SLIM: a three-dimensional baroclinic finite-element model

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Modeling and Computations of Shallow Water Coastal Flows

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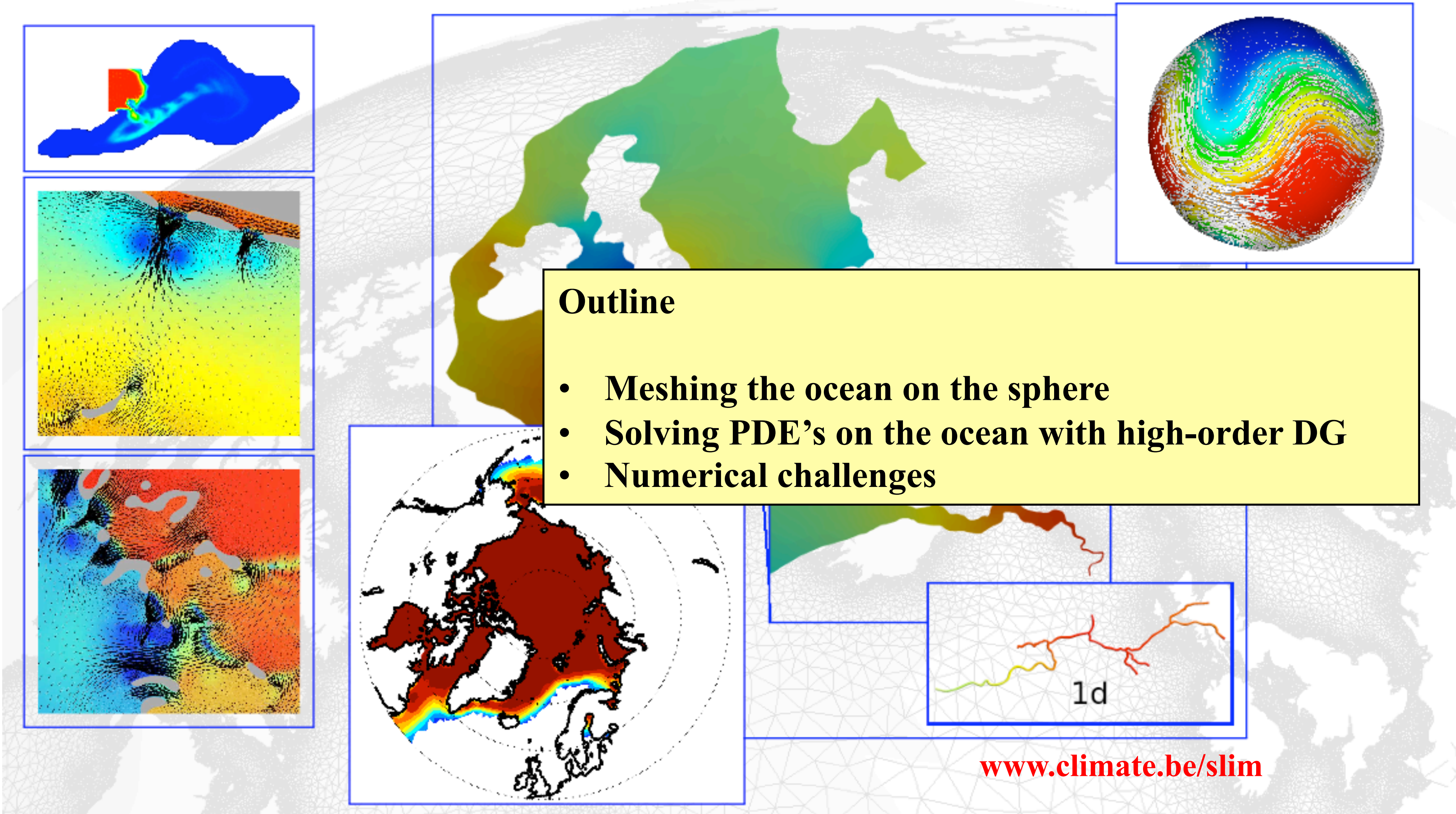
Slim : a multi-scale model for the ocean, coaslines and rivers

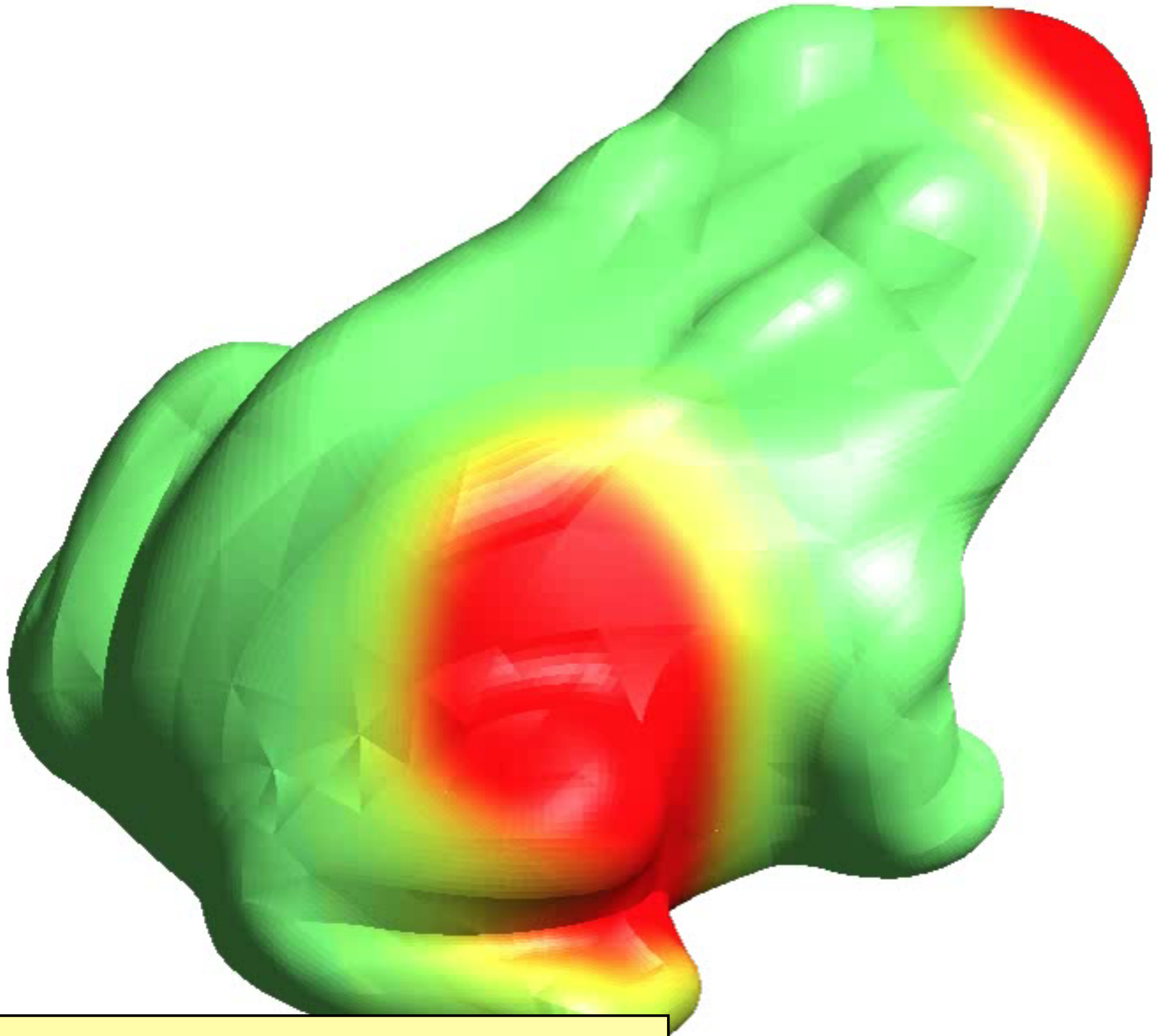


Outline

- **Meshing the ocean on the sphere**
- **Solving PDE’s on the ocean with high-order DG**
- **Numerical challenges**

www.climate.be/slim

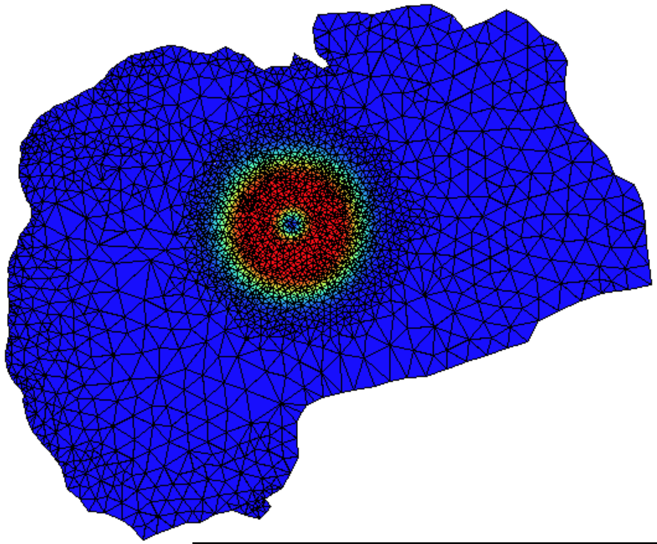
Gravity waves on a froggy planet



Building a general method for irregular manifolds

- **The method is independent of the manifold**
- **It must be easy to implement**
- **It must be robust to handle such a funny benchmark**

Are adaptive unstructured-grid models coming of age ?



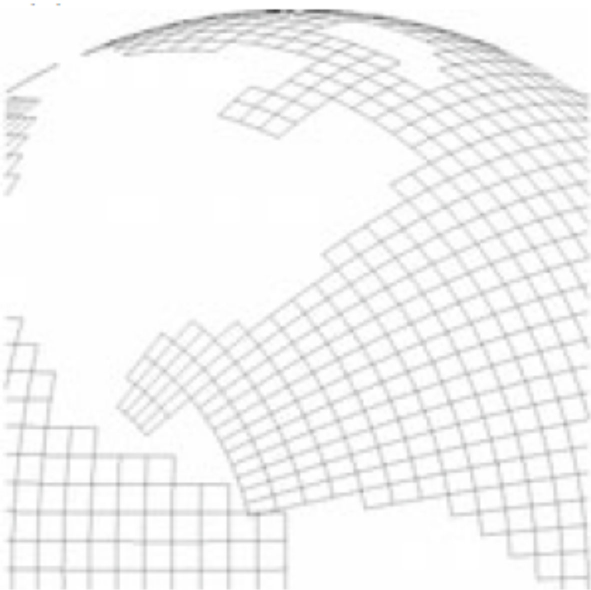
Reduced-gravity simulation of a baroclinic eddy in the Gulf of Mexico.

This simulation is several orders of magnitude cheaper than a constant resolution one of the same accuracy ! (Bernard, 2007)

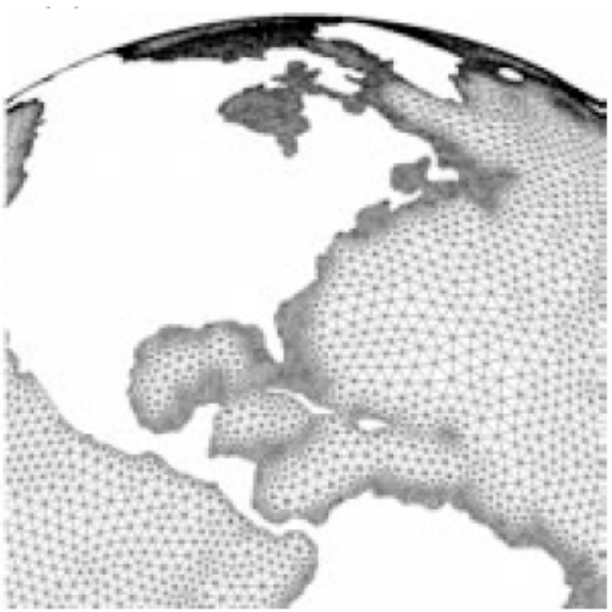
- **Numerical models of marine systems should be able to explicitly represent the broadest possible range of scales.**
- **Increasing the resolution everywhere is not the best option as this often results in a very inefficient use of the computational resources.**
- **The idea is to increase the resolution where and when it is needed !**

Structured grid …

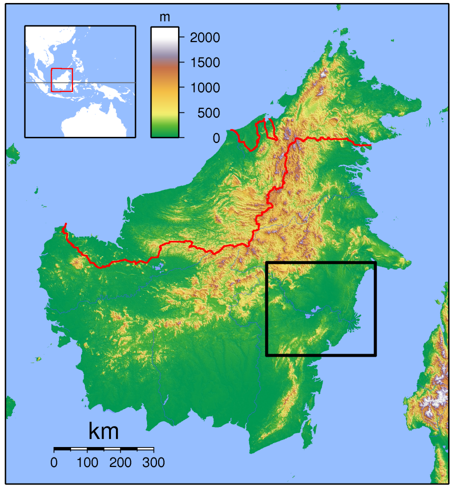
- Finite differences are easy to implement
- Programming is easy
- Well known in the world of oceanography
- Bad representation of the coastlines
- Difficult to enhance locally the resolution
- Poles singularity

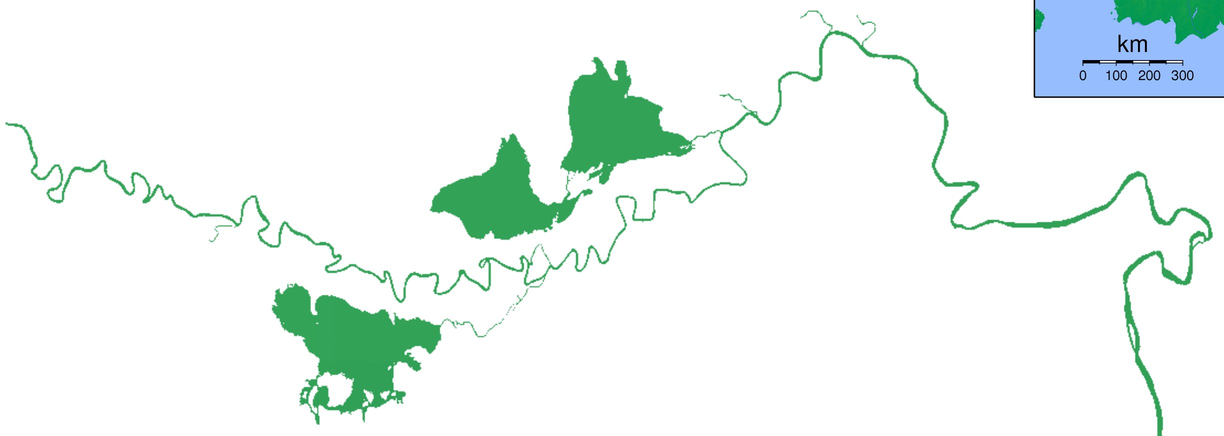


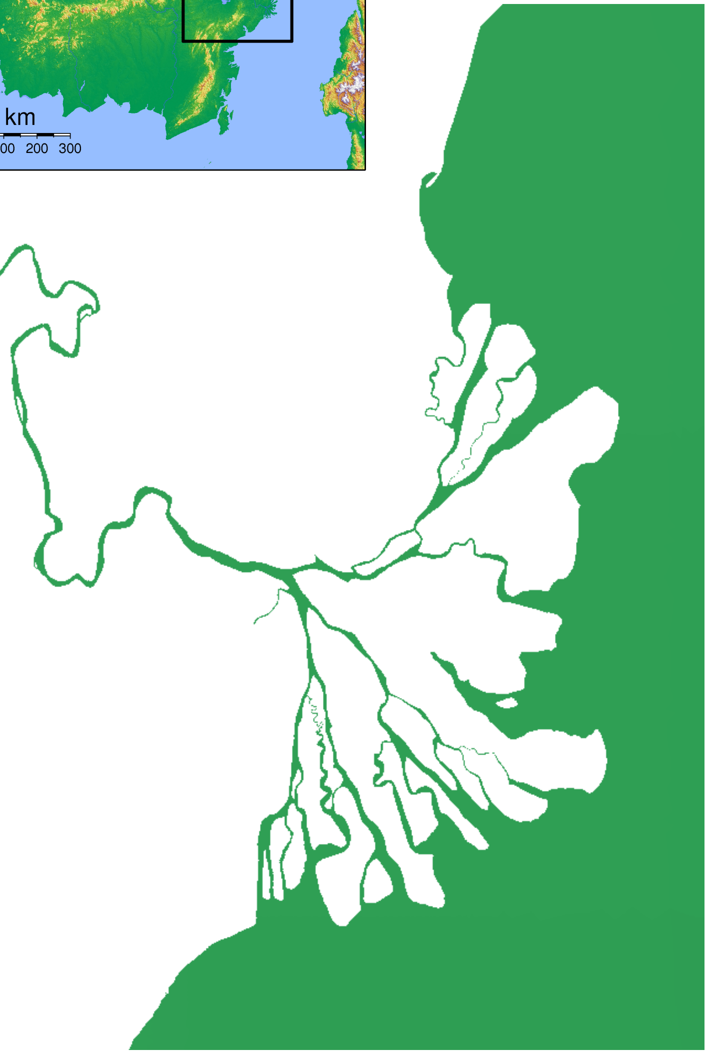
…versus unstructured grid



- Numerical methods are more complicated
- Programming is more complicated
- Not well known in the world of oceanography
- Accurate representation of the coastlines
- Enhancing the resolution is flexible
- No singular points

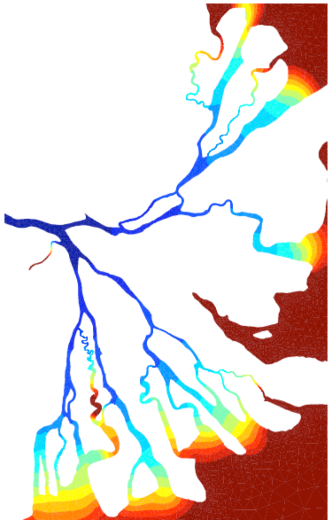


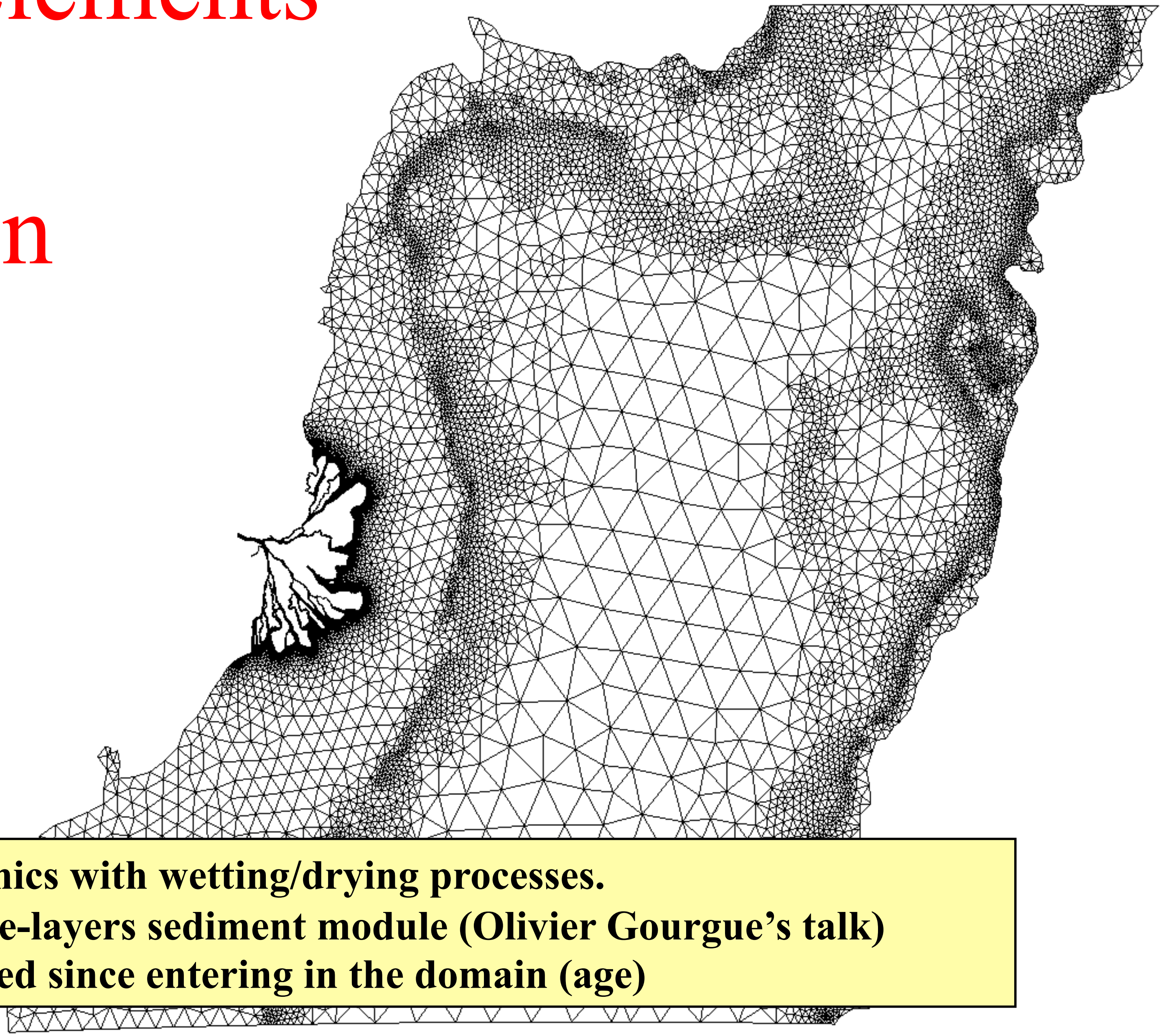




Mahakam river,
a highly multi-scaled
motivating example

72% of the elements are in 1.4% of the domain

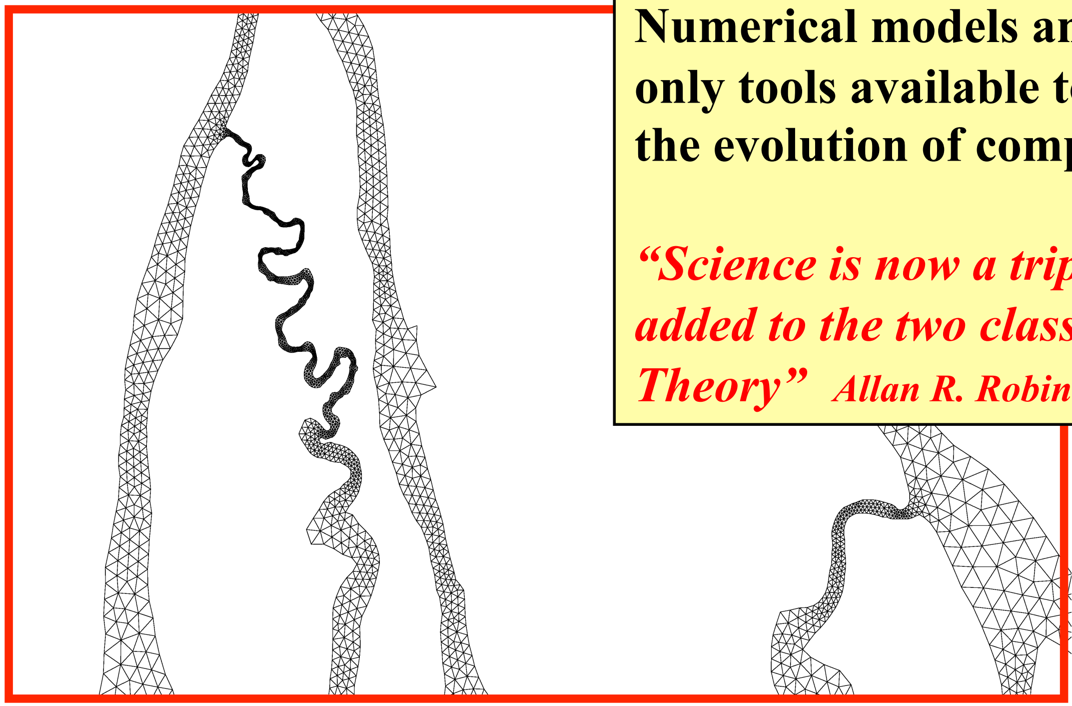


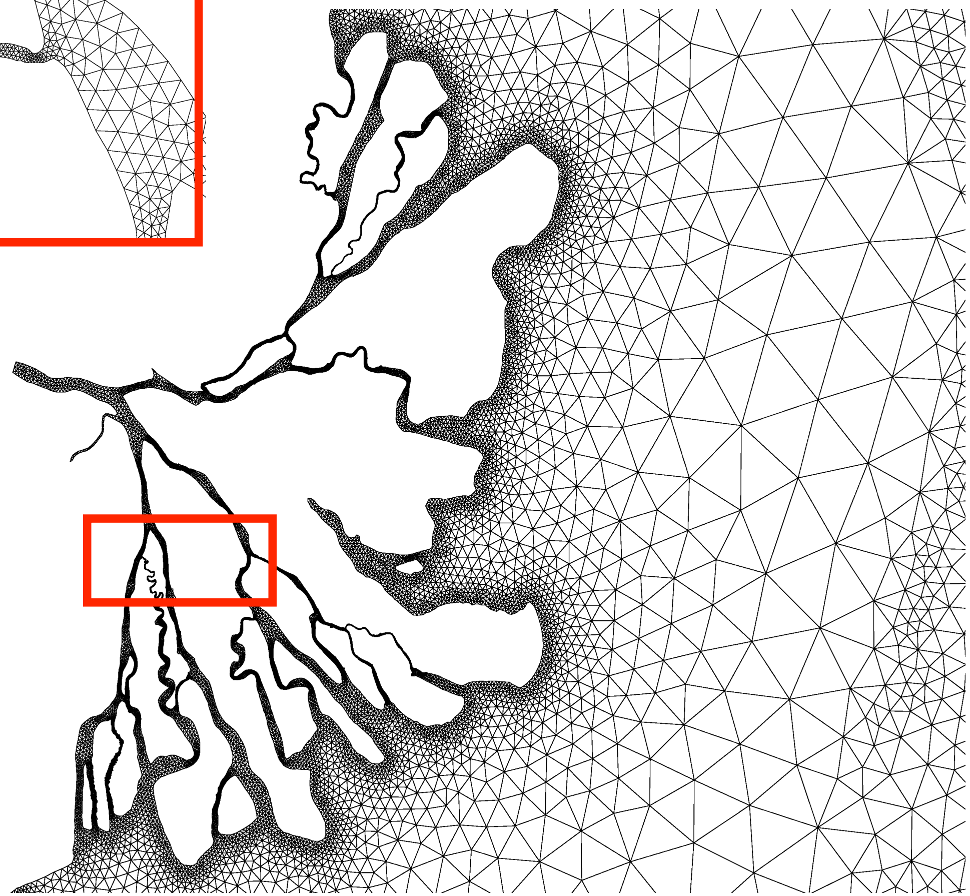


- **Validated hydrodynamics with wetting/drying processes.**
- **Development of a three-layers sediment module (Olivier Gourgue's talk)**
- **Computing time elapsed since entering in the domain (age)**

Numerical models and computer simulations are the only tools available to understand in detail and predict the evolution of complex environmental systems.

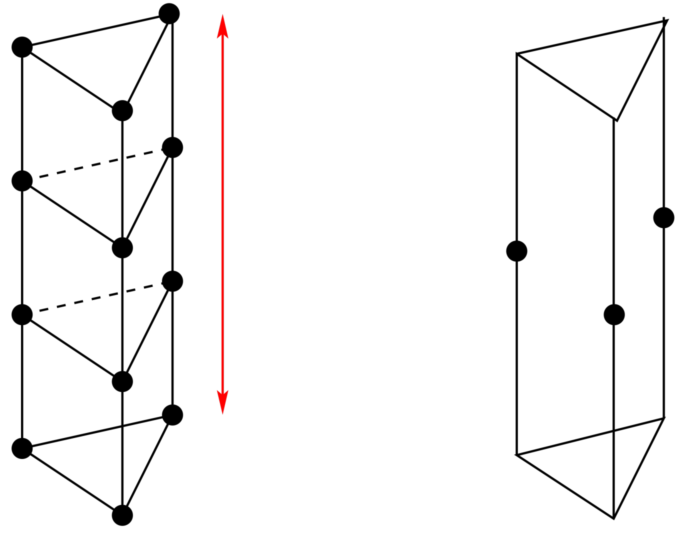
"Science is now a tripartite endeavour, with Simulation added to the two classical components, Experiment and Theory"* *Allan R. Robinson

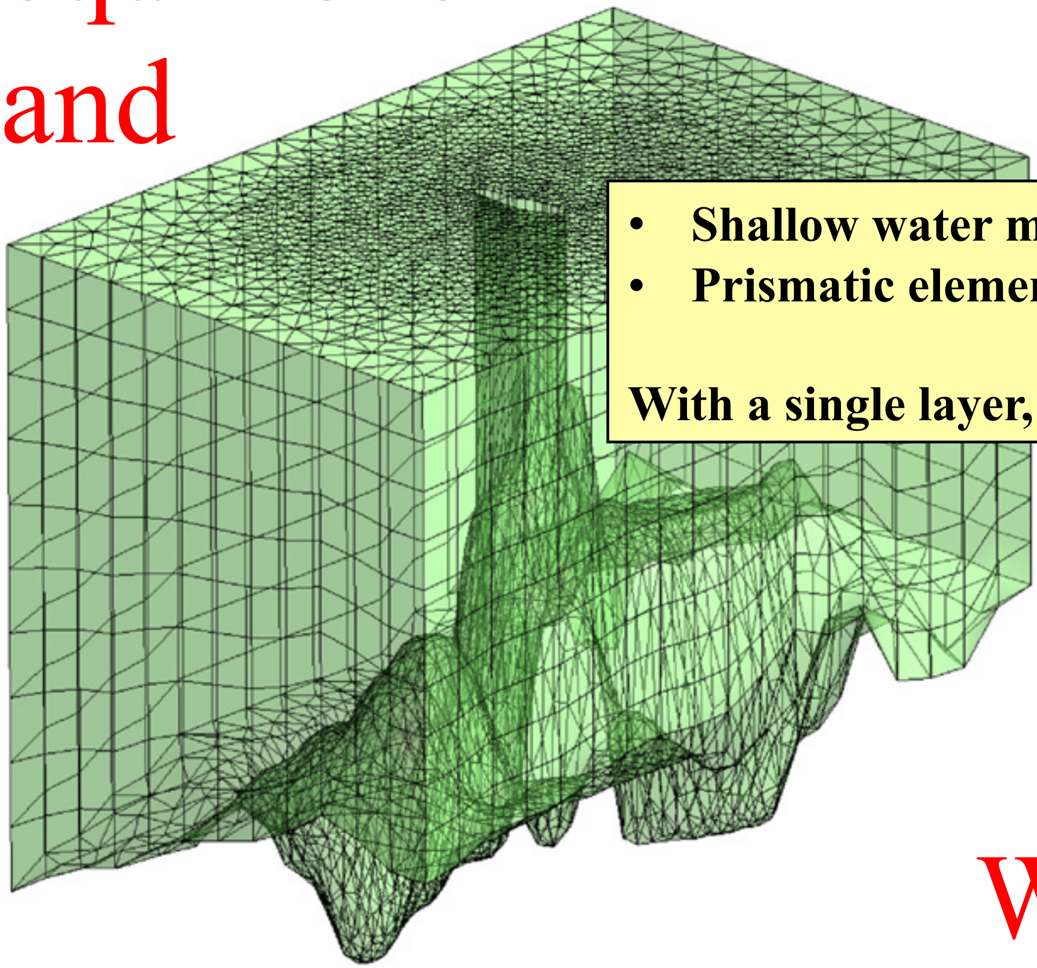




Size of the smallest element is 7 m

The hydrostatic Boussinesq equations and





- **Shallow water model is the depth-integrated 3D model**
- **Prismatic elements appear as a natural choice**

With a single layer, we solve the shallow water model !

… the Shallow Water Equations

A lot of physical processes inside the Shallow Water Equations

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \left((h + \eta)\, \boldsymbol{u} \right) = 0,$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot (\nabla \boldsymbol{u}) + f \boldsymbol{k} \times \boldsymbol{u} + g \nabla \eta = \frac{1}{H} \nabla \cdot \left(H\nu (\nabla \boldsymbol{u}) \right) + \frac{\boldsymbol{\tau}^s + \boldsymbol{\tau}^b}{\rho H}.$$

Waves equation
Equal-order discretization !

$P_1 - P_1$

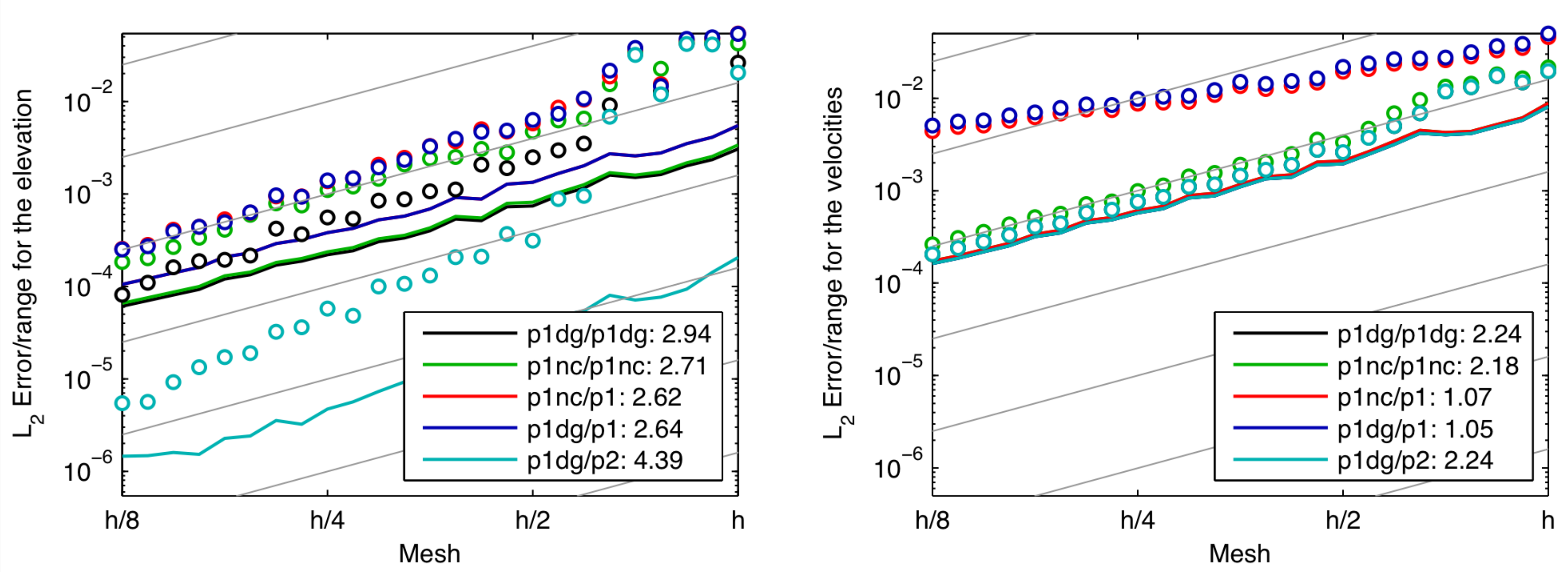
Geostrophy equilibrium
Exactly satisfied ?

$P_1^{DG} - P_2^{DG}$

Stokes problem:
LBB condition occurs !

$P_2 - P_1$

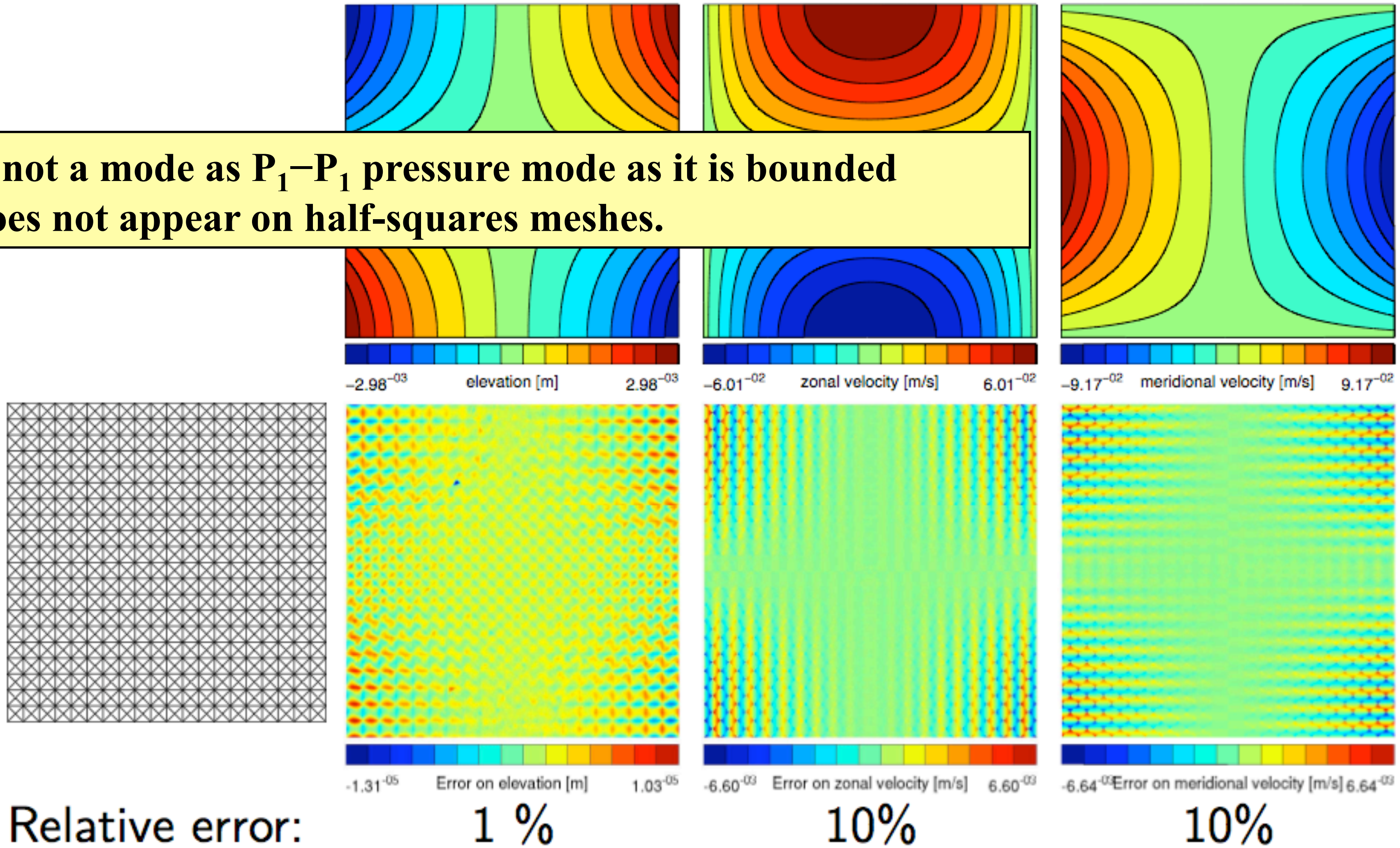
P_1^{NC}-P_1 inviscid computations look pretty nice …



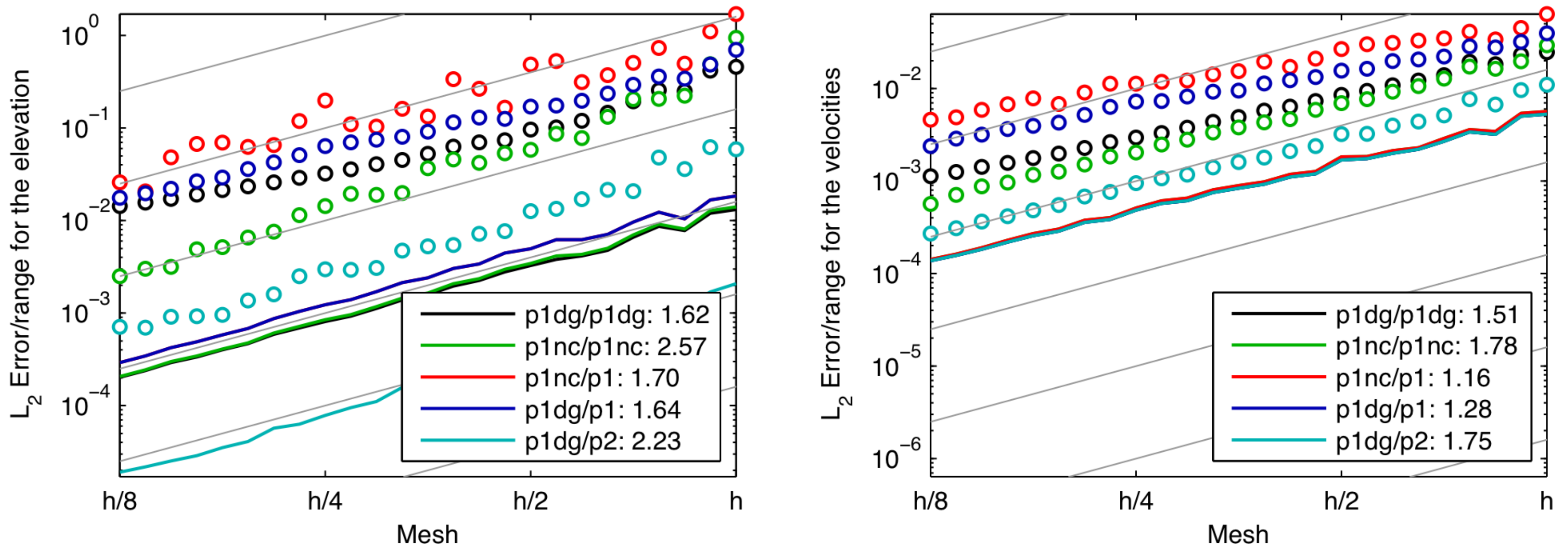
… but exhibit only a first-order convergence!

Structured noise is observed !

- **it is not a mode as P_1−P_1 pressure mode as it is bounded**
- **it does not appear on half-squares meshes.**

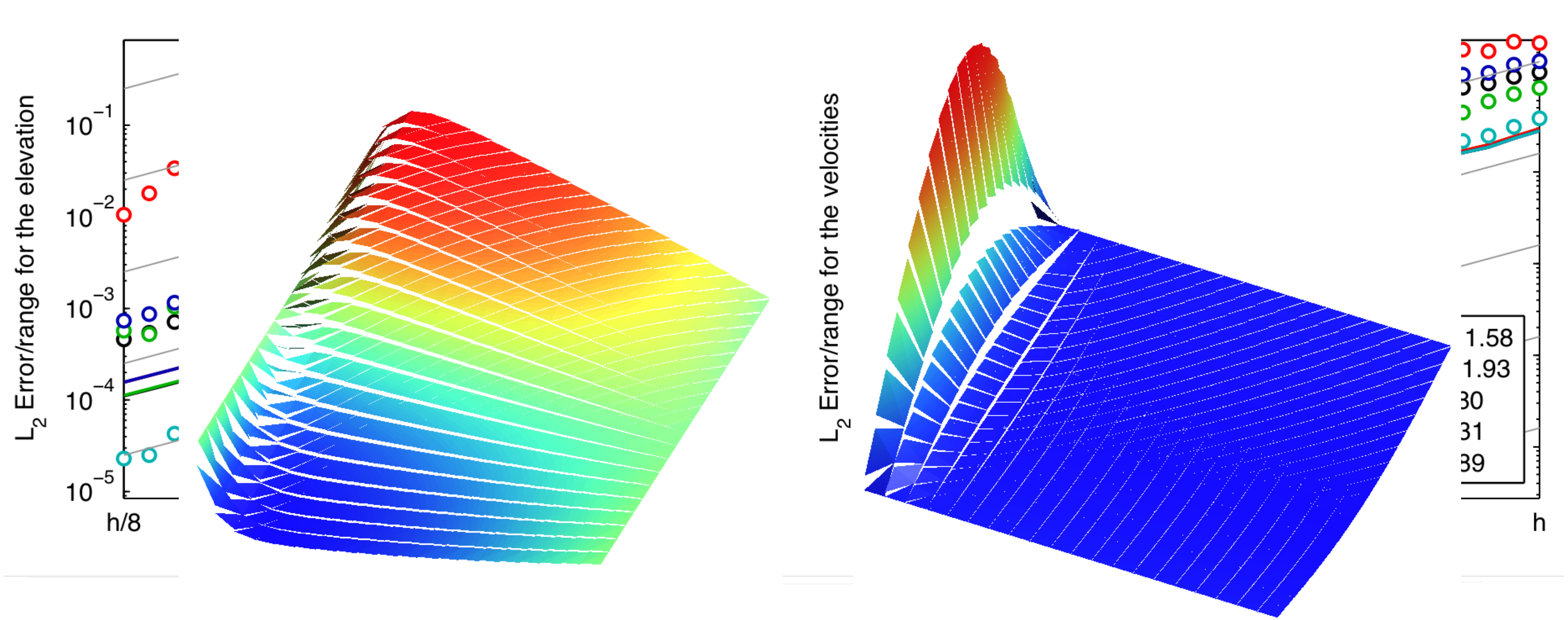


P_1^{DG}-P_2 wins the accuracy award!



- **Second-order convergence for all benchmarks.**
- **Higher order quadrature rules are required.**
- **Consistency requires to use P_2 tracers !**
- **Efficient iterative solution strategy ?**

Coriolis issue for P_1^{DG}-P_1^{DG}



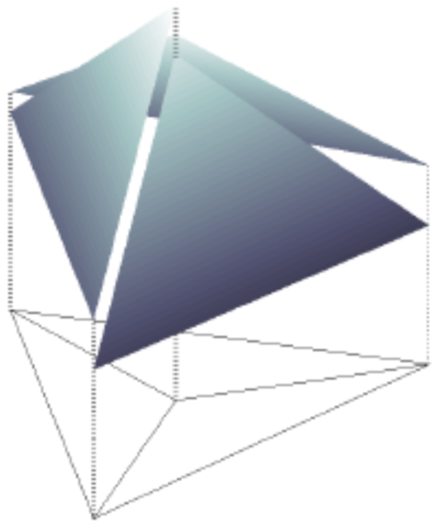
- **Half an order of accuracy is lost with Coriolis**
- **Coriolis term has no corresponding interface term**
- **Only normal velocity jumps are removed by the Riemann solver**
- **Tangent velocity jumps amplified by Coriolis term and not damped**

The Galerkin Discontinuous Method

Finite Volumes
- Natural treatment of wave-like terms
- Low order on unstructured meshes

Continous Finite Elements
- Optimal for second-order terms
- High order interpolation spaces

Best of both approaches !
- Wave terms handled in the finite volume spirit
- Second-order terms accurately handled with IP formulation
- High order interpolation spaces



The Galerkin Discontinuous Method

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}, \eta) = \boldsymbol{f}(\boldsymbol{u}, \eta)$$

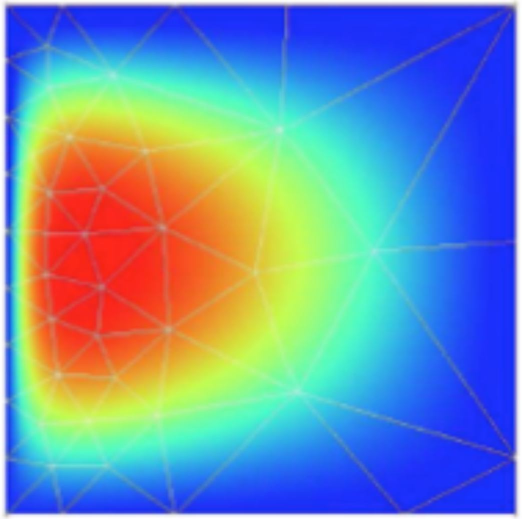
The classical DG weak formulation reads:

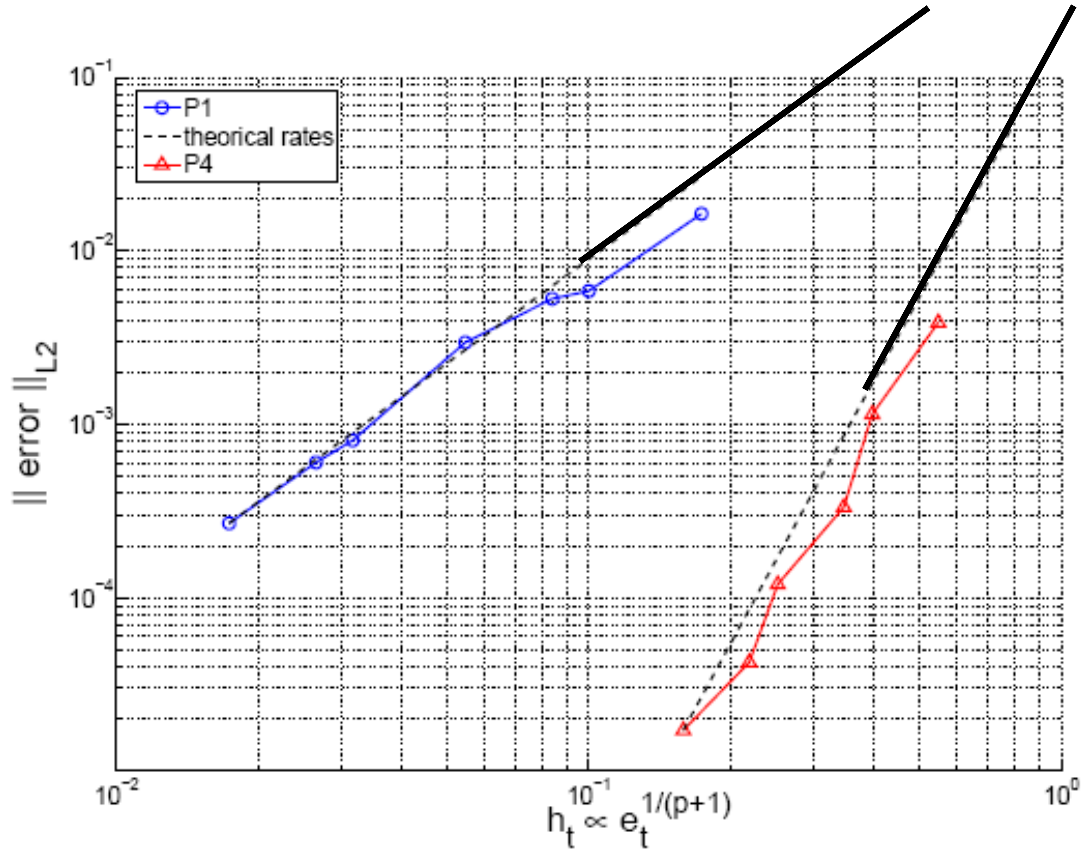
$$\frac{\partial}{\partial t} < \boldsymbol{u}^h \cdot \hat{\boldsymbol{u}}^h >_{\Omega_e} \quad - \quad < \boldsymbol{\sigma}(\boldsymbol{u}^h, \eta^h) \cdot \nabla \hat{\boldsymbol{u}}^h >_{\Omega_e}$$

$$+ \quad \ll \boldsymbol{\sigma}^*(\boldsymbol{u}^h, \eta^h) \cdot \boldsymbol{n} \cdot \hat{\boldsymbol{u}}^h \gg_{\partial\Omega_e} \quad = \quad \mathbf{0}$$

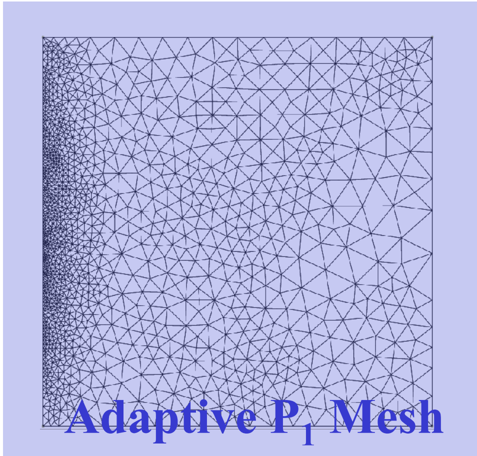
- **Bloc-diagonal global matrices**
- **Transfer between elements through the flux on the edges**
- **A weak collocated formulation can be also derived**
- **Upwinding by the flux evaluation (Riemann's solver)**

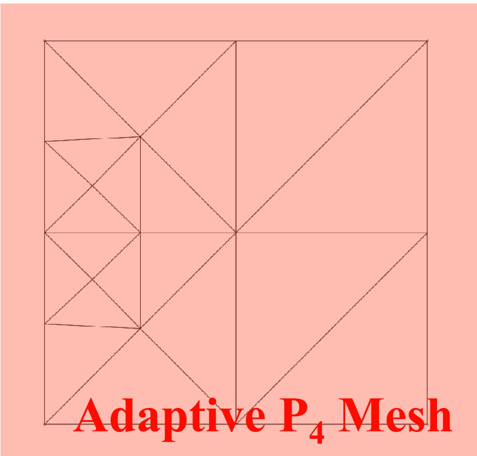
Theoretical rates of convergence are obtained for the analytical Stommel problem

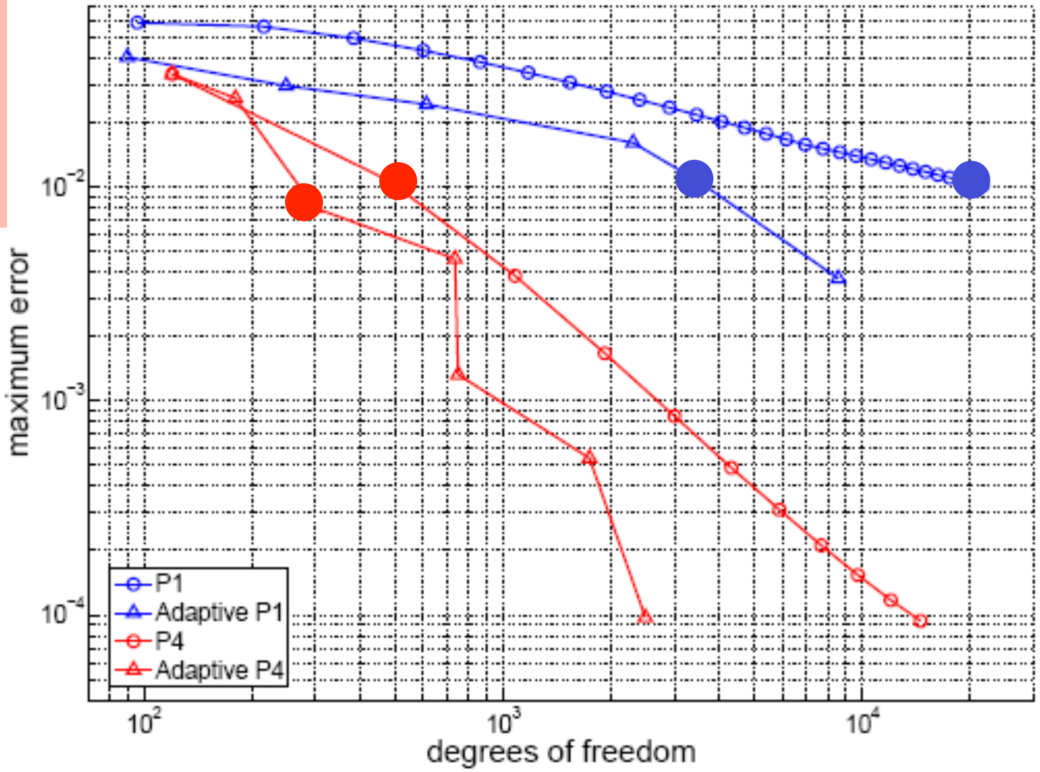




How does it converge ?







P_1^{DG}-P_1^{DG} is currently used because it is fast

Implicit time marching

- **Implicit scheme needs linear solver**
- **DG + ILU(0) GMRES solution strategy is efficient**

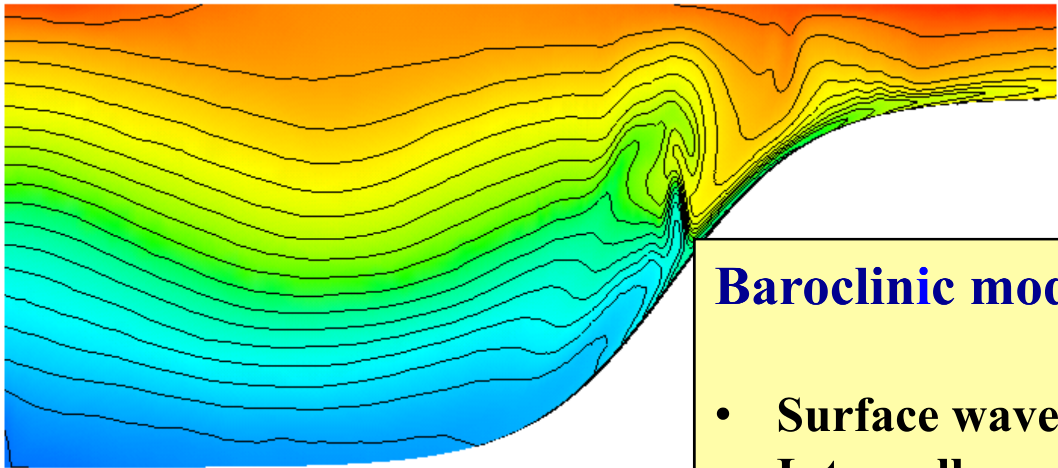
Explicit time marching

- **Finite volume limiters can be applied**
- **Conservative wetting and drying procedures are available**
- **Not more expensive than non-conforming linear element**

3D : baroclinic effects take place!

Barotopic model

- **Surface waves and advection**
- **Subcritical for large scale problems**



Baroclinic model

- **Surface waves, advection and internal waves**
- **Internally supercritical flows are common**
- **Internal waves break occur**
- **Density current fronts are supercritical**
- **Specific limiters are needed**

Internal waves couple…

- **Tracers are advected by the vertical velocity**
- **Vertical velocity is deduced from the horizontal velocity**
- **Pressure gradient is a source term for the horizontal momentum**
- **Pressure gradient is deduced from the density gradient**
- **Density gradient is linked to the tracers by an equation of state**

Interface terms must take into account this physics at least for subcritical flows !

$$w \rightarrow S, T$$
$$\mathbf{u} \rightarrow w$$
$$p \rightarrow \mathbf{u}$$
$$\rho \rightarrow p$$
$$S, T \rightarrow \rho$$

…momenttum,
mass and tracers.

Lax-Friedrichs flux is the key ingredient …

$$\{F\} + \lambda_{\mathrm{max}}[u]$$

Deriving a Riemann solver would be quite difficult because the equations are not in a convervative form.

- **We add to the centered scheme a jump penalty term proportional to estimated maximum internal wave speed.**
- **Those terms are added only in prognostic equations related to baroclinic effects: momentum and tracer equations.**
- **The continuity equation does not have such interface terms.**

Elevation can be viewed as the 2d counterpart of vertical velocity

$$\boxed{\frac{\partial w}{\partial z} + \nabla_h \cdot \boldsymbol{u} = 0}$$

By integrating the equation over the vertical

$$w|_\eta - w|_{-h} + \int_{-h(x)}^{\eta(x)} \nabla_h \cdot \boldsymbol{u} \, dz = 0$$

$$\frac{\partial \eta}{\partial t} + \underbrace{\boldsymbol{u}|_\eta \nabla_h \eta - \boldsymbol{u}|_{-h} \nabla_h(-h) + \int_{-h(x)}^{\eta(x)} \nabla_h \cdot \boldsymbol{u} \, dz}_{\nabla_h \cdot \int_{-h}^{\eta} \boldsymbol{u} \, \mathrm{d}z} = 0$$

$$\boxed{\frac{\partial \eta}{\partial t} + \nabla_h \cdot ((h + \eta)\boldsymbol{U}) = 0}$$

Vertical velocity must not be smoothed

The vertical velocity w is a diagnostic field

- **It is only required to obtain the vertical advection.**
- **It acts as the volume term of a divergence integrated by parts.**
- **It is balanced by the interface terms related to internal waves**

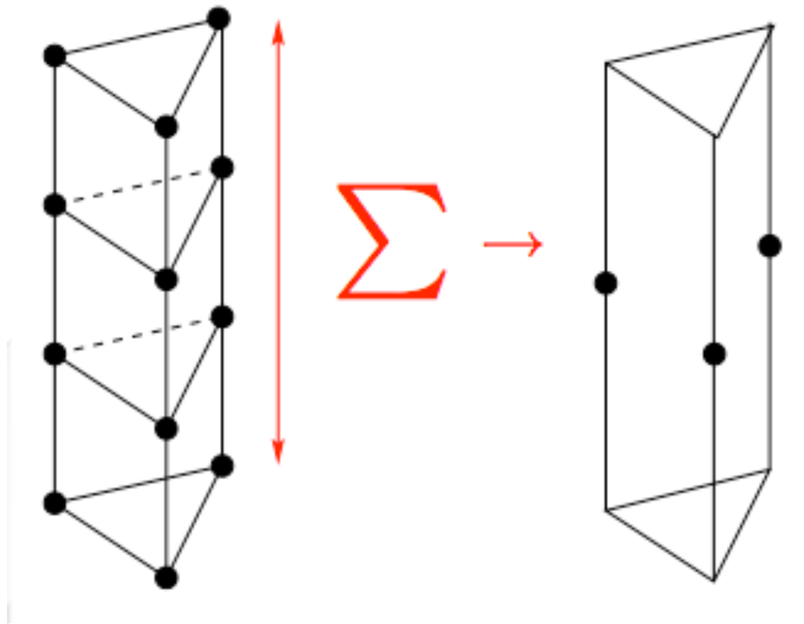
$$< \psi \nabla \cdot \mathbf{F} > = \ll \psi \mathbf{F} \cdot \mathbf{n} \gg - < \nabla \psi \cdot \mathbf{F} >$$

Semi-implicit (IMEX) Runge-Kutta schemes

- **The time step can easily be changed.**
- **High order versions are available.**
- **The linear system for 3d momentum has a block structure corresponding to the columns of dof 's.**

	Implicit	Explicit	Constrained
2d	Coriolis waves	Bottom friction horizontal diffusion advection	
3d	Vertical processes Coriolis	Bottom friction horizontal diffusion advection	waves

Implicit mode splitting procedure



- **The 3D dof 's of a whole vertical line are aggregated into a single 2D dof.**
- **It can be viewed as a restriction on the functional space: the 2D mode corresponds to a single layer.**

$$\int_{-h}^{\eta} \frac{\partial \boldsymbol{u}}{\partial t} + f\boldsymbol{k} \times \boldsymbol{u} + \dots dz = (h+\eta)\left(\frac{\partial \boldsymbol{U}}{\partial t} + f\boldsymbol{k} \times \boldsymbol{U} + \dots\right)$$

Implicit mode splitting procedure

$$f(u_i) = 0$$

$$\sum w_i u_i = U$$

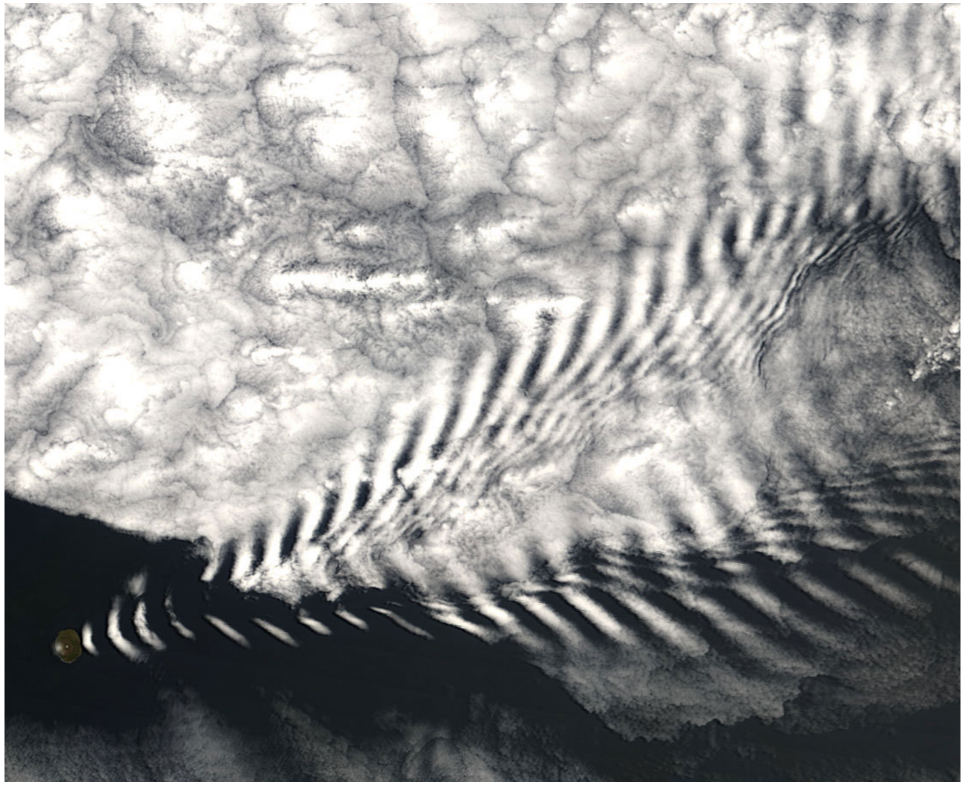
Lagrange multipliers ensure compatibility.

- **Requiring compatibily add too much equations.**
- **Incorporating Lagrange multipliers allows us to weakly impose the compatbility between the 2D and 3D velocity fields**

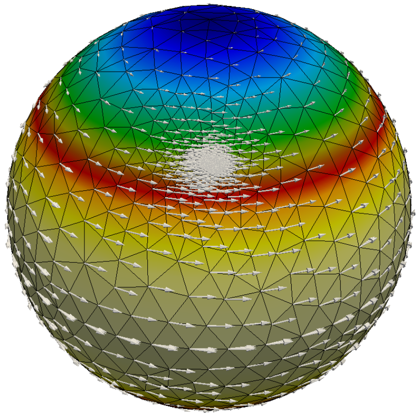
$$f(u_i) + w_i \lambda = 0$$

$$\sum w_i u_i = U$$

Internal waves in the lee of a moderately tall seamount

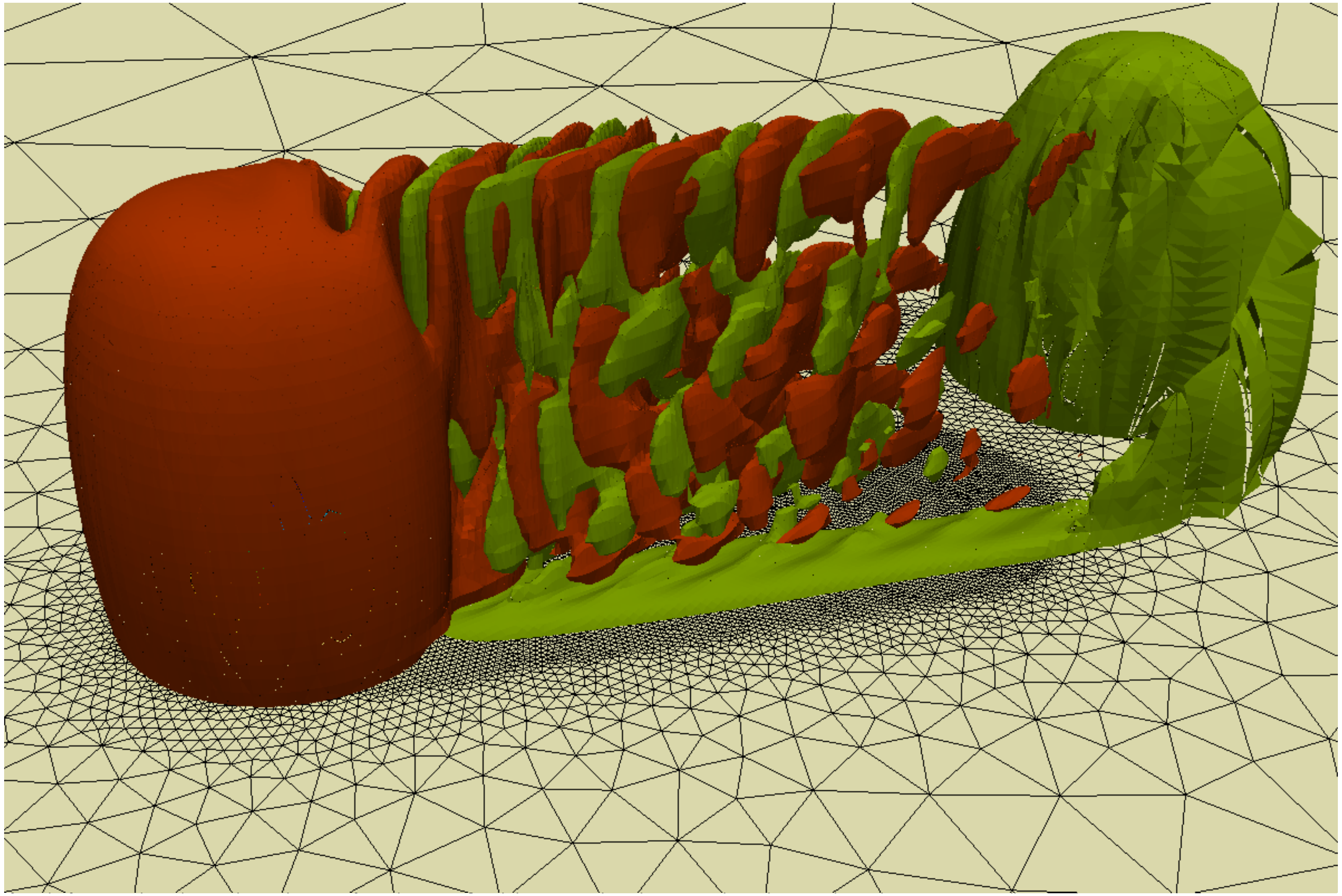


Cloud waves in the lee of Amsterdam island (NASA image from J. Schmalz)

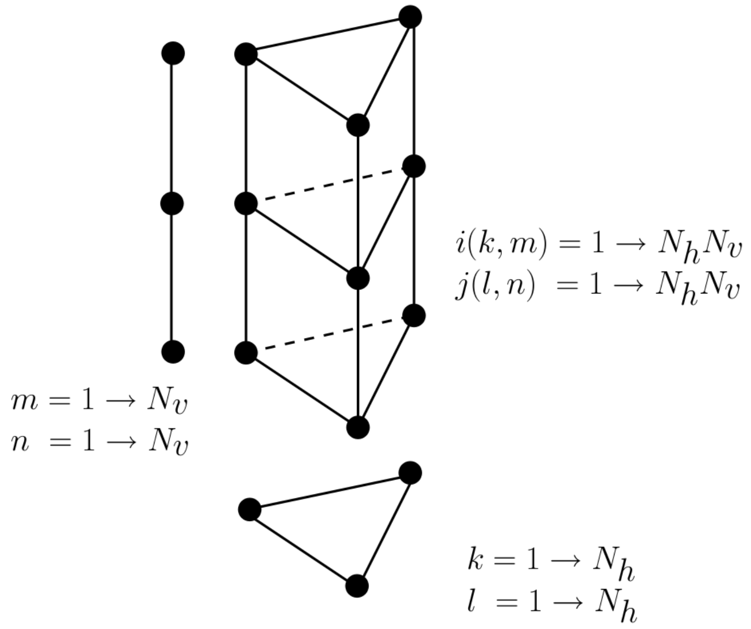


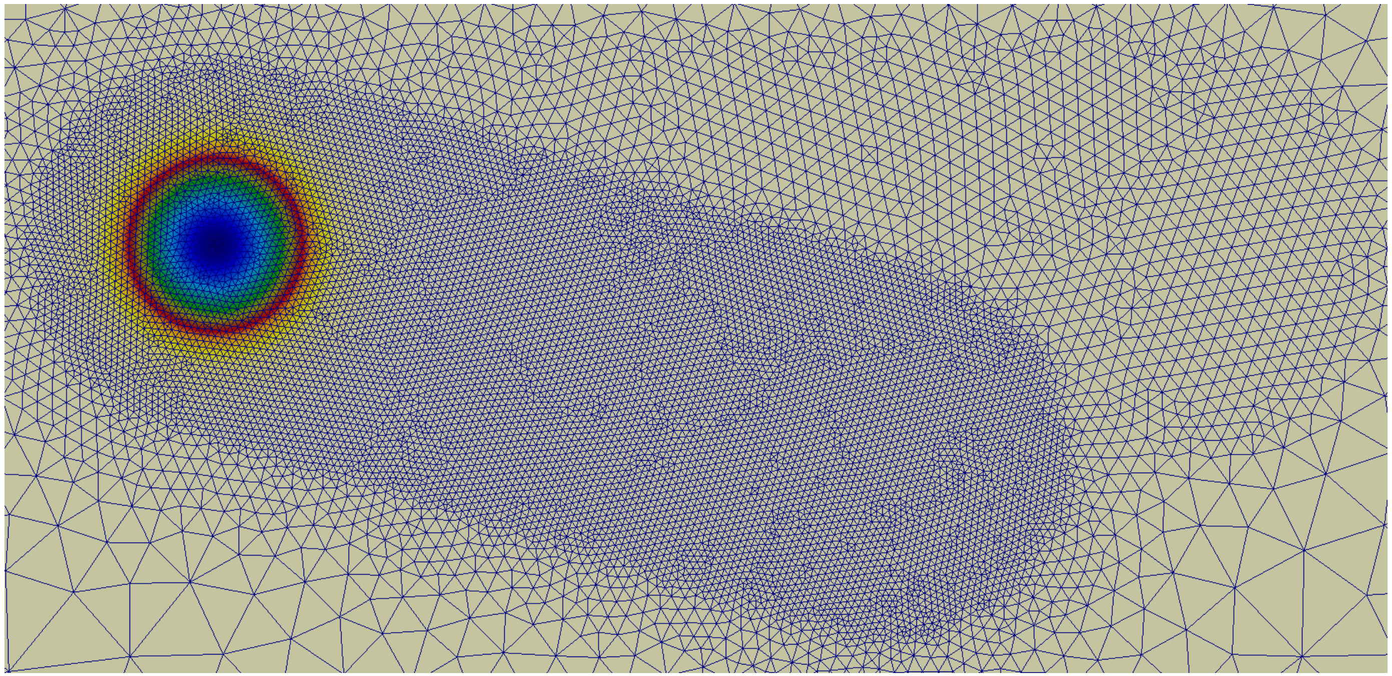
The computation starts with a global zonal geostrophic equilibrium ignoring the seamount as in Williamson testcase 5

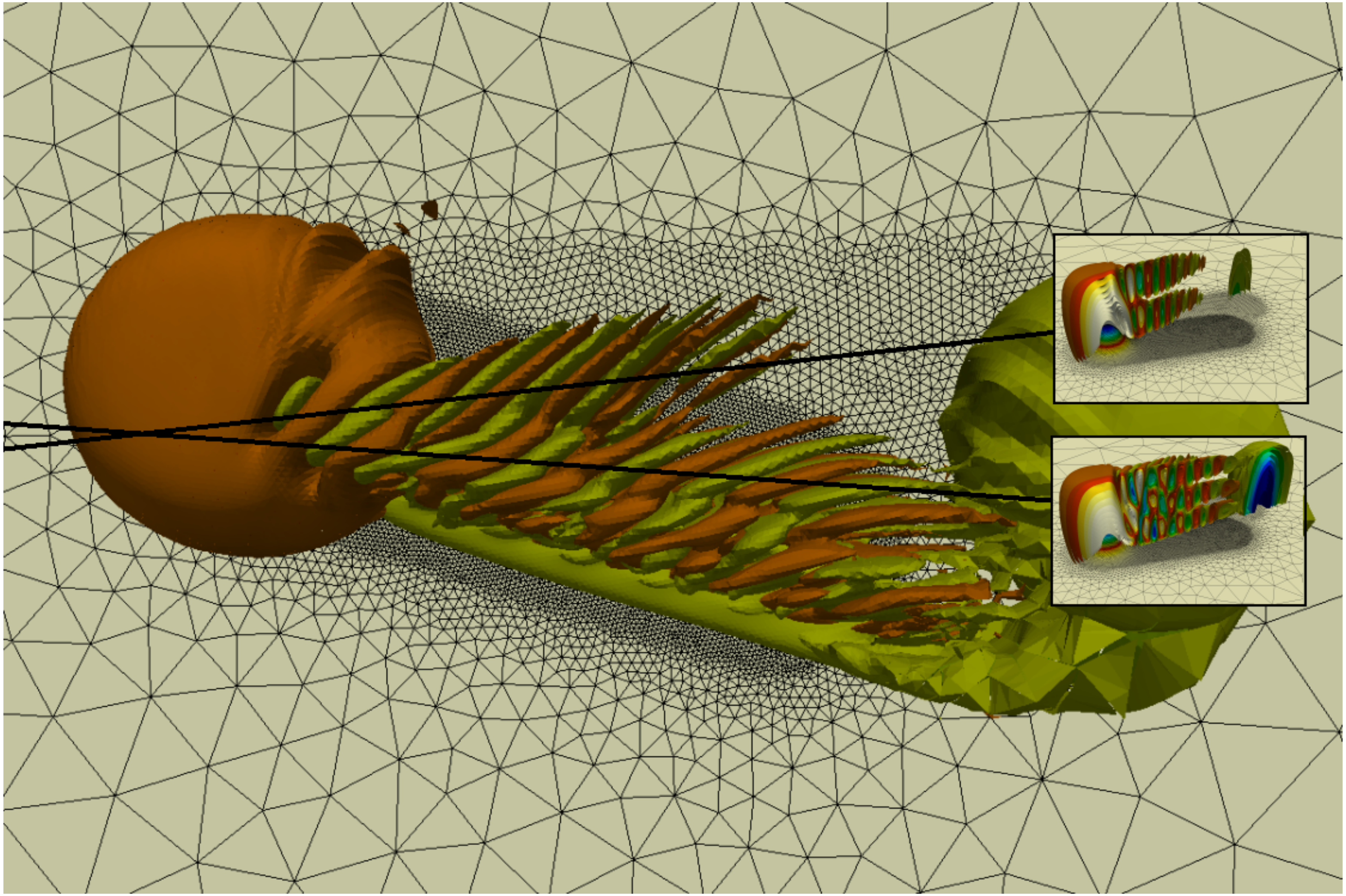
7 days evolution
of density deviation field



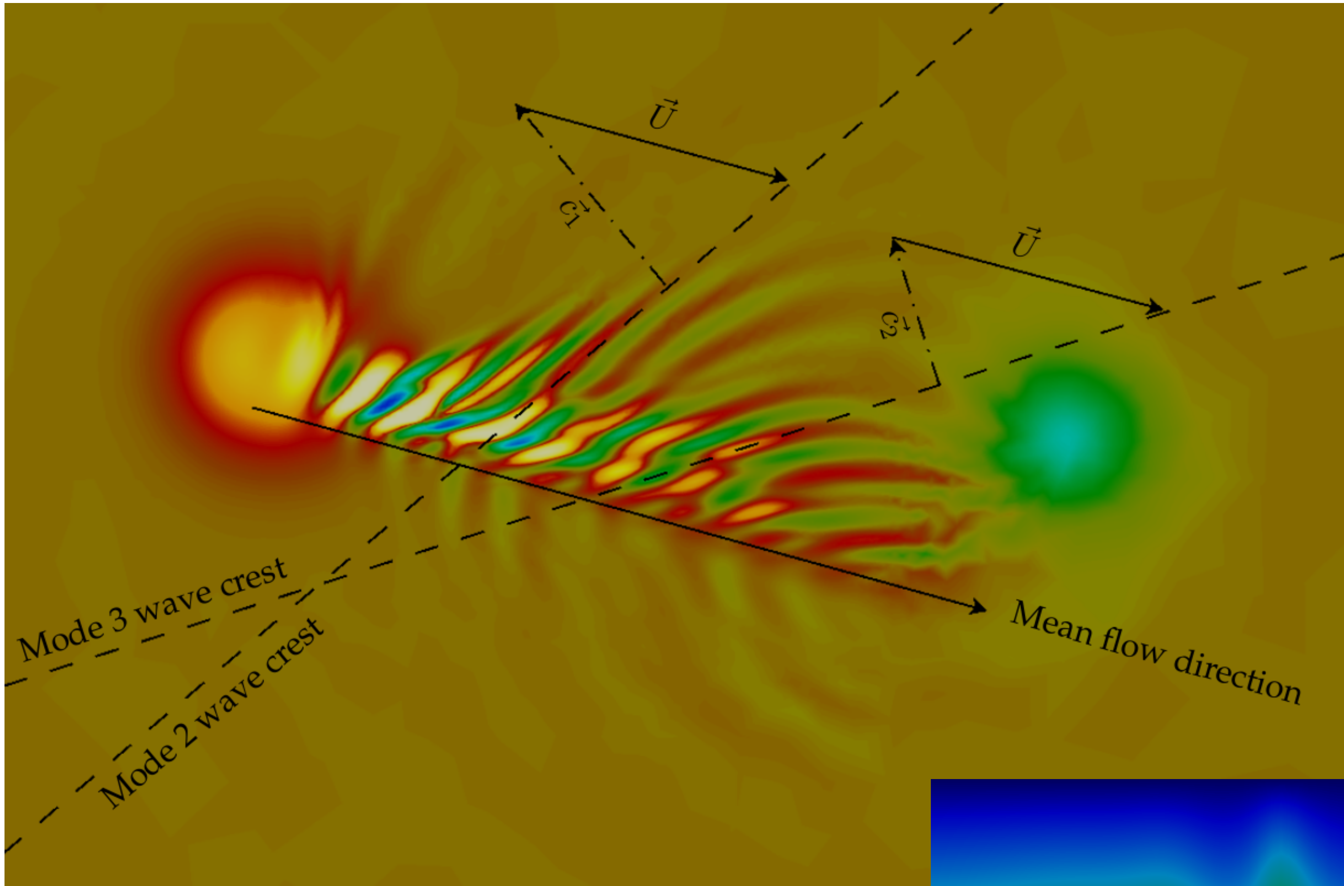
Mesh of 23562 triangles extruded into 25 σ layers



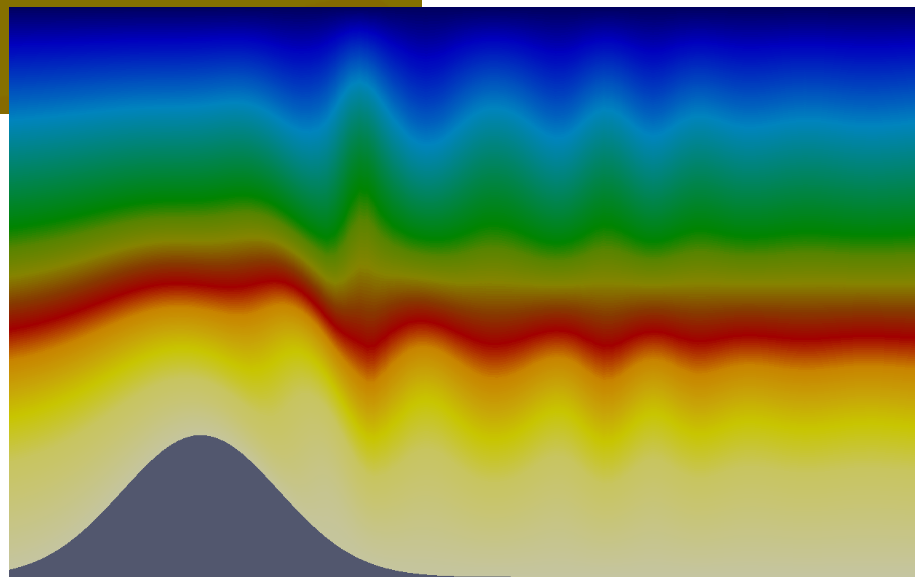




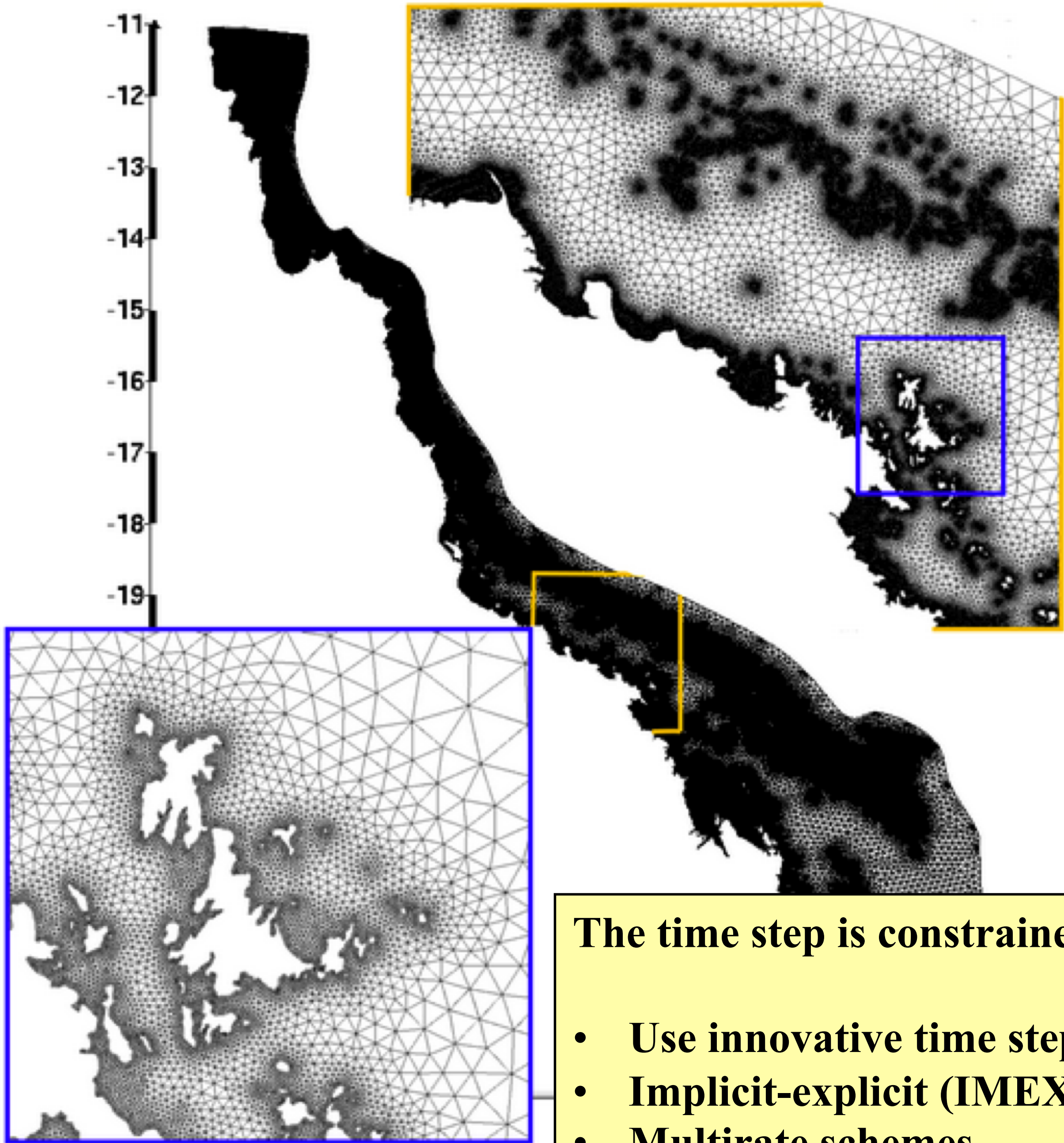
Two well separated modes at day 7



Cut in the density field at day 7



The time stepping issue



- ***890,000 triangles***
- ***Smallest element : 7 m***
- ***Largest element : 3,300 m***
- ***99.9 % > 60m***

The time step is constrained by the smallest element .

- **Use innovative time stepping procedures**
- **Implicit-explicit (IMEX) schemes**
- **Multirate schemes**

Reduce cost by 1000 !
Use high performance computers !

10 Gflops
2 processors

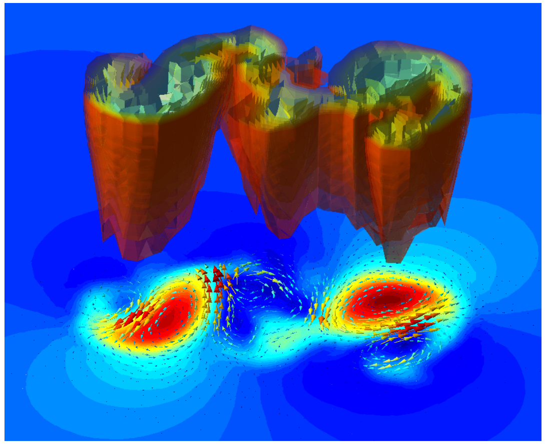
1.759 Pflops
224,162 processors

- **Exploit single precision BLAS/LAPACK for the efficient implementation of the explicit and implicit discontinuous Galerkin methods (Jonathan Lambrechts talk).**
- **Implement new time-integration procedures adapting the time step to the physical processes (Bruno Seny's talk).**
- **Introduce multi-level methods for the implicit linear and non-linear solvers with multigrid methods as a preconditioner for stiff, non-linear and non-positive-definite systems (Samuel Melchior's talk).**

Each route could reduce the computational cost by one order of magnitude.

2D conclusions

- DG is the most compelling solution
- Both implicit and explicit procedures are needed
  - Implicit for long term simulations
  - Explicit allows to use simple limiters
- P_2 on curved meshes would be faster and more accurate with the same number of dofs
  - Efficient limiters for P_2 are not obvious to derive



3D conclusions
The long way to realistic models

- **An accurate DG discretization on the sphere withe a flexible implicit mode splitting has been developed**

- **It should work with limiters for supercritical flows**

- **Mode splitting may not be the best solution**
  Multigrid implicit scheme, aware of the physics, is also attractive