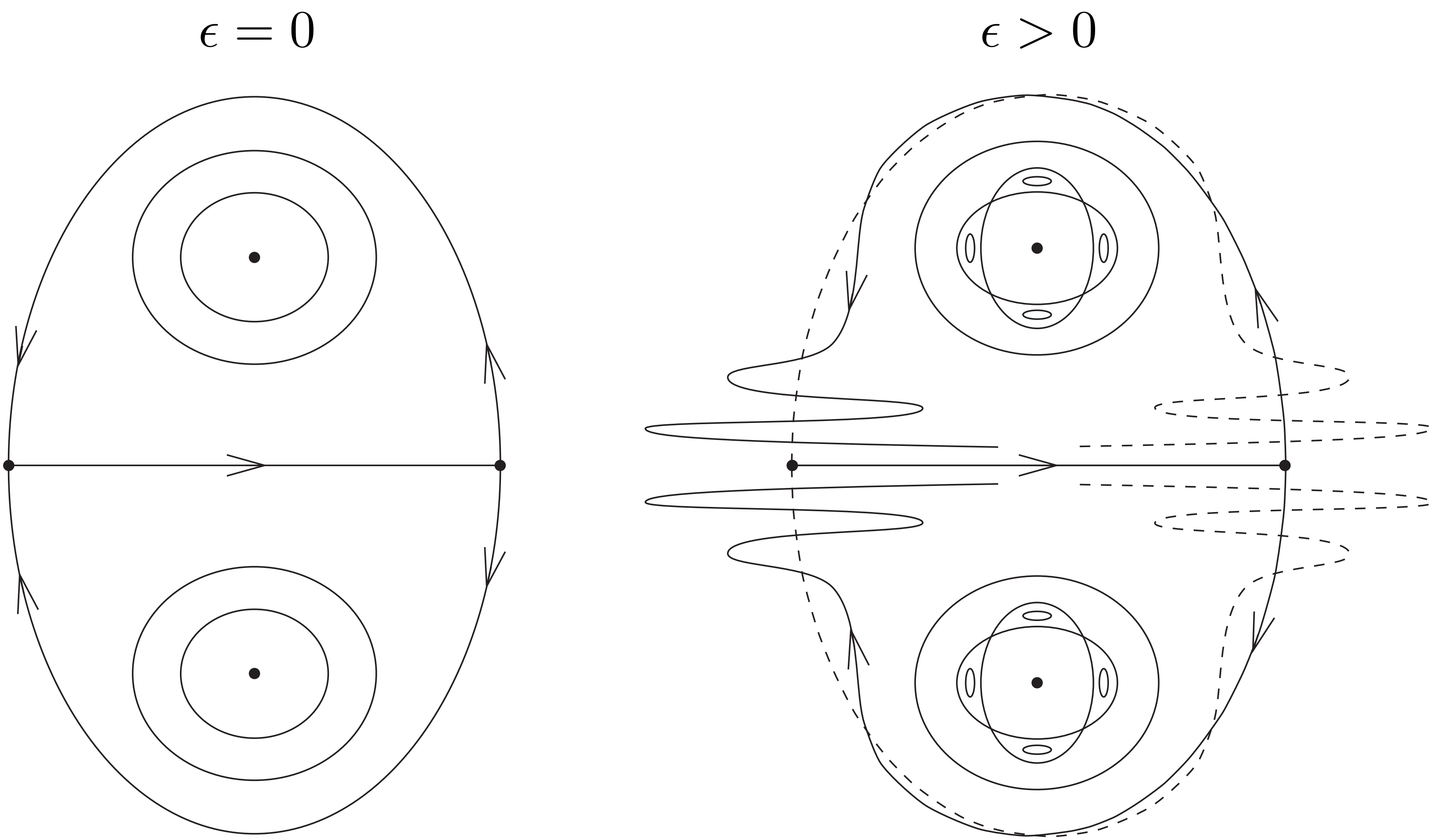
Rom-Kedar, Leonard & Wiggins (1990) , Ide & Wiggins (1995)

$$\psi(x,y,t) \; = \; \psi_0(x,y) \; + \; \epsilon \, \psi_1(x,y,t)$$

ψ_0 : counter-rotating pair of point vortices

ψ_1 : time-periodic strain field

Poincaré section (schematic)



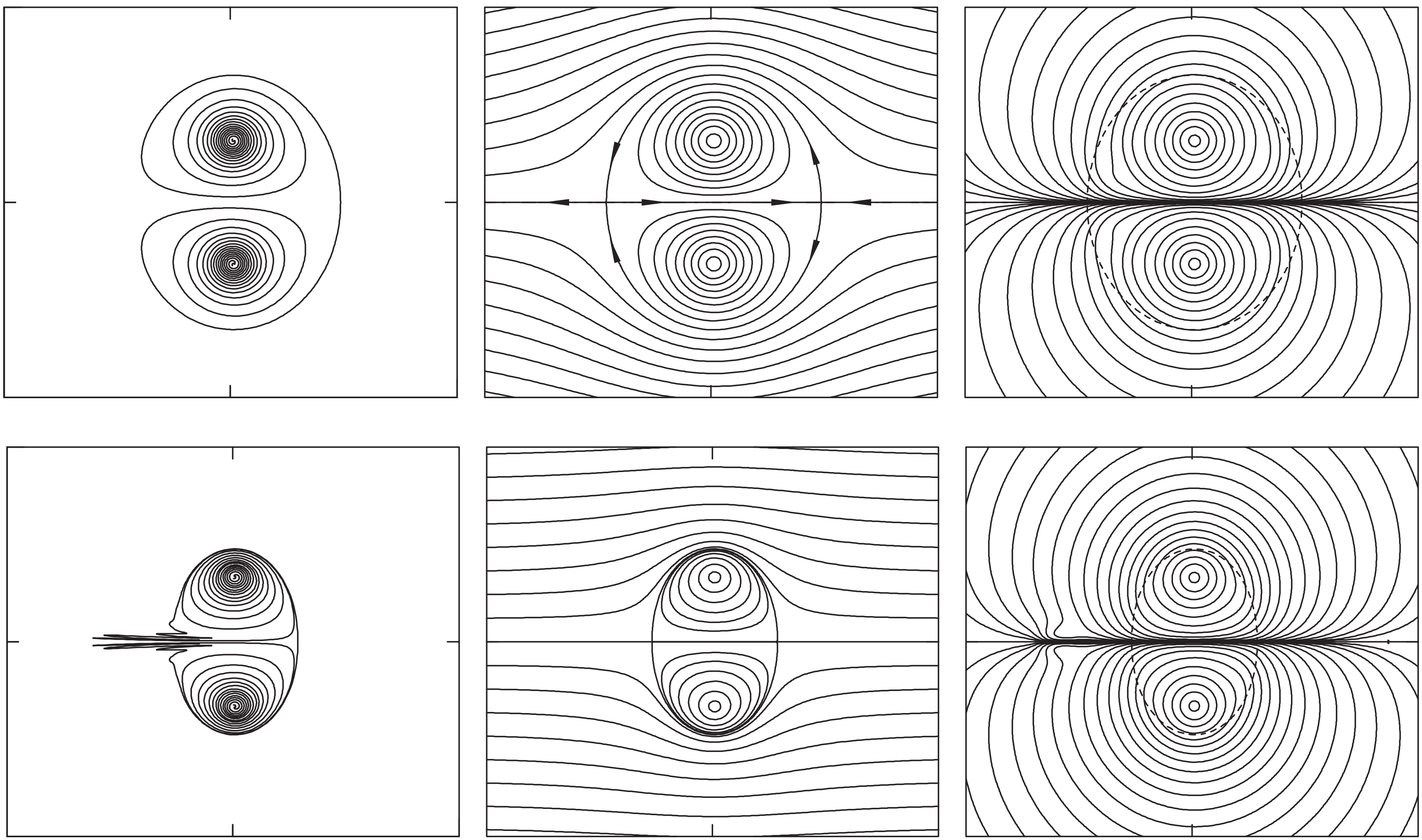
hyperbolic points $\longrightarrow$ heteroclinic orbits $\longrightarrow$ heteroclinic tangle

elliptic points $\longrightarrow$ periodic orbits $\longrightarrow$ KAM curves , resonances

vortex sheet

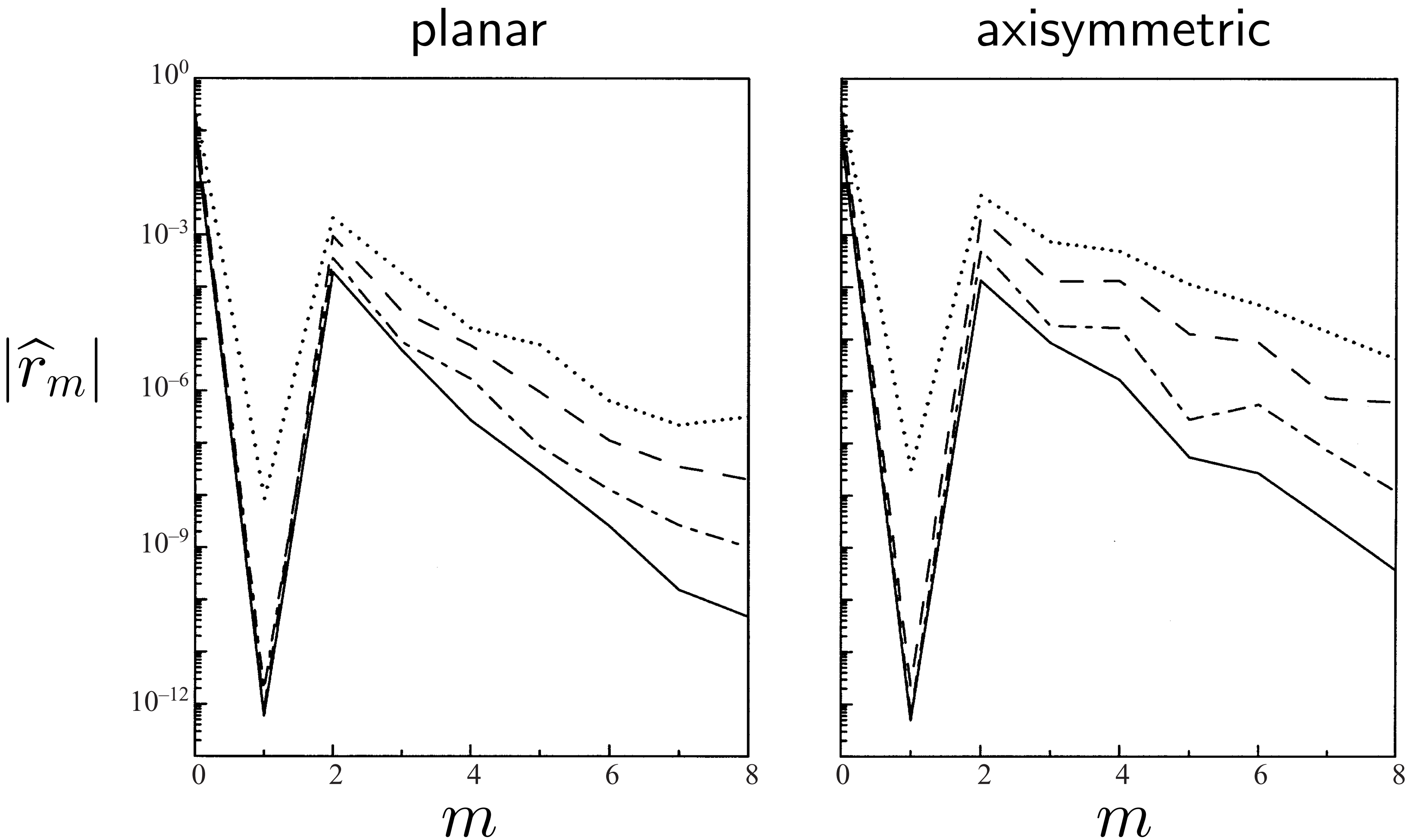
quasisteady state , $t = 40$, $\psi \approx \psi_0 + \epsilon\, \psi_1(t)$

sheet streamlines vorticity

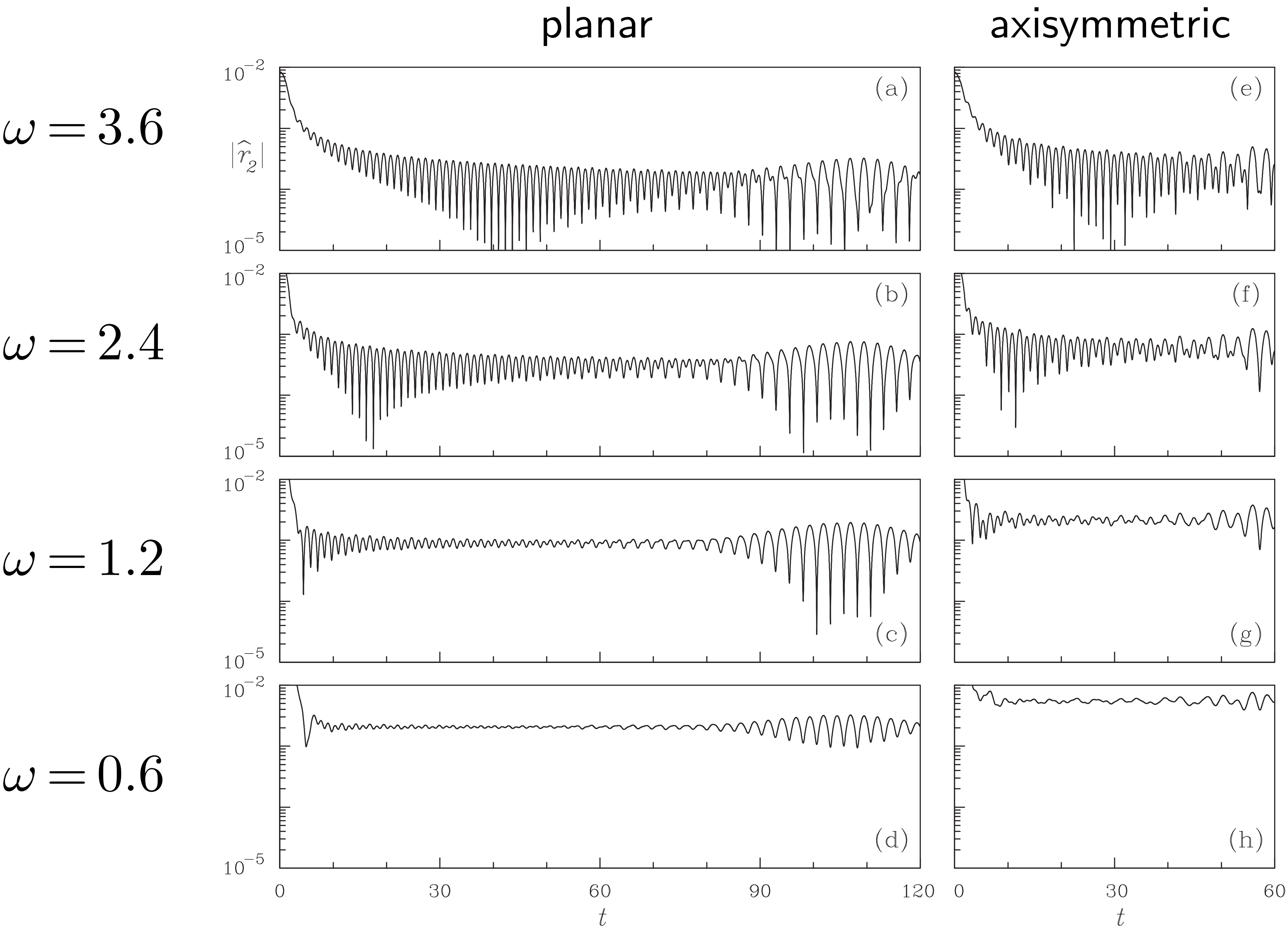


deformation of vorticity contours : $t = 40$

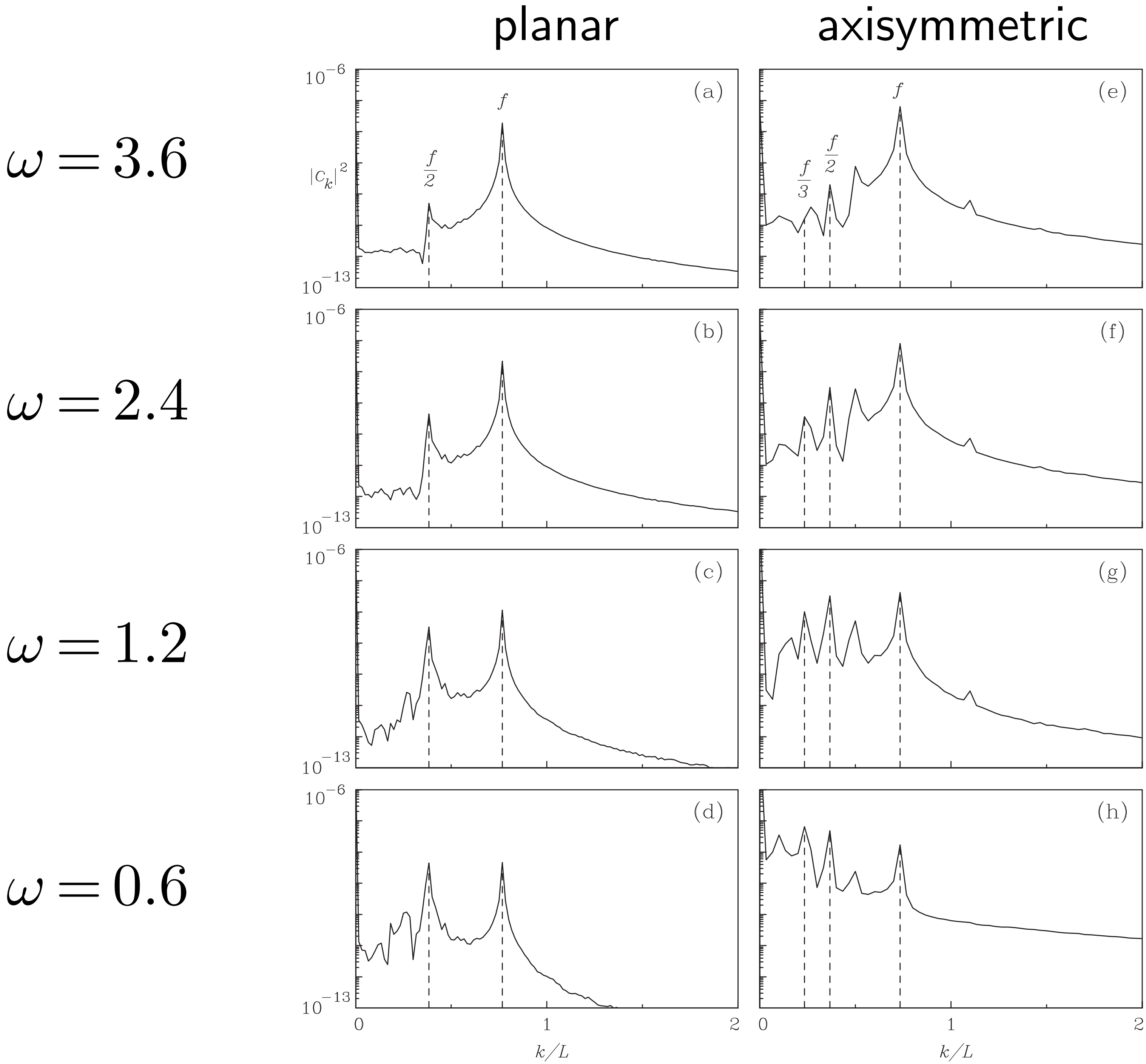
$$\omega = 3.6/2.4/1.2/0.6 \quad , \quad r(\theta) = \sum_m \widehat{r}_m \, \mathrm{e}^{\mathrm{i}m\theta}$$



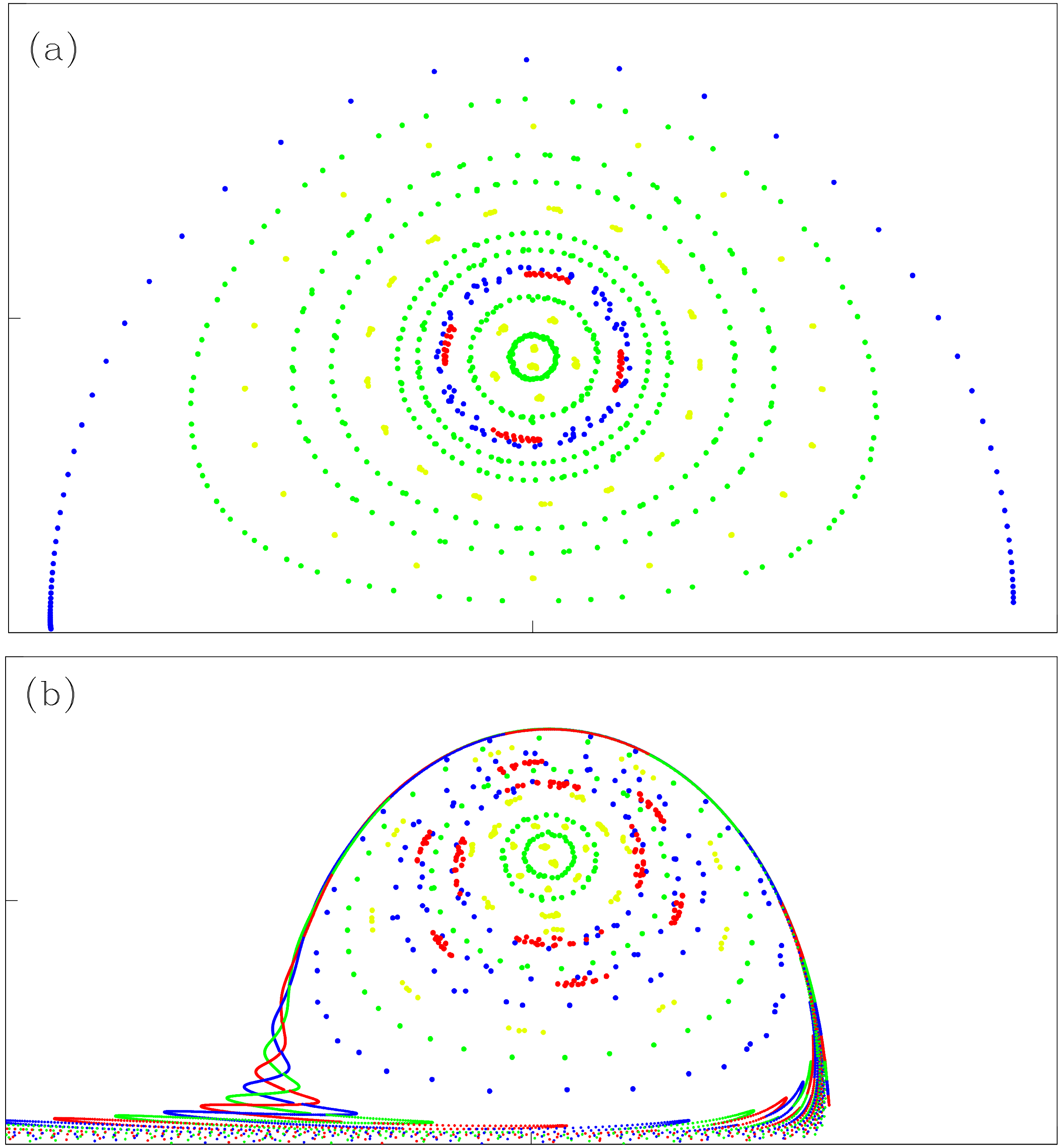
amplitude of elliptic mode : $|\widehat{r}_2(t)|$ oscillates



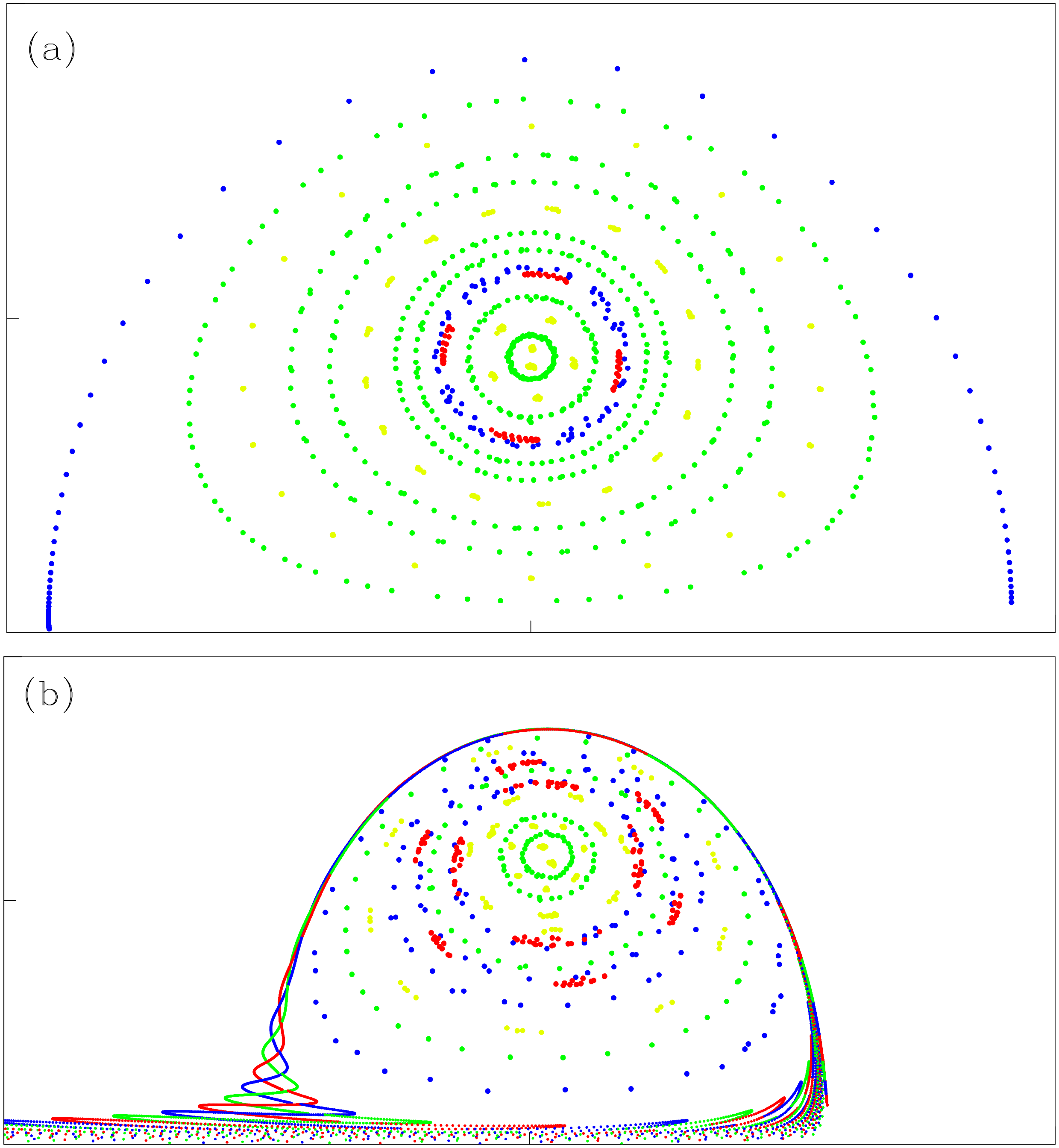
power spectrum



Poincaré section : numerical evidence of chaos in vortex sheet flow

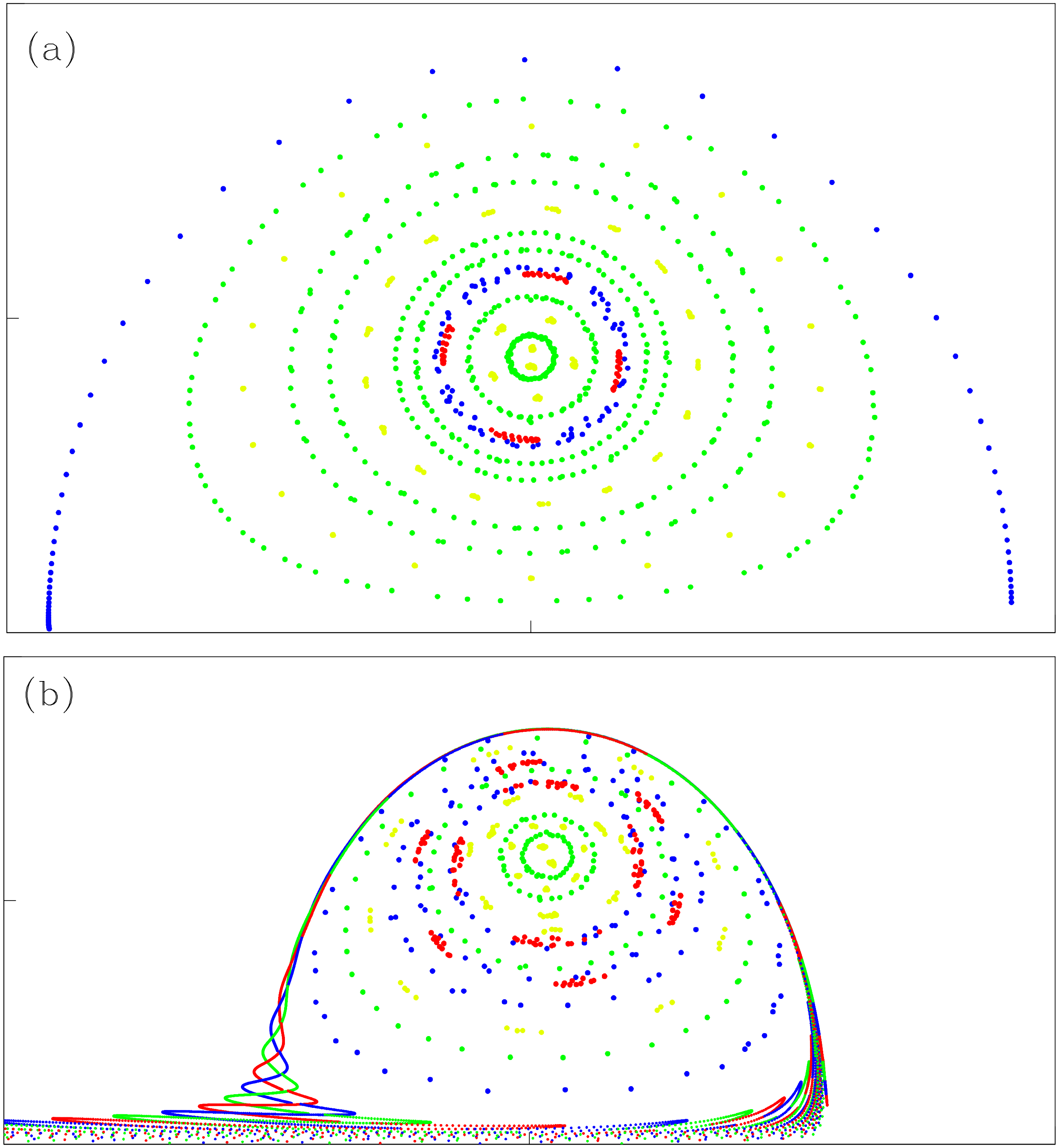


Poincaré section : numerical evidence of chaos in vortex sheet flow



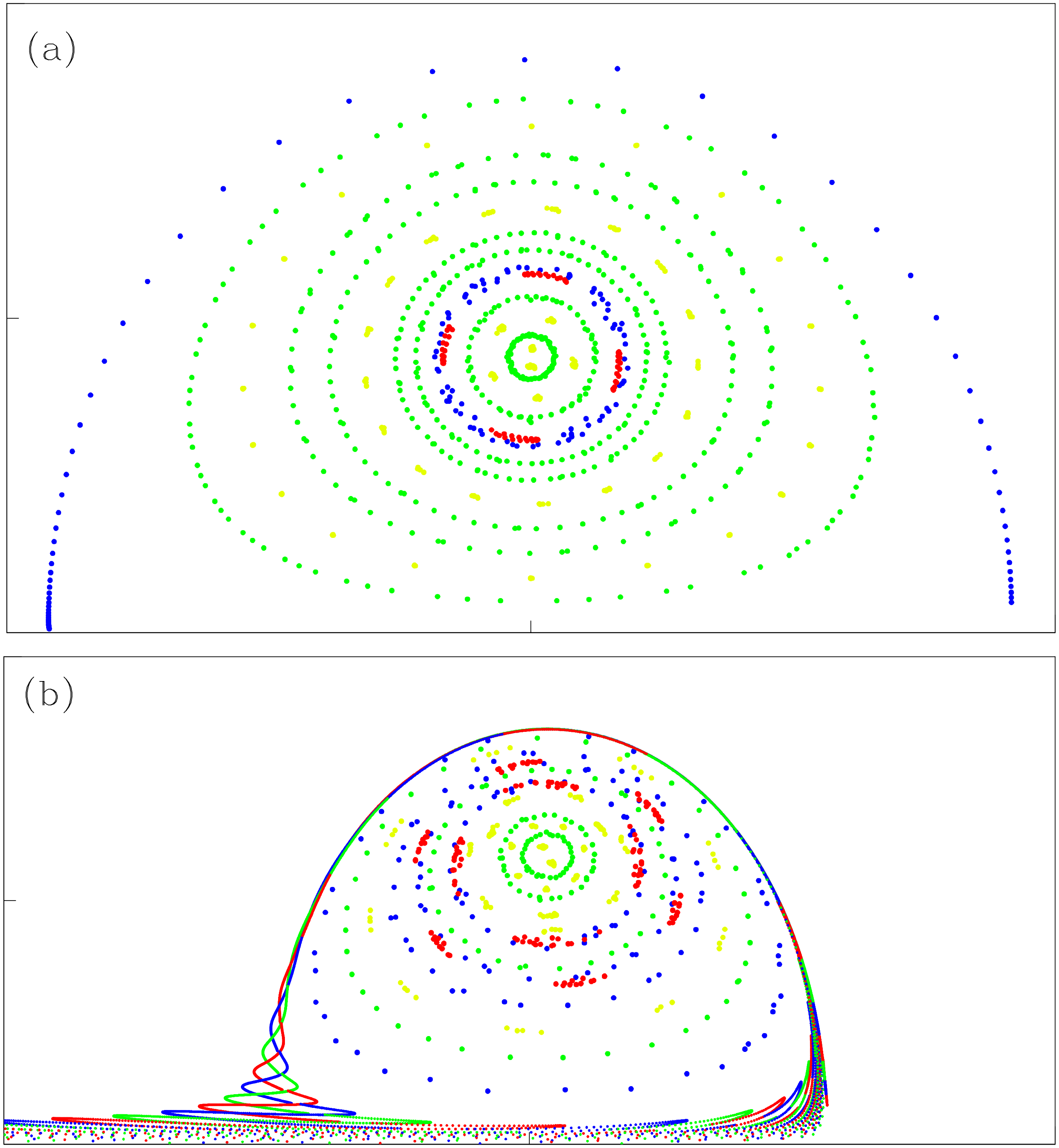
question - ?

Poincaré section : numerical evidence of chaos in vortex sheet flow



question - where does the oscillation come from?

Poincaré section : numerical evidence of chaos in vortex sheet flow

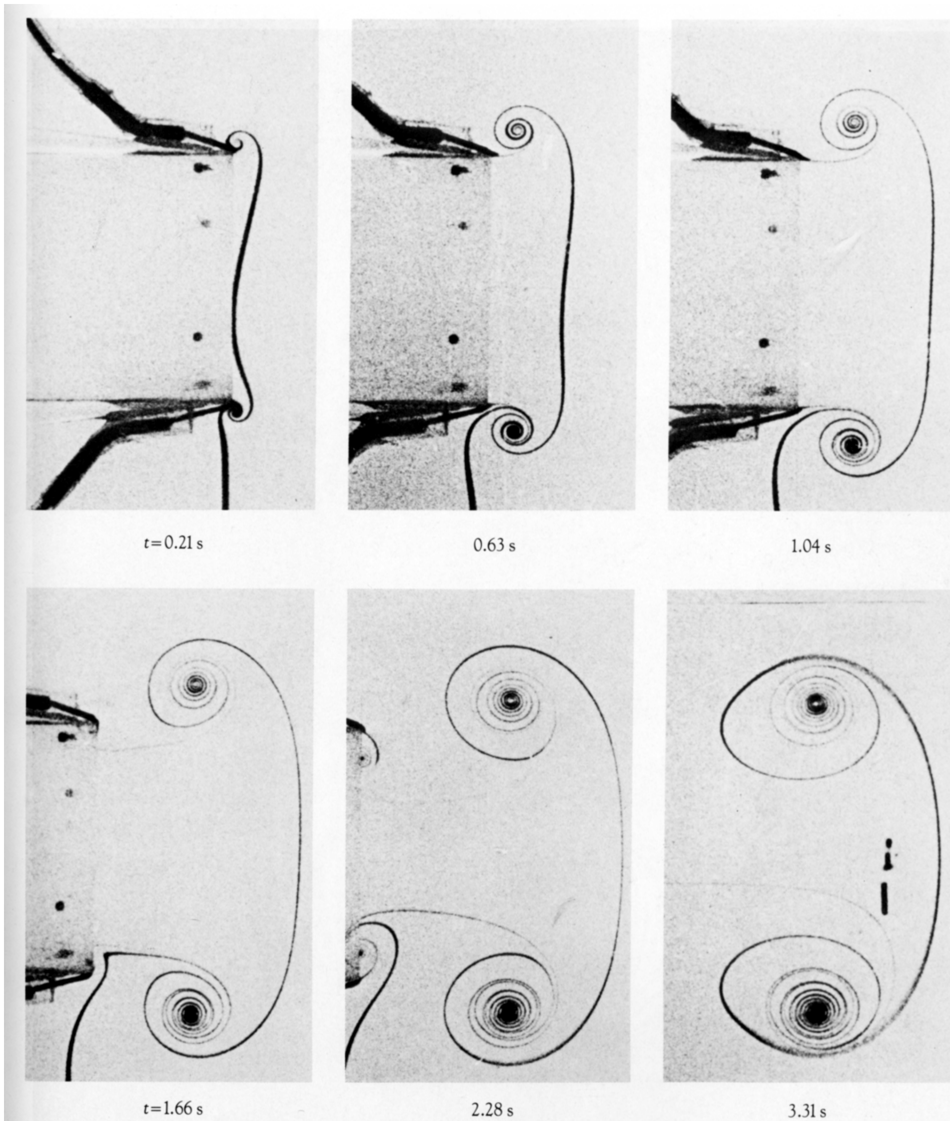


question - where does the oscillation come from? . . . Kida (1981)

validation : experiment/simulation

validation : experiment/simulation

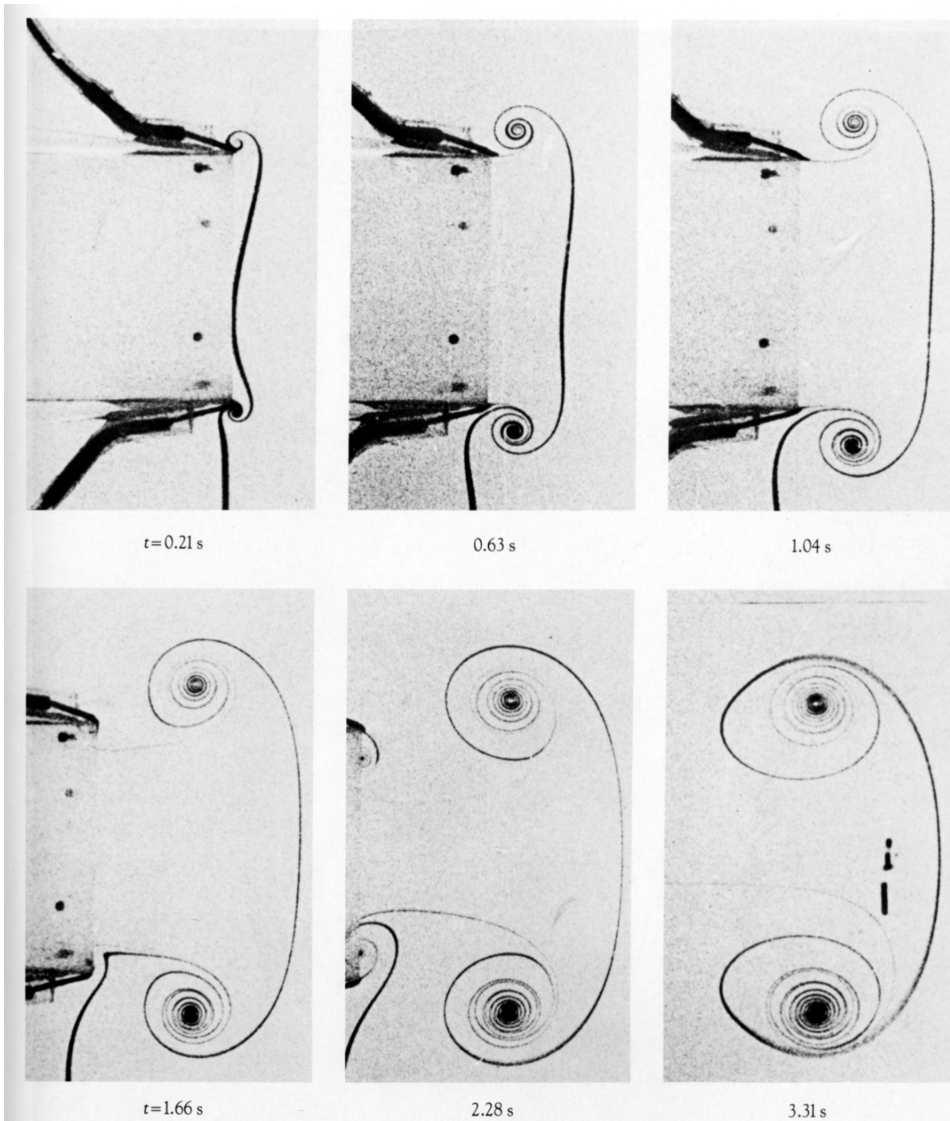
Didden (1979)



Van Dyke, "An Album of Fluid Motion"

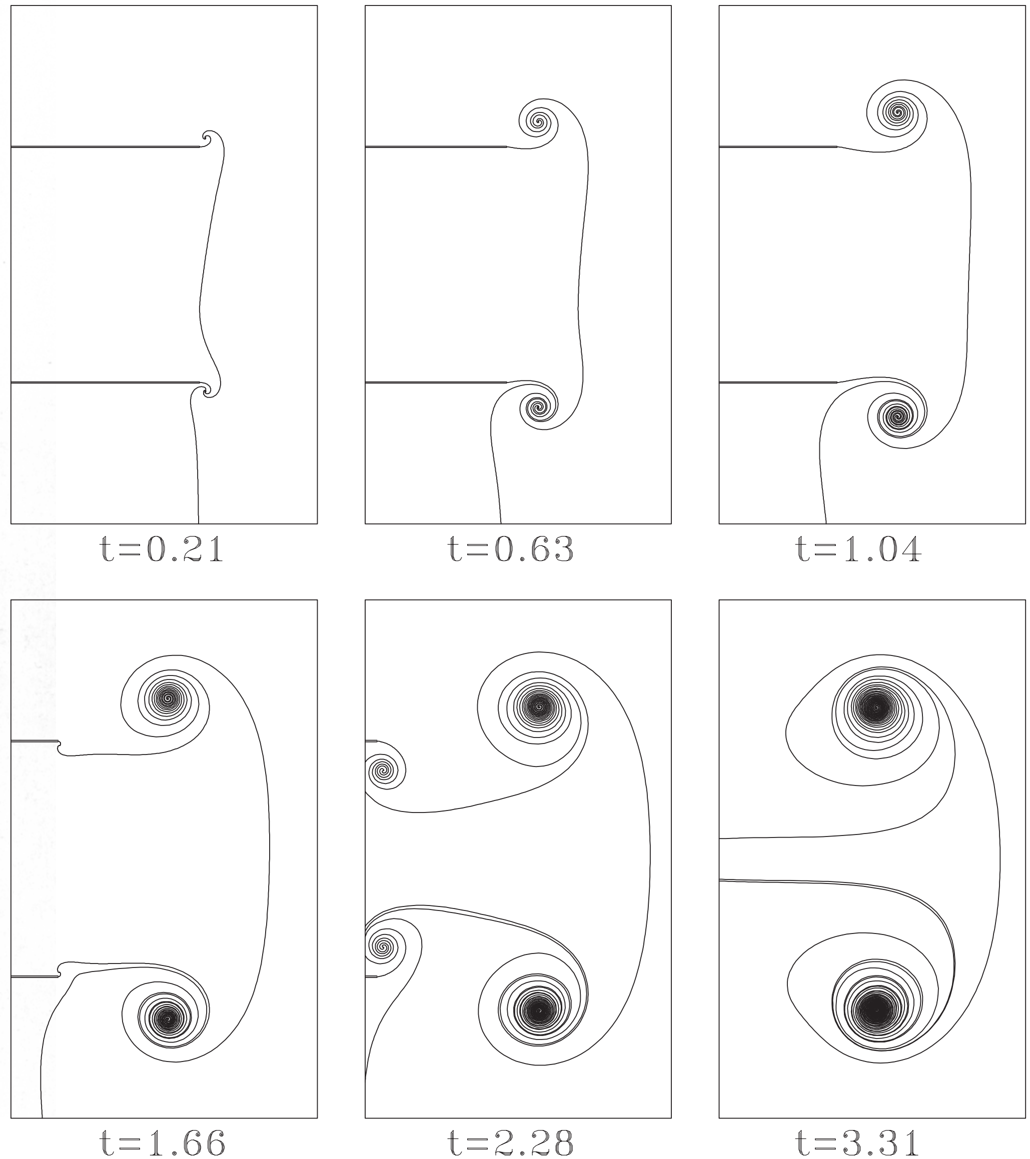
validation : experiment/simulation

Didden (1979)



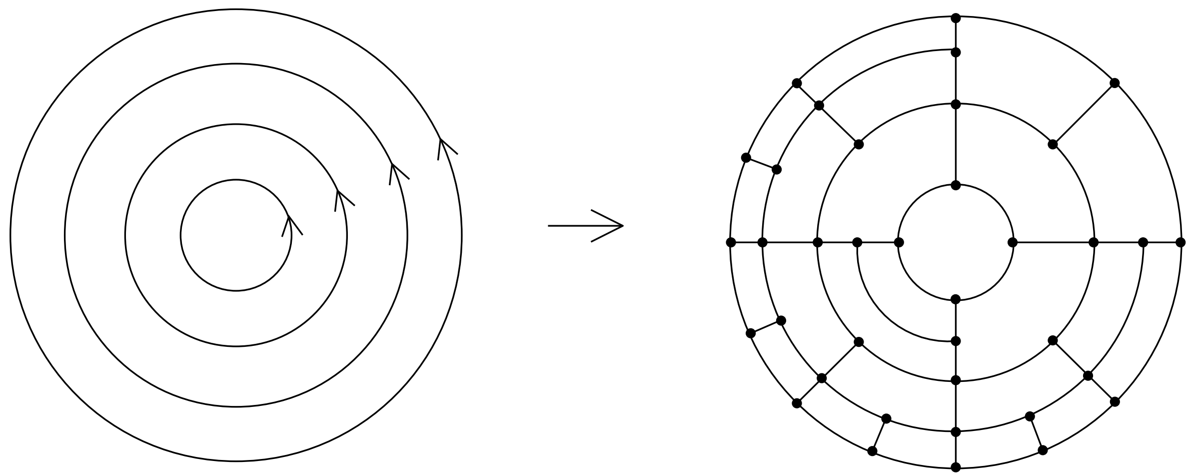
Van Dyke, "An Album of Fluid Motion"

Nitsche-K (1994)

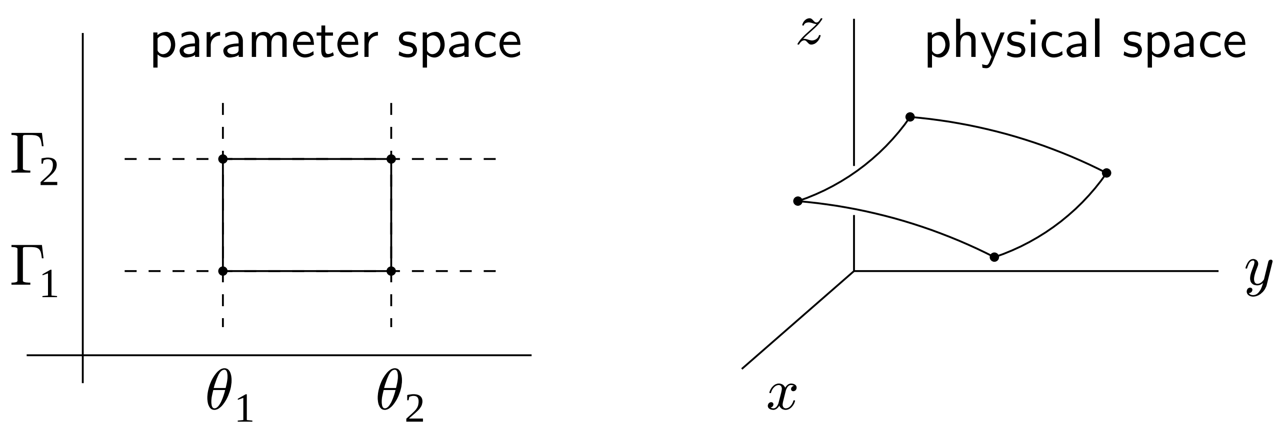


3. vortex ring simulations in 3D

circular disk vortex sheet → particle/panel discretization



• a typical panel



• ODEs : $\dfrac{dx_i}{dt} = \displaystyle\sum_{j=1}^{N} K_\delta(x_i - x_j) \times w_j$, $K_\delta(x) = \dfrac{x}{4\pi(|x|^2 + \delta^2)^{3/2}}$

ODEs

$$\frac{dx_i}{dt} = \sum_{j=1}^{N} K_\delta(x_i, x_j) \times w_j \quad , \quad i = 1 : N$$

N-body problem

a) direct summation : $O(N^2)$

particle-particle

b) treecode : $O(N \log N)$

particle-cluster

Barnes-Hut (1986) , Lindsay-K (2001) , Li-Johnston-K (2009)

c) fast multipole method : $O(N)$

cluster-cluster

Greengard-Rokhlin (1987)

example

oblique collision of two vortex rings

experiments

T.T. Lim (1989)

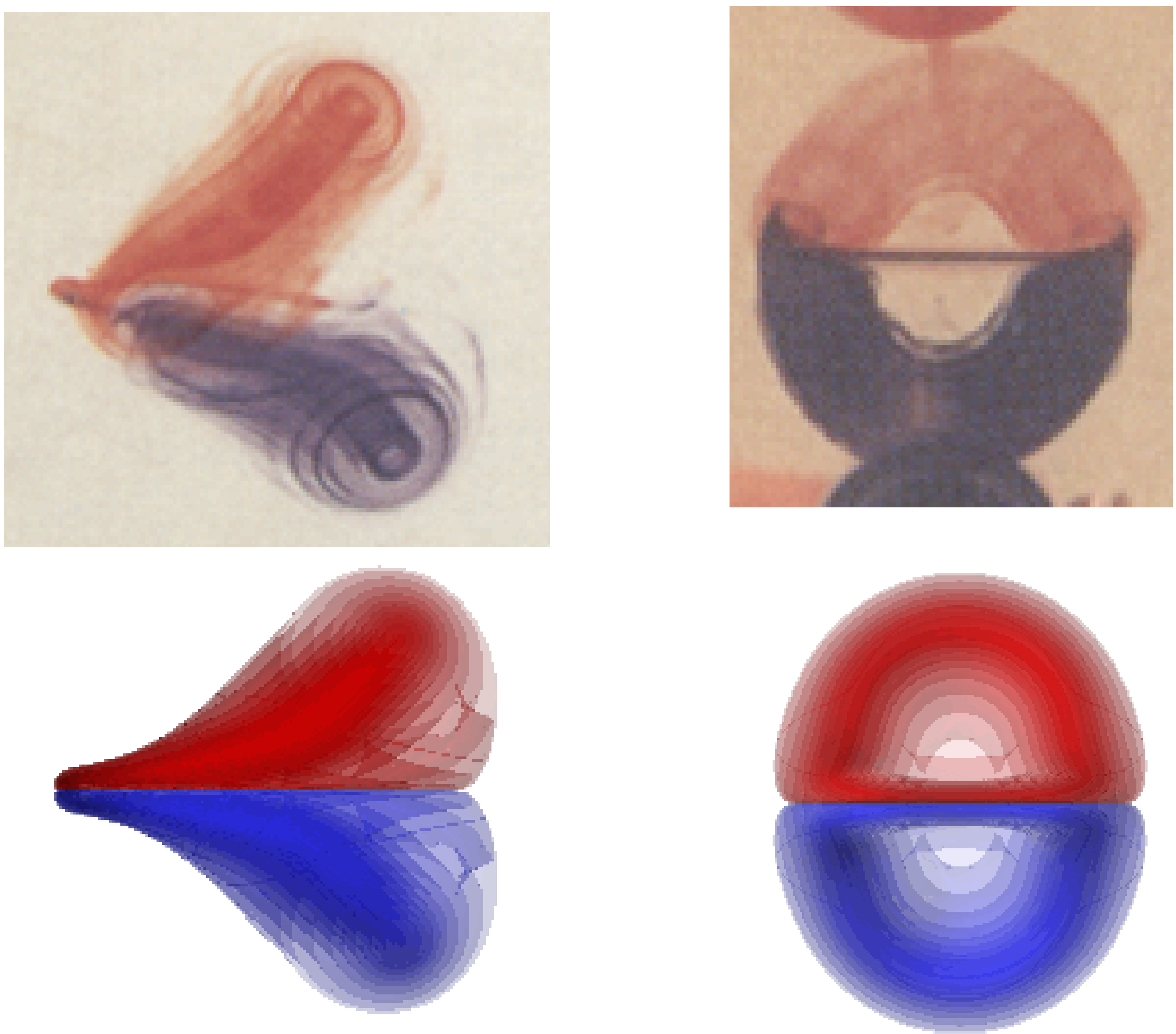
simulations

Leon Kaganovskiy (2006)

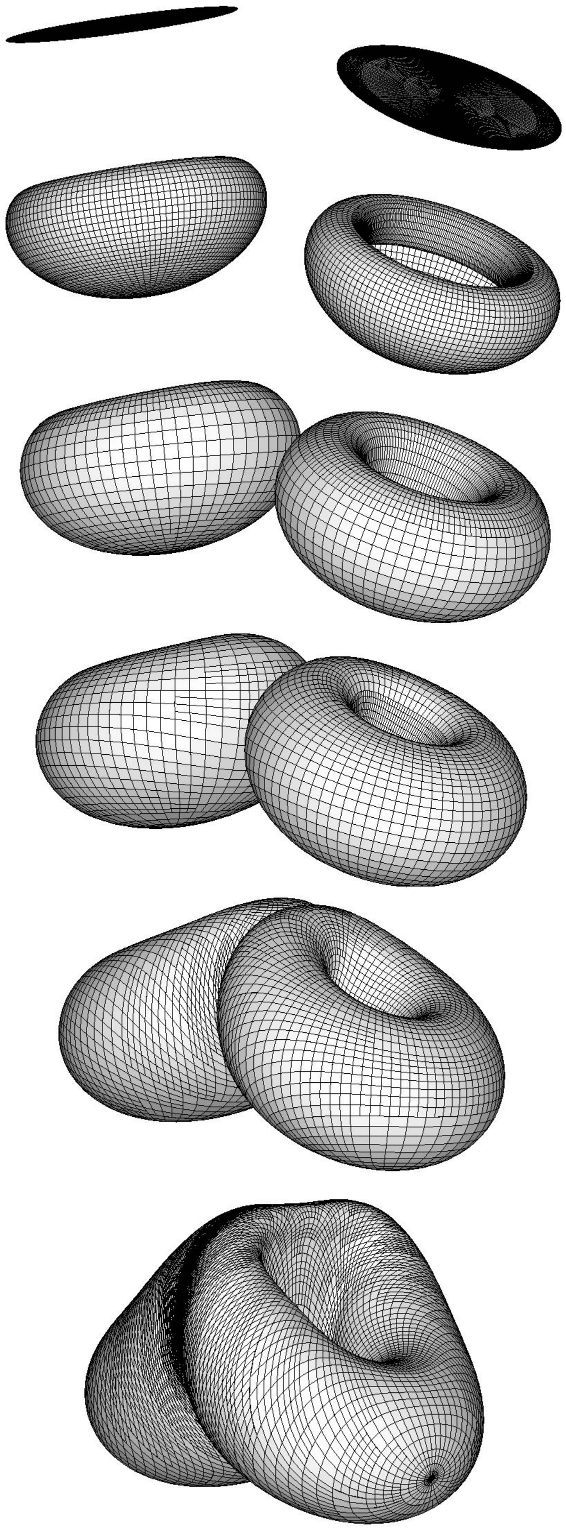
$N_0 \approx 7{,}500 \;\rightarrow\; N_f \approx 1{,}000{,}000$

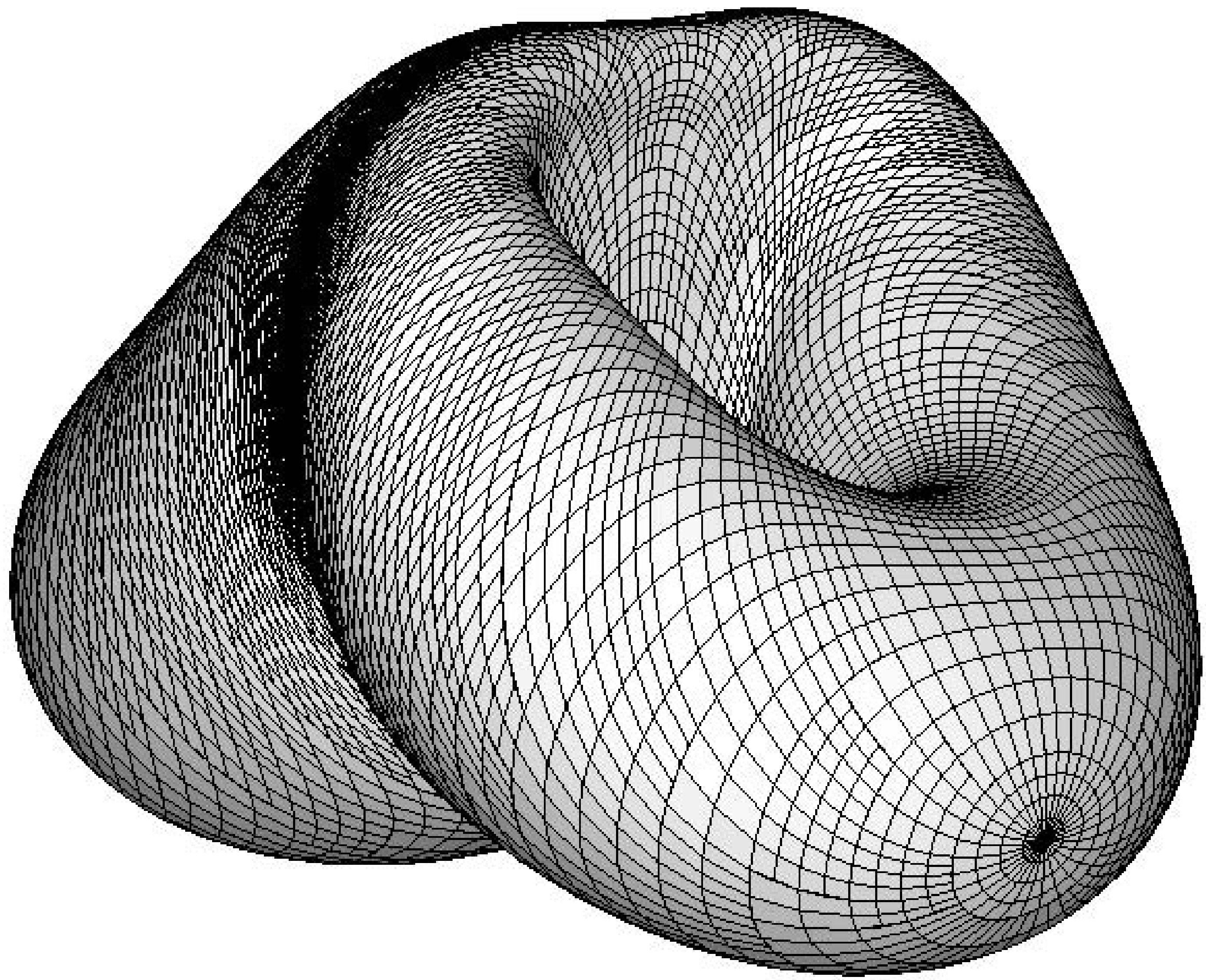
oblique collision of two vortex rings : T.T. Lim (1989)

experiment/simulation

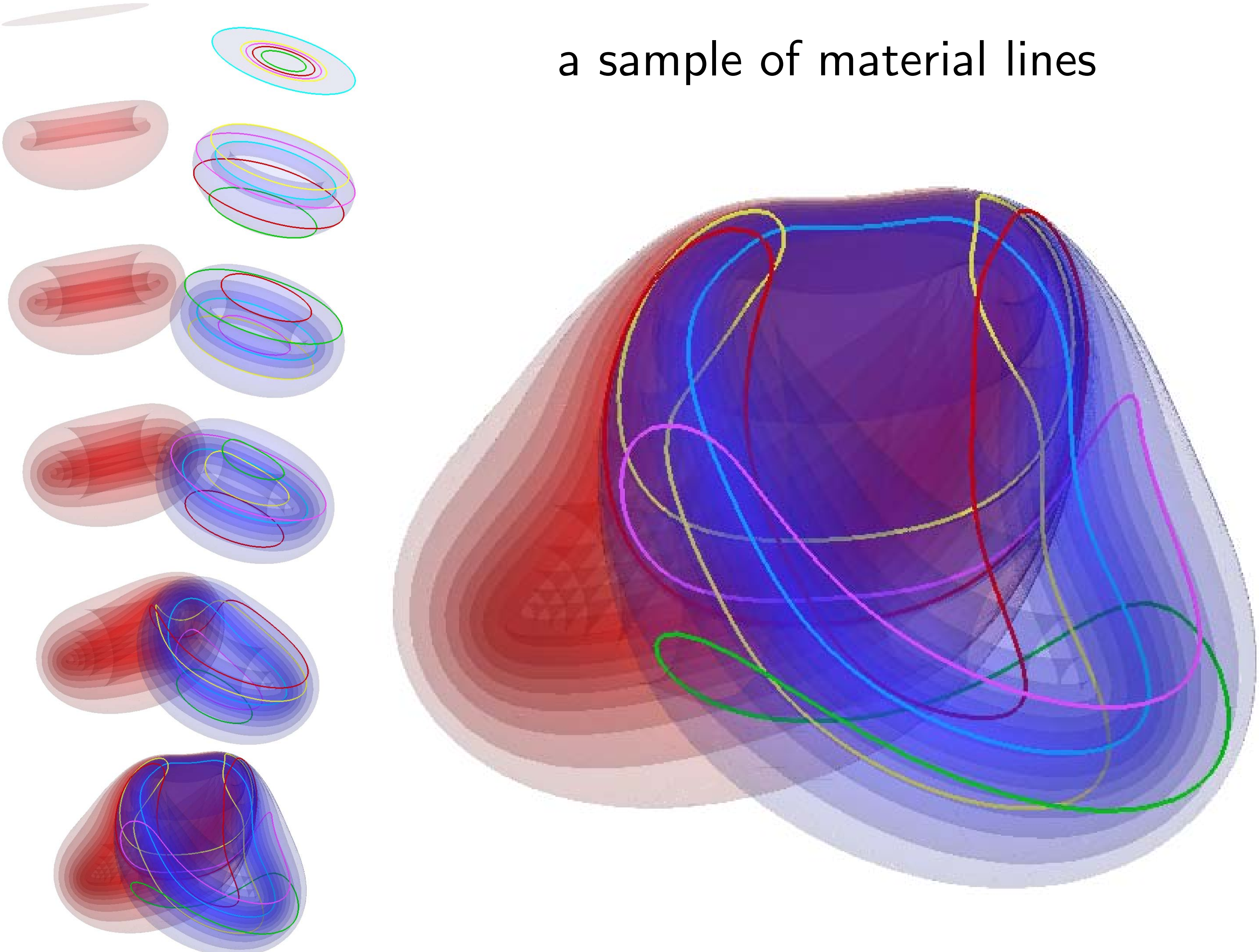


particle/panel mesh
adaptive refinement





a sample of material lines



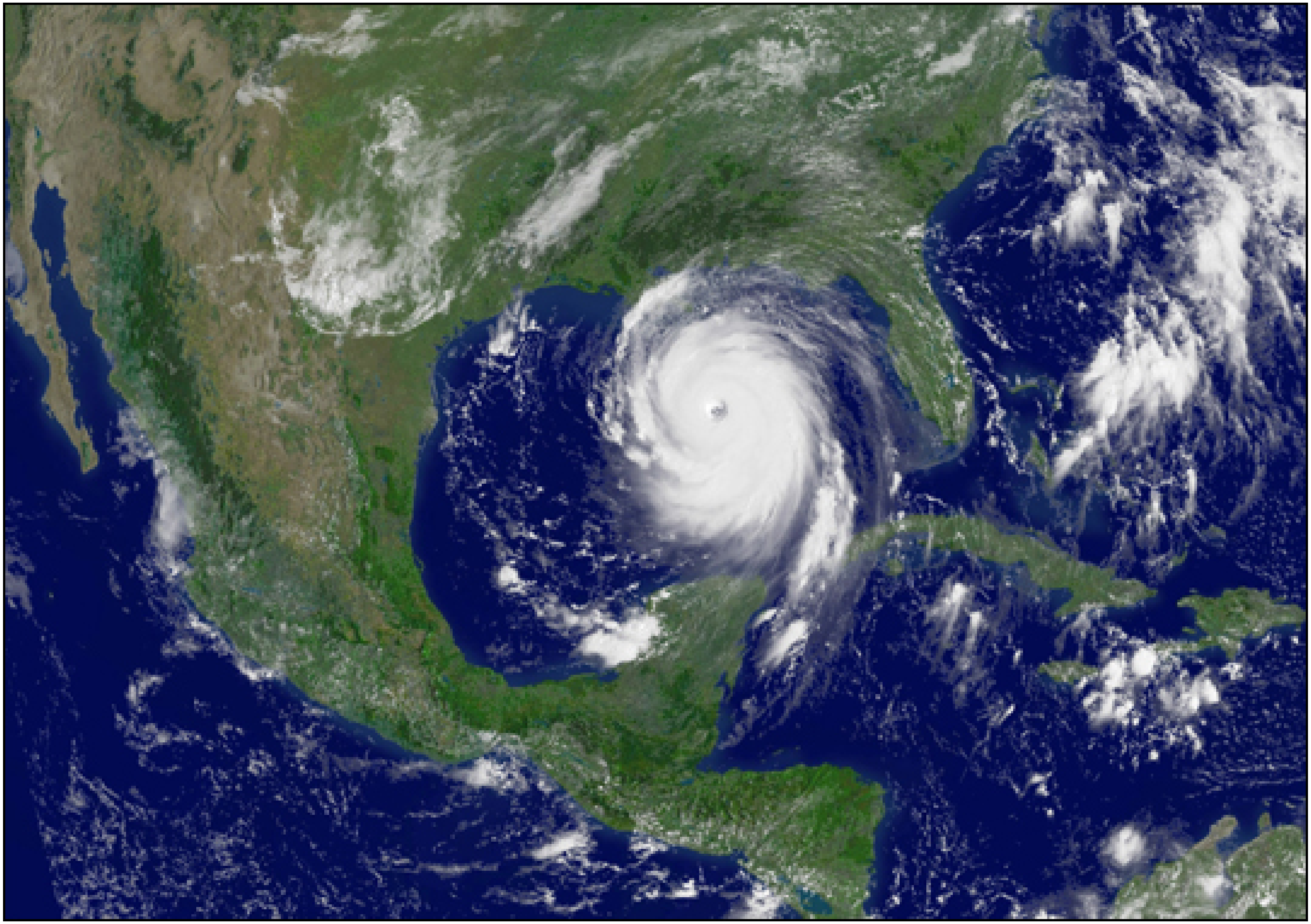
4. barotropic vorticity equation on a sphere

- Poisson equation on a sphere
- Lagrangian vortex method
- preliminary results (Lei Wang 2010 thesis)

  ex 1 : Rossby-Haurwitz wave

  ex 2 : vortex patch

motivation : weather/climate



Hurricane Katrina (image by NOAA)

barotropic vorticity equation on a sphere

relative vorticity : $\zeta = \nabla \times u \ , \ \nabla \cdot u = 0$

absolute vorticity : $\eta = \zeta + 2\Omega z$

$$\Delta\psi = -\zeta$$

$$u = \nabla\psi \times x$$

$$\eta_t + u \cdot \nabla\eta = 0$$

previous work (partial list!)

- Charney-Fjörtoft-von Neumann (1950) : finite-difference
- Sadourny-Arakawa-Mintz (1968) : finite-difference/hexagonal grid
- Dritschel-Polvani (1992) : contour dynamics
- Newton (2000) : point vortex model
- Levy-Nair-Tufo (2009) : spectral element/discontinuous Galerkin

Poisson equation on a sphere

$$\Delta\psi = -\zeta$$

θ : colatitude , λ : longitude

$$\Delta\psi = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\lambda^2}\right)$$

- finite-difference , finite-element , spectral

- Green's function : Kimura-Okamoto (1987) , . . .

$$G(x, x') = -\frac{1}{4\pi}\log(1 - x \cdot x')$$

$$\psi(x) = \int_{S^2} G(x, x')\zeta(x')dS'$$

barotropic vorticity equation on a sphere

Lagrangian form

flow map : $x(a,t)$

$$\frac{\partial x}{\partial t}(a,t) = \int_{S^2} \nabla G(x,x') \times x \cdot \zeta(x',t) dS'$$

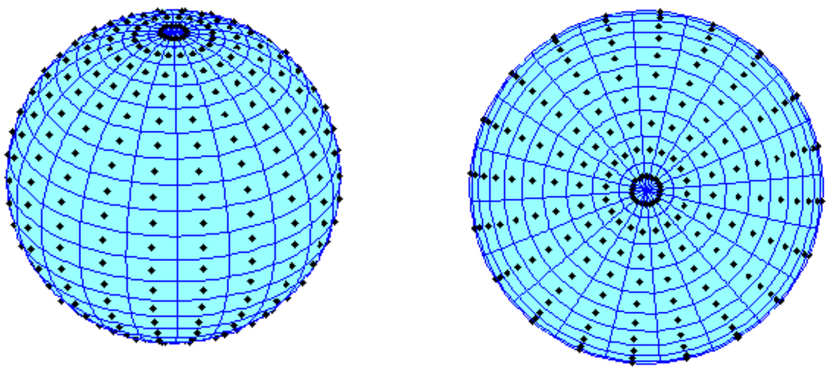
panels : $S^2 = \cup_{j=1}^{N} A_j$

particles : $x_j(t)$

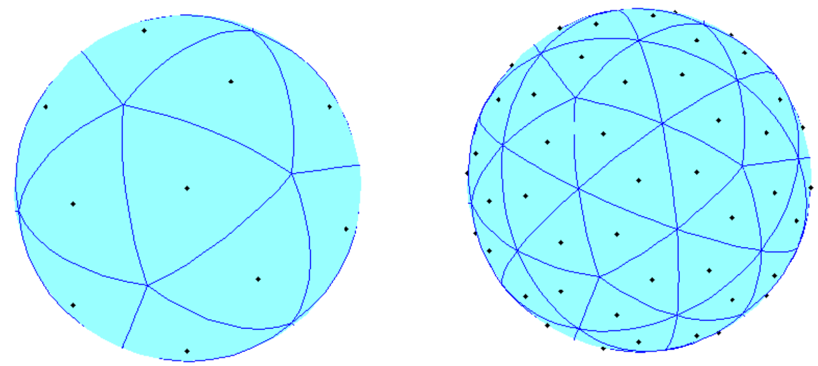
$$\frac{dx_i}{dt} = -\frac{1}{4\pi} \sum_{j=1}^{N} \frac{x_i \times x_j}{1 - x_i \cdot x_j + \delta^2} \zeta_j \, |A_j|$$

$\zeta_j = \zeta_{j0} + 2\Omega(z_{j0} - z_j)$: conservation of absolute vorticity

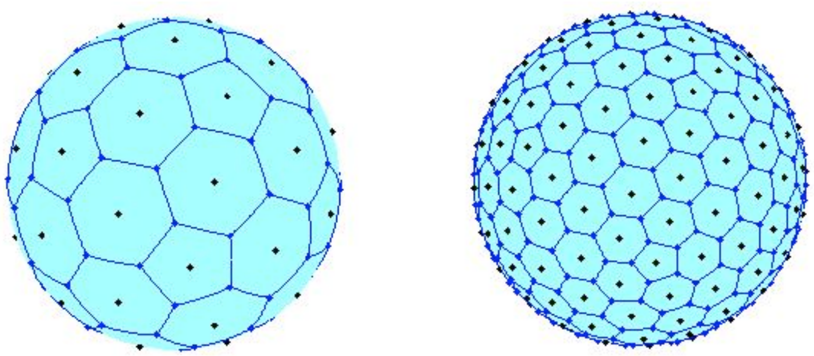
panel discretization of sphere



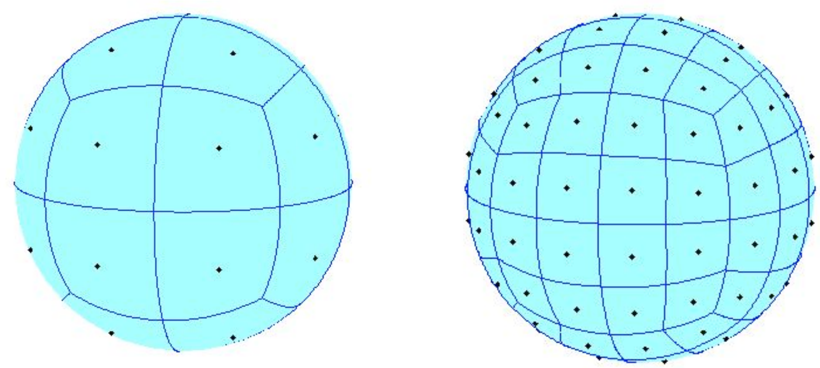
latitude-longitude



icosahedral/triangle



icosahedral/hexagon

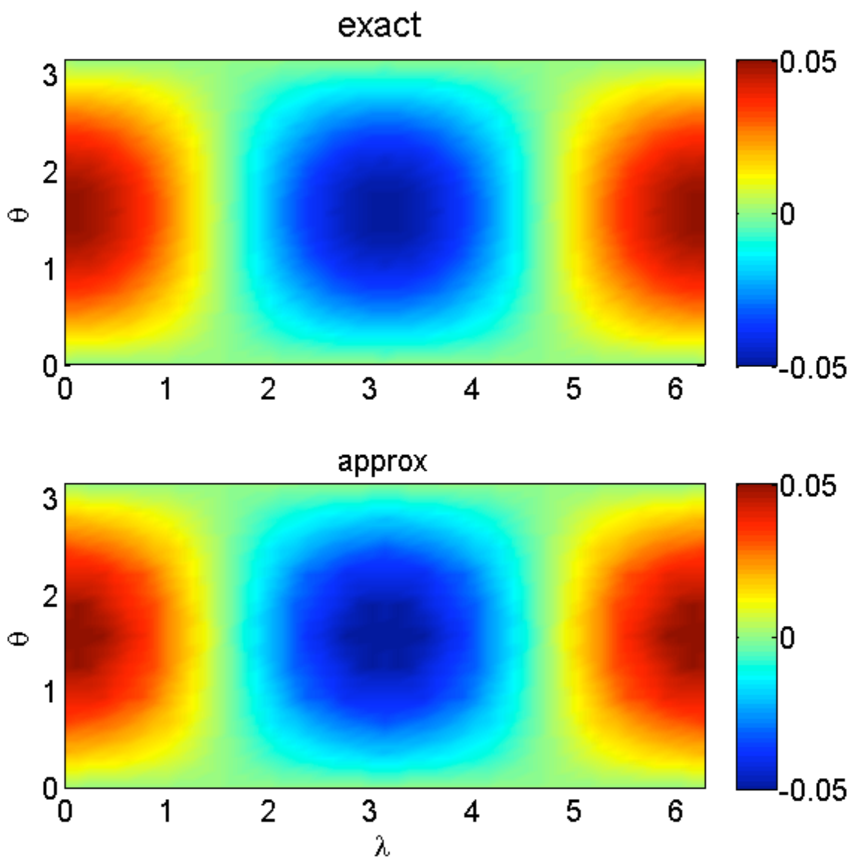


cubed-sphere

ex 1 : Rossby-Haurwitz wave

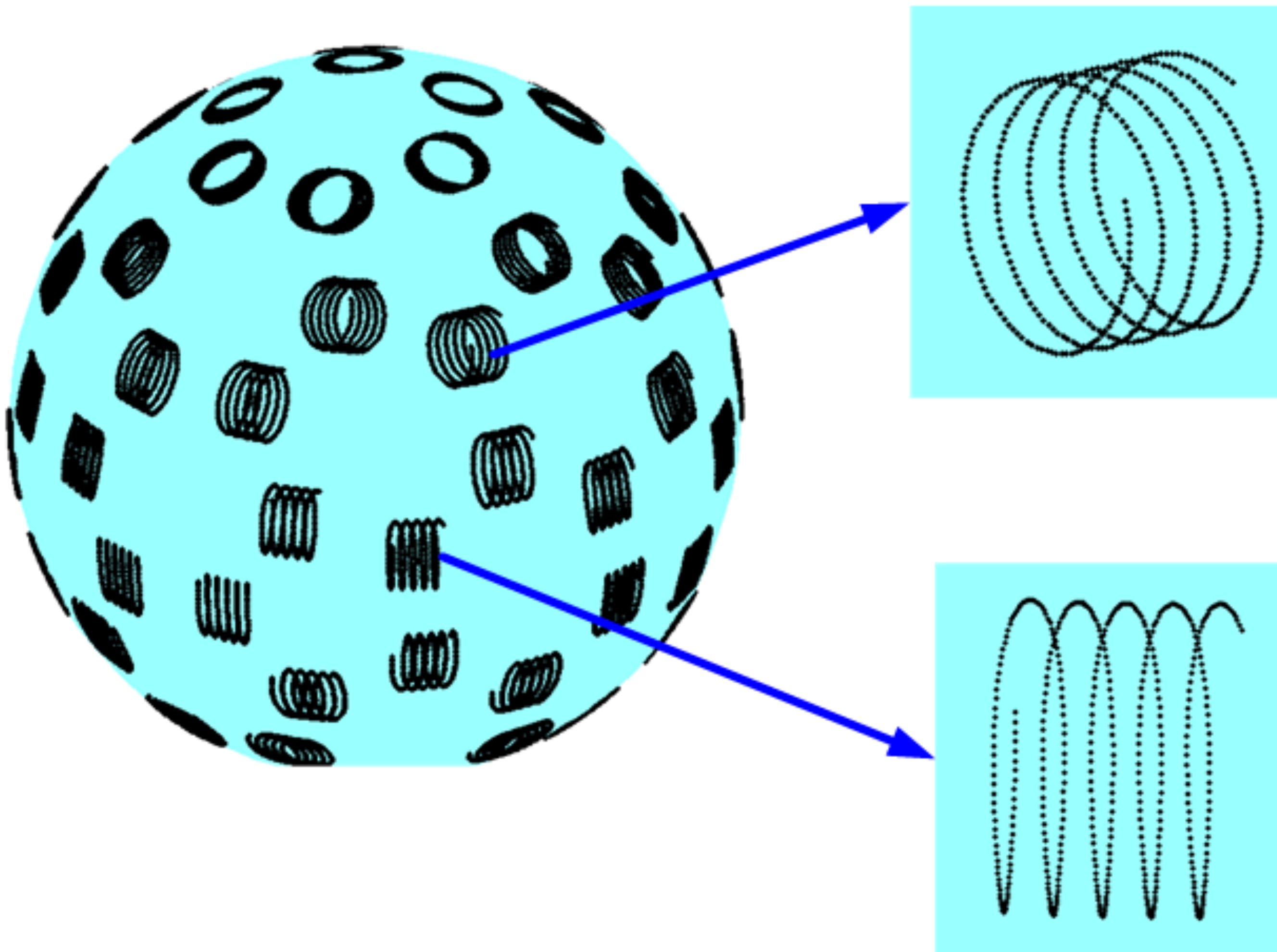
$$\psi_{exact}(\lambda, \theta, t) = \epsilon \sin\theta \cos(\lambda + \Omega t)$$

$$\psi_{approx}(x, t) = -\frac{1}{4\pi} \sum_{j=1}^{N} \log(1 - x \cdot x_j(t) + \delta^2) \zeta_j(t) |A_j|$$

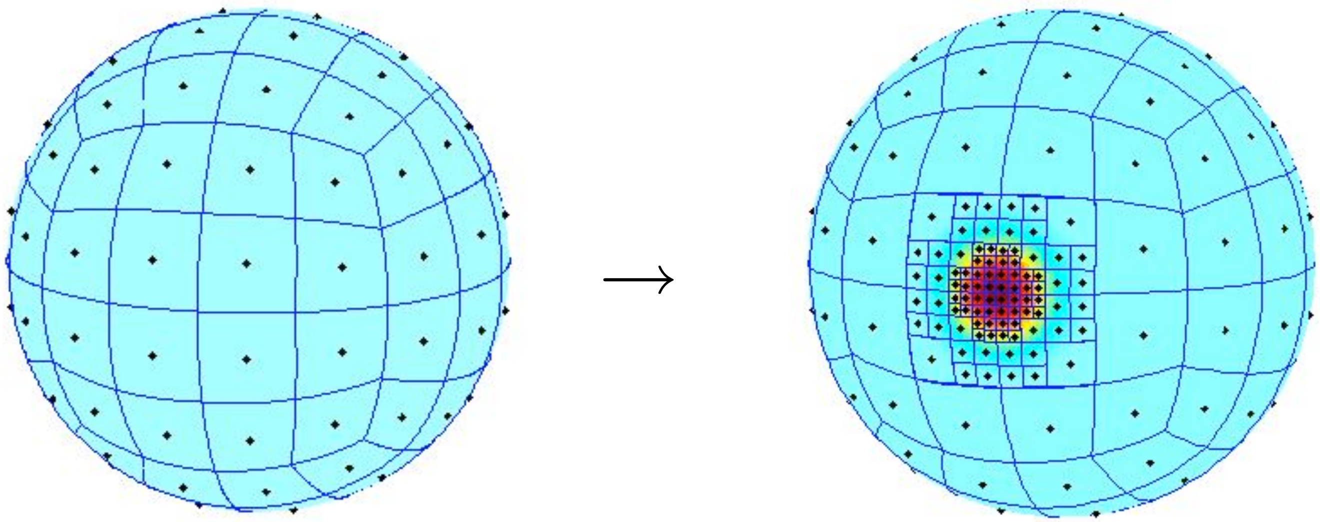


ex 1 : Rossby-Haurwitz wave

particle trajectories : cycloids



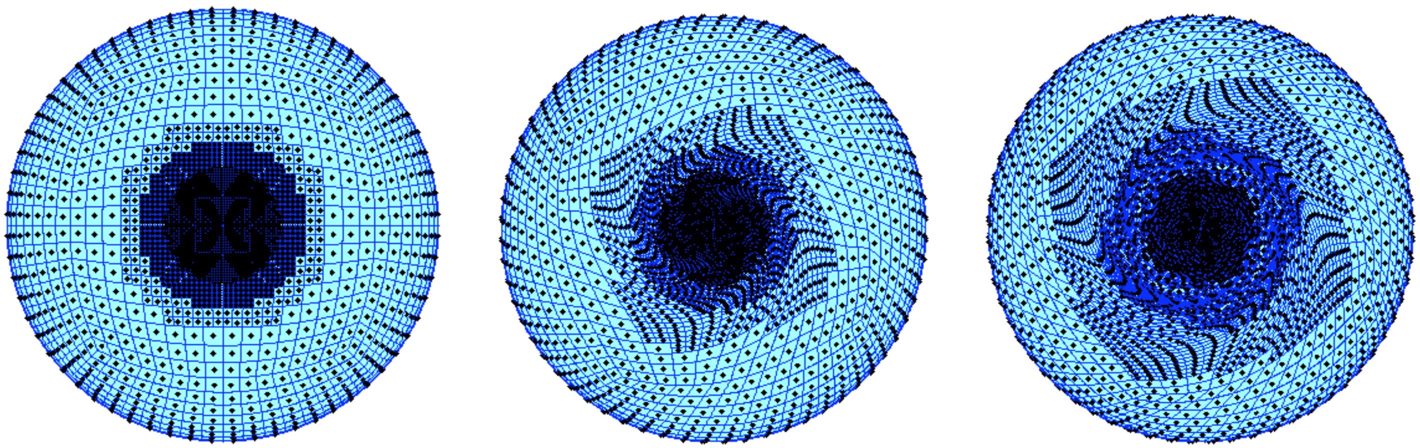
adaptive panel refinement



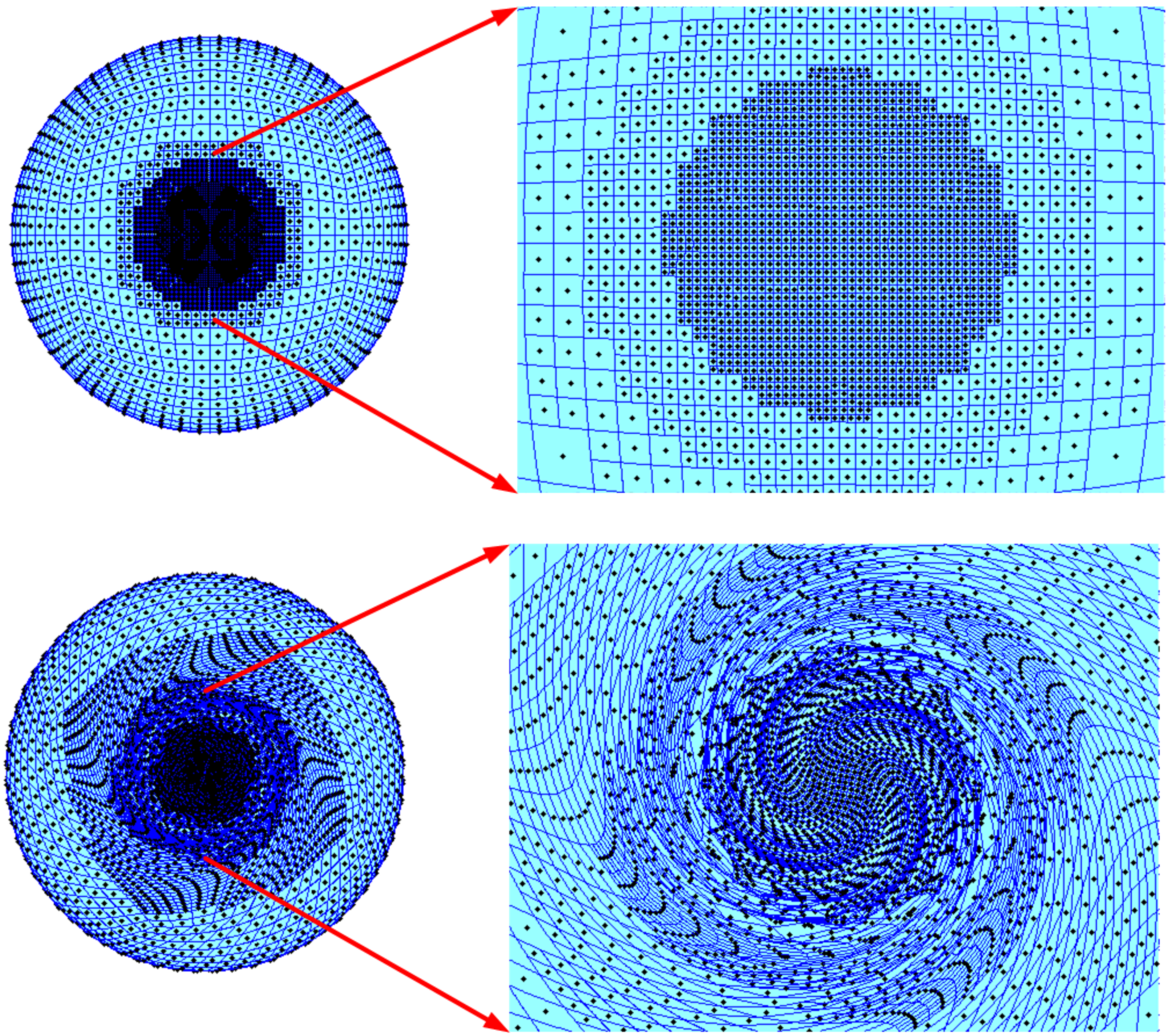
ex 2 : vortex patch : $\Omega = 0$

parameters : $\delta = 0.02\,,\ \epsilon_\Gamma = 0.0002\,,\ \epsilon_d = 0.02$

$(a)\quad t = 0, N = 3504 \qquad t = 2\pi, N = 3600 \qquad t = 4\pi, N = 4188$

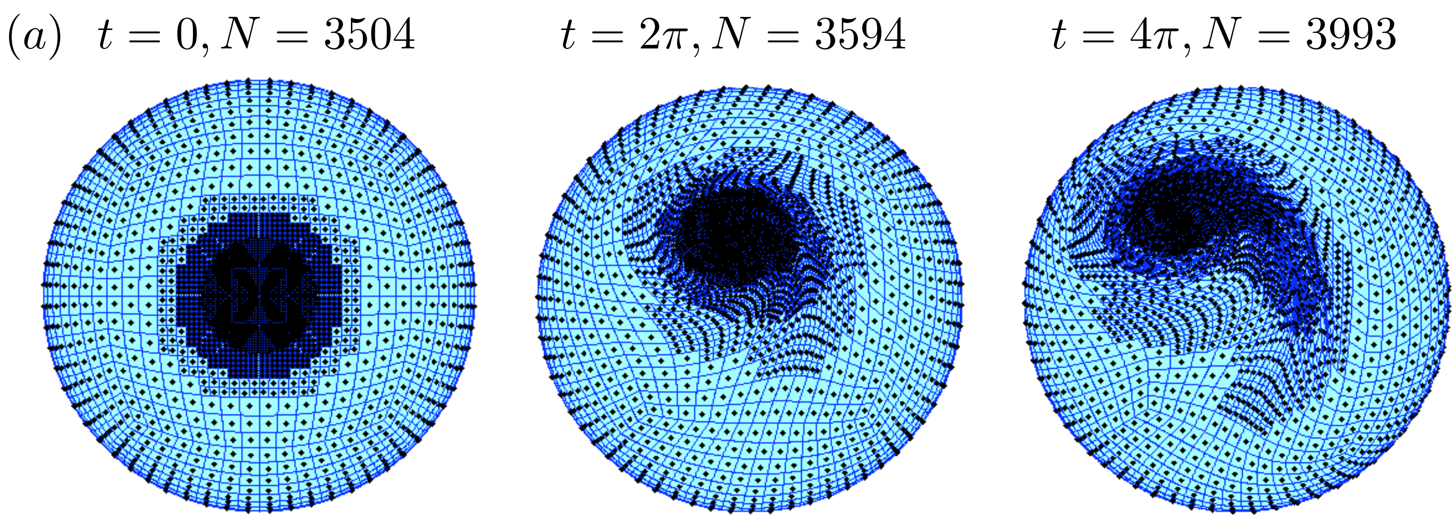


ex 2 : vortex patch : $\Omega = 0$



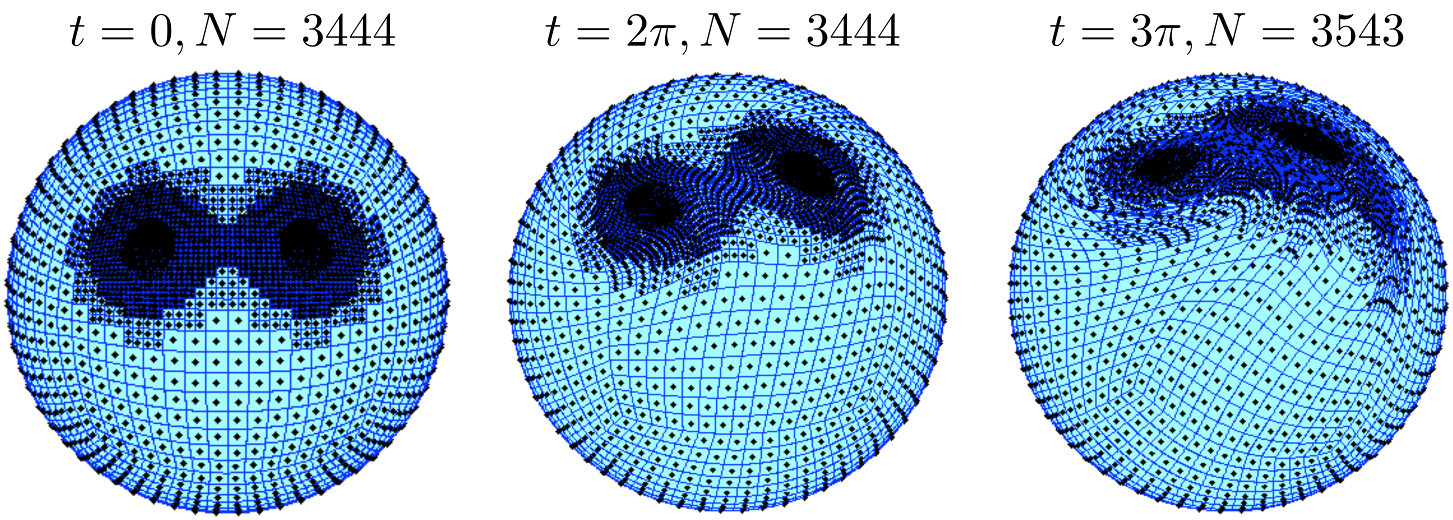
ex 2 : vortex patch : $\Omega = 1/2$

parameters : $\delta = 0.02\,,\ \epsilon_\Gamma = 0.0002\,,\ \epsilon_d = 0.02$



ex 2 : two vortex patches : $\Omega = 1/2$

parameters : $\delta = 0.02$, $\epsilon_\Gamma = 0.0004$, $\epsilon_d = 0.04$



summary

Lagrangian particle method for incompressible flow

regularized kernel : $1/r \to 1/\sqrt{r^2+\delta^2}$

treecode : $O(N^2) \to O(N \log N)$

adaptive refinement

current/future work (Pete Bosler)

improve numerical method : remeshing

extension to shallow water equations : vorticity/divergence

other applications

radial basis functions (Lei Wang)

Poisson-Boltzmann equation/bioelectrostatics (Weihua Geng)

charge transport in solar cells (Lunmei Huang)