On the Born-Infeld Abelian Higgs cosmic strings with symmetric vacuum

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Abstract

We study a system of elliptic equations from the Abelian Born-Infeld system coupled with the Einstein equations under the boundary condition of the symmetric vacuum(nontopological type). When the total string number satisfies $1 \leq N < \frac{1}{4\pi G}$, where G is the gravitational constant, we construct a family of solutions to the system. The qualitative properties of the solutions are quite different from the solutions with the boundary condition of the broken vacuum symmetry.

1 Introduction

We consider the following problem for (u, η):

$$
(P)\begin{cases}
\Delta u = \dfrac{2e^{\eta}(e^u - 1)}{\sqrt{1 - \frac{1}{b^2}(e^u - 1)^2}} + 4\pi \displaystyle\sum_{j=1}^{m} n_j \delta(z - z_j), \\
\Delta[\eta + 8\pi G(e^u - u)] = -32\pi^2 G \displaystyle\sum_{j=1}^{m} n_j \delta(z - z_j), \\
e^u \to 0 \quad \text{and} \quad e^{\eta} \to 0 \quad \text{as} \quad |z| \to 0,
\end{cases}
$$

where we denoted $z = x_1 + ix_2 \in \mathbb{C} = \mathbb{R}^2$, and $G > 0$ is the gravitational constant. The problem (P) represent physically the equilibrium configuration of the Abelian Born-Infeld cosmic strings. More precisely, it generates

a static Einstein equations coupled with the Abelian Born-Infeld electrodynamic equations under the assumption of translation symmetry in one spatial direction(say in x_3 direction). For more details on the derivation of the above system starting from the Lagrangian of the self-gravitating Abelian Born-Infeld theory by applying the Bogoml'nyi method combined with the Taubes' reduction argument[17], as well as the physical backgrounds, we refer to [23]. Parenthetically, we note that the Born-Infeld field theory, originated in [2], has now become a very hot issue of research in the theoretical physics, mainly due to its natural connection to the superstring theory(e.g. [10, 18], and so many articles in the LANL archive). Also in the physics of cosmology there are many studies on the cosmic strings(See e.g.[9, 11, 19, 20] and references therein). The Abelian Born-Infeld cosmic string model incorporates the Born-Infeld field theory into the area of cosmic string theories.
The boundary condition in (P), in particular, means that we are assuming symmetric vacuum near infinity. The above system with the different boundary condition, namely with the broken vacuum symmetry boundary condition, $e^u \to 1 \quad \text{and} \quad e^\eta \to 0 \quad \text{as} \quad |z| \to 0$, as well as the similar problem on a compact surface have been studied extensively by Y. Yang[22, 23, 21]. Our boundary condition above is different from that of those studies. We remark that this symmetric vacuum boundary condition is not allowed in order to generate finite energy solution of the system in the case of flat(Minkowskian) Abelian Born-Infeld theory[23]. On the other hand, our symmetric vacuum boundary condition resembles the nontopological boundary condition in the Chern-Simons Higgs and the related theories studied in [15, 8, 4, 5, 6, 3](The periodic version of the nontopological solutions are studied in [16, 14, 13]). Our aim in this paper is to construct solutions of (P). Moreover, our construction provides very precise information of the qualitative properties of solutions, including the asymptotes of u, η near infinity. The following is our main Theorem.

Theorem 1.1 *Suppose $\{n_j\}_{j=1}^m \subset \mathbb{N} \cup \{0\}$, and $\{z_j\}_{j=1}^m \in \mathbb{R}^2$ be given. We set $N = \sum_{j=1}^m n_j$. Assume*

$$1 \le N < \frac{1}{4\pi G}, \tag{1.1}$$

and $b > 1$. Then, there exists a constant $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$ there exists a family of solutions to (P), (u_1, u_2). Moreover, the solutions we constructed have the following representations:

$$\begin{aligned} u(z) &= \ln \rho^{I}_{\varepsilon,\delta^*_\varepsilon}(z) + \varepsilon^2 w_1(\varepsilon|z|) + \varepsilon^2 u^*_{1,\varepsilon}(\varepsilon z), & (1.2)\\ \eta(z) &= \ln \rho^{II}_{\varepsilon,\delta^*_\varepsilon}(z) + \varepsilon^2 w_2(\varepsilon|z|) + \varepsilon^2 u^*_{2,\varepsilon}(\varepsilon z) & (1.3) \end{aligned}$$

with

$$\rho^{I}_{\delta,\varepsilon}(z) = \frac{\varepsilon^{2N+2}\prod_{j=1}^{m}|z-z_j|^{2n_j}}{(1+|\varepsilon z+\delta|^2)^{\frac{2}{a}}}, \tag{1.4}$$

$$\rho^{II}_{\delta,\varepsilon}(z) = \frac{4\varepsilon^2}{a\lambda_1(1+|\varepsilon z+\delta|^2)^2}, \qquad \delta = \delta_1 + i\delta_2 \in \mathbb{C}, \tag{1.5}$$

where and hereafter we denote

$$a = 8\pi G. \tag{1.6}$$

In (1.2) and (1.3), the function $\varepsilon \mapsto \delta^*_\varepsilon$ *is a continuous function in a neighborhood of* 0*, and* $|\delta^*_\varepsilon| \to 0$ *as* $\varepsilon \to 0$*. The radial functions* w_1, w_2 *have the following asymptotic behaviors.*

$$w_1(|z|) = -C_1 \ln|z| + O(1), \tag{1.7}$$

$$w_2(|z|) = -C_2 \ln|z| + O(1) \tag{1.8}$$

as $|z| \to \infty$ *with the constants* C_1, C_2 *defined by*

$$C_1 = \frac{8[(a+1)\lambda_1+\lambda_2]N!(1-aN)}{a^2\lambda_1\prod_{k=1-N}^{2}\left(\frac{2}{a}+k\right)} \tag{1.9}$$

$$C_2 = \frac{8[(a+1)\lambda_1+\lambda_2]N!(1-aN)}{a\lambda_1\prod_{k=1-N}^{2}\left(\frac{2}{a}+k\right)} (= aC_1). \tag{1.10}$$

The functions $u^*_{1,\varepsilon}, u^*_{2,\varepsilon}$ *satisfy*

$$\sup_{z\in\mathbb{R}^2} \frac{|u^*_{1,\varepsilon}(\varepsilon z)| + |u^*_{2,\varepsilon}(\varepsilon z)|}{\ln(e+|z|)} \le o(1) \qquad \text{as } \varepsilon \to 0. \tag{1.11}$$

Remark 1.1: From the second equation of (P) we obtain

$$\Delta\left[\eta + 8\pi G(e^u - u) + 8\pi G\sum_{j=1}^{m} n_j \ln|z-z_j|^2\right] = 0.$$

Hence,

$$\eta = 8\pi G(u - e^u) - 8\pi G\sum_{j=1}^{m} n_j \ln|z-z_j|^2 + h(z),$$

where $h(z)$ is a harmonic function. Under the boundary conditions for u and η, we can assume $h(z) \equiv c_0$ for a constant c_0. Hence,

$$e^\eta = c_0\left(\prod_{j=1}^{m}|z-z_j|^{n_j}\right)^{-16\pi G} e^{8\pi G(u-e^u)}.$$

Thus, substituting the asymptotic formula from (1.2)

$$u(z) = \left[2N - \frac{4}{a} - C_1\varepsilon^2 + o(\varepsilon^2)\right]\ln|z| + O(1) \quad \text{as } |z| \to \infty \text{ and } \varepsilon \to 0$$

we obtain

$$e^{\eta(z)} = O\left(|z|^{-4-C_2\varepsilon^2+o(\varepsilon^2)}\right) \quad \text{as } |z| \to \infty \text{ and } \varepsilon \to 0, \tag{1.12}$$

which is consistent with (1.3), and shows extremely weak dependence(via C_1) on the total vortex number N of decay properties near infinities of the conformal factor e^η. This is in contrast with the case of broken vacuum symmetry boundary condition, $e^u \to 1$ and $e^\eta \to 0$ as $|z| \to \infty$[23], where we have the decay of the conformal factor,

$$e^{\eta(z)} = O(|z|^{-16\pi GN}) \qquad \text{as } |z| \to \infty,$$

which shows strong dependence of the decay on the total string number N.

Remark 1.2: We recall the result in Section 10.5 of [23] that the 2 surface $\mathcal{M}_2 = (\mathbb{R}^2, e^\eta\delta_{jk})$ is complete if and only if

$$\int_{\mathbb{R}^2} e^{\frac{1}{2}\eta}dx = \infty.$$

According to the representation formula (1.3), this, in turn, is equivalent to

$$\int_0^\infty (1+r)^{-1-\frac{C_2}{2}\varepsilon^2+o(\varepsilon^2)}dr = \infty.$$

We observe, however, from (1.10) that $C_2 > 0 (< 0)$ if $aN < (>) 1$. Thus we conclude that *the 2 surface $\mathcal{M}_2 = (\mathbb{R}^2, e^\eta\delta_{jk})$ is complete(incomplete) if $aN < (>) 1$*. Since we do not know the sign of $v^*_{2,\varepsilon}$, the case $aN = 1$ is inconclusive for the completeness of $\mathcal{M}_2$.
After this work is completed the author find that there is an interesting paper by F. Lin and Y. Yang[12] on the sigma model coupled with the Born-Infeld system and the gravitation, for which the reduced semilinear elliptic system is different from that of (P), and the corresponding mathematical analysis is completely different from ours in the next section.

2 Proof of the Main Theorem

In (P) the second equation added to the first equation times $a = 8\pi G$ gives

$$\Delta(\eta + ae^u) = \frac{2ae^\eta(e^u - 1)}{\sqrt{1 - \frac{1}{b^2}(e^u-1)^2}}. \tag{2.1}$$

Replacing the second equation of (P) by (2.1), we obtain the following equivalent system to (P).

$$\Delta u = \frac{2e^{\eta}(e^u - 1)}{\sqrt{1 - \frac{1}{b^2}(e^u - 1)^2}} + 4\pi \sum_{j=1}^{m} n_j \delta(z - z_j), \tag{2.2}$$

$$\Delta(\eta + ae^u) = \frac{2ae^{\eta}(e^u - 1)}{\sqrt{1 - \frac{1}{b^2}(e^u - 1)^2}}. \tag{2.3}$$

We consider the following 'principal part' of the system, (2.2)-(2.3).

$$\Delta u_0 = -\lambda_1 e^{\eta_0} + 4\pi \sum_{j=1}^{m} n_j \delta(z - z_j), \tag{2.4}$$

$$\Delta \eta_0 = -a\lambda_1 e^{\eta_0}, \tag{2.5}$$

where λ_1 is defined in (1.6). As a family of solution (2.5) we have

$$\eta_0(z) = \ln \rho^{II}_{\delta,\varepsilon}(z) \tag{2.6}$$

with $\rho^{II}_{\delta,\varepsilon}(z)$ defined in (1.5). In order to solve (2.4) we rewrite it as

$$\Delta \left(au_0 - a \sum_{j=1}^{m} n_j \ln c_0 |z - z_j|^2 \right) = -a\lambda_1 e^{\eta_0}, \tag{2.7}$$

where c_0 is an arbitrary positive constant. Comparing (2.7) with (2.5), we find that

$$au_0 - a \sum_{j=1}^{m} n_j \ln c_0 |z - z_j|^2 = \eta_0 + h(z) \tag{2.8}$$

for a harmonic function, $h(z)$. We choose $h(z) \equiv 0$. Then, substituting η_0 in (2.6) into (2.8), and solving it for u_0, and choosing the constant c_0 in as

$$c_0 = \frac{\varepsilon^{2N}}{8} (a\lambda_1)^{\frac{1}{a}},$$

we find that

$$u_0(z) = \ln \rho^{I}_{\delta,\varepsilon}(z), \tag{2.9}$$

with $\rho^{I}_{\delta,\varepsilon}(z)$ defined in (1.4). We set

$$g^{I}_{\delta,\varepsilon}(z) = \frac{1}{\varepsilon^2} \rho^{I}_{\delta,\varepsilon}\left(\frac{z}{\varepsilon}\right), \qquad g^{II}_{\delta,\varepsilon}(z) = \frac{1}{\varepsilon^2} \rho^{II}_{\delta,\varepsilon}\left(\frac{z}{\varepsilon}\right),$$

and define $\rho_1(r), \rho_2)r)$ by

$$\rho_1(r) = \frac{r^{2N}}{(1 + r^2)^{\frac{2}{a}}} = \lim_{\varepsilon + |\delta| \to 0} g^{I}_{\delta,\varepsilon}(z), \quad \rho_2(r) = \frac{8}{a\lambda_1(1 + r^2)^2} = \lim_{\varepsilon + |\delta| \to 0} g^{II}_{\delta,\varepsilon}(z).$$

We make transforms from (u,η) to (u_1,u_2) as follows

$$\begin{aligned} u(z) &= \ln \rho^{I}_{\delta,\varepsilon}(z) + \varepsilon^2 w_1(\varepsilon z) + \varepsilon^2 u_1(\varepsilon z) \\ \eta(z) &= \ln \rho^{II}_{\delta,\varepsilon}(z) + \varepsilon^2 w_2(\varepsilon z) + \varepsilon^2 u_2(\varepsilon z) \end{aligned} \tag{2.10}$$

where w_1, w_2 are the radial functions, $w_j(z) = w_j(|z|)$, $j = 1,2$ to be determined below. Then, (2.2)-(2.3) can be transformed into the functional equation, $P = (P_1, P_2) = 0$, where

$$P_1(u_1,u_2,\delta,\varepsilon) =$$
$$\Delta u_1 - \frac{2g^{I}_{\delta,\varepsilon}(z) g^{II}_{\delta,\varepsilon}(z) e^{\varepsilon^2(u_1+u_2+w_1+w_2)} - \frac{2g^{II}_{\delta,\varepsilon}(z)}{\varepsilon^2} e^{\varepsilon^2(u_2+w_2)}}{\sqrt{1-\frac{1}{b^2}[\varepsilon^2 g^{I}_{\delta,\varepsilon}(z) e^{\varepsilon^2(u_1+w_1)} - 1]^2}} - \frac{\lambda_1 g^{II}_{\delta,\varepsilon}(z)}{\varepsilon^2} + \Delta w_1,$$

$$P_2(u_1,u_2,\delta,\varepsilon) = \Delta\left[u_2 + a g^{I}_{\delta,\varepsilon}(z) e^{\varepsilon^2(u_1+w_1)}\right]$$
$$- \frac{2a g^{I}_{\delta,\varepsilon}(z) g^{II}_{\delta,\varepsilon}(z) e^{\varepsilon^2(u_1+u_2+w_1+w_2)} - \frac{2a g^{II}_{\delta,\varepsilon}(z)}{\varepsilon^2} e^{\varepsilon^2(u_2+w_2)}}{\sqrt{1-\frac{1}{b^2}[\varepsilon^2 g^{I}_{\delta,\varepsilon}(z) e^{\varepsilon^2(u_1+w_1)} - 1]^2}} - \frac{a\lambda_1 g^{II}_{\delta,\varepsilon}(z)}{\varepsilon^2} + \Delta w_2.$$

Now we introduce the functions spaces used in [4]. Let us fix $\alpha \in (0, \frac{1}{2})$ throughout this paper. Following [1], we introduce the Banach spaces X_α and Y_α as

$$X_\alpha = \{u \in L^2_{loc}(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} (1+|x|^{2+\alpha})|u(x)|^2 dx < \infty\}$$

equipped with the norm $\|u\|^2_{X_\alpha} = \int_{\mathbb{R}^2}(1+|x|^{2+\alpha})|u(x)|^2dx$, and

$$Y_\alpha = \{u \in W^{2,2}_{loc}(\mathbb{R}^2) \mid \|\Delta u\|^2_{X_\alpha} + \left\|\frac{u(x)}{1+|x|^{1+\frac{\alpha}{2}}}\right\|^2_{L^2(\mathbb{R}^2)} < \infty\}$$

equipped with the norm $\|u\|^2_{Y_\alpha} = \|\Delta u\|^2_{X_\alpha} + \left\|\frac{u(x)}{1+|x|^{1+\frac{\alpha}{2}}}\right\|^2_{L^2(\mathbb{R}^2)}$. We recall the following propositions proved in [4].

Proposition 2.1 *Let Y_α be the function space introduced above. Then we have the followings.*

(i) If $v \in Y_\alpha$ is a harmonic function, then $v \equiv$ constant.

(ii) There exists a constant $C_1 > 0$ such that for all $v \in Y_\alpha$

$$|v(x)| \le C_1 \|v\|_{Y_\alpha} \ln(e+|x|), \qquad \forall x \in \mathbb{R}^2.$$

Proposition 2.2 *Let $\alpha \in (0, \frac{1}{2})$, and let us set*

$$L = \Delta + \rho : Y_\alpha \to X_\alpha,$$

where $\rho = \frac{8}{(1+r^2)^2}$. We have

$$KerL = Span\{\varphi_+, \varphi_-, \varphi_0\}, \tag{2.11}$$

where we denoted

$$\varphi_+ = \frac{r}{1+r^2}\cos\theta, \qquad \varphi_- = \frac{r}{1+r^2}\sin\theta, \qquad \varphi_0 = \frac{1-r^2}{1+r^2}.$$

Moreover, we have

$$ImL = \{f \in X_\alpha | \int_{\mathbb{R}^2} f\varphi_\pm = 0\}. \tag{2.12}$$

We can check easily that P is a well defined continuous mapping from $B_{\varepsilon_0} \subset (Y_\alpha)^2 \times \mathbb{C} \times \mathbb{R}_+$ into $(X_\alpha)^2$, where $B_{\varepsilon_0} = \{\|u_1\|_{Y_\alpha} + \|u_2\|_{Y_\alpha} + |\delta| \le \varepsilon < \varepsilon_0\}$ for sufficiently small ε_0. In order to have $g^I_{\delta,\varepsilon}(z) \to O(1)$ as $|z| \to 0$ we impose

$$aN < 2,$$

which is equivalent to (1.1).

We now extend continuously $P(0,0,0,\varepsilon)$ to $\varepsilon = 0$ by imposing the condition that $\lim_{\varepsilon\to 0} P(0,0,0,\varepsilon) = 0$. In order to compute the limit $\lim_{\varepsilon\to 0} P(0,0,0,\varepsilon)$ we note the fact

$$\begin{aligned} \frac{2}{\sqrt{1-\frac{1}{b^2}(x-1)^2}} - \lambda_1 &= \frac{2}{\sqrt{1-\frac{1}{b^2}(x-1)^2}} - \frac{2}{\sqrt{1-\frac{1}{b^2}}} \\ &= -\lambda_2 x + O(x^2) \end{aligned} \tag{2.13}$$

as $x \to 0$, where λ_2 is defined in (1.6). Using this fact we obtain

$$\lim_{\varepsilon\to 0} P_1(0,0,0,\varepsilon) = -(\lambda_1+\lambda_2)\rho_1\rho_2 + \lambda_1\rho_2 w_2 + \Delta w_1,$$

and

$$\lim_{\varepsilon\to 0} P_2(0,0,0,\varepsilon) = a\Delta\rho_1 - a(\lambda_1+\lambda_2)\rho_1\rho_2 + a\lambda_1\rho_2 w_2 + \Delta w_2.$$

Hence, the condition $\lim_{\varepsilon\to 0} P(0,0,0,\varepsilon) = 0$ implies the following linear system for w_1, w_2.

$$\Delta w_1 + \lambda_1\rho_2 w_2 - (\lambda_1+\lambda_2)\rho_1\rho_2 = 0, \tag{2.14}$$

$$\Delta w_2 + a\lambda_1\rho_2 w_2 - a(\lambda_1+\lambda_2)\rho_1\rho_2 + a\Delta\rho_1 = 0. \tag{2.15}$$

We establish the following lemma about asymptotic behaviors of the solutions $w_1, w_2 \in Y_\alpha$.

Lemma 2.1 *Let C_1, C_2 be the numbers introduced in (1.9), (1.10) respectively. Then, there exist radial solutions $w_1(|z|)), w_2(|z|)$ of (2.14)-(2.15), belong to Y_α, and satisfy the asymptotic formula in (1.7) and (1.8) respectively.*

Proof: From (2.14)$\times a-$(2.15) we obtain

$$\Delta(aw_1 - w_2 - a\rho_1) = 0.$$

We seek w_1, w_2 with $aw_1 - w_2 - a\rho_1 \in Y_\alpha$. Then, it follows that $aw_1 - w_2 - a\rho_1$ =constant by ([1], Proposition 1.1). We choose this constant$=0$. Then, $\rho_2 w_2 = a\rho_2 w_1 - a\rho_1\rho_2$. Substituting this into (2.14) we obtain the following reduced system for w_1, w_2.

$$\Delta w_1 + a\lambda_1\rho_2 w_1 = [(a+1)\lambda_1 + \lambda_2]\rho_1\rho_2, \tag{2.16}$$

$$w_2 = aw_1 - a\rho_1. \tag{2.17}$$

Let us set $f(r) = [(a+1)\lambda_1 + \lambda_2]\rho_1\rho_2$.

Then, it is found in [1, 4] that the ordinary differential equation(with respect to r), (2.16) has a solution $w_1(r) \in Y_\alpha$ given by

$$w_1(r) = \varphi_0(r)\left\{\int_0^r \frac{\phi_f(s) - \phi_f(1)}{(1-s)^2}ds + \frac{\phi_f(1)r}{1-r}\right\} \tag{2.18}$$

with

$$\phi_f(r) := \left(\frac{1+r^2}{1-r^2}\right)^2 \frac{(1-r)^2}{r}\int_0^r \varphi_0(t)tf(t)dt,$$

where $\phi_f(1)$ and $w_1(1)$ are defined as limits of $\phi_f(r)$ and $w_1(r)$ as $r \to 1$. From the formula (2.18) we find that

$$w_1(r) = \varphi_0(r)\int_2^r \left(\frac{1+s^2}{1-s^2}\right)^2 \frac{I(s)}{s}ds + (\text{bounded function of } r) \tag{2.19}$$

as $r \to \infty$, where

$$I(s) = [(a+1)\lambda_1 + \lambda_2]\int_0^s \varphi_0(t)t\rho_1(t)\rho_2(t)dt.$$

Since $\varphi_0(r) \to -1$ as $r \to \infty$, (1.7) follows if we show

$$\begin{aligned} I &= I(\infty) = [(a+1)\lambda_1 + \lambda_2]\int_0^\infty \varphi_0(r)r\rho_1(r)\rho_2(r)dr \\ &= \frac{8[(a+1)\lambda_1 + \lambda_2]N!(1-aN)}{a^2\lambda_1\prod_{k=1-N}^{2}\left(\frac{2}{a}+k\right)} = C_1. \end{aligned}$$

Indeed, substituting $r^2 = t$ in the integrand of I, we have

$$
\begin{aligned}
I &= \frac{4[(a+1)\lambda_1+\lambda_2]}{a\lambda_1}\int_0^\infty \frac{(1-t)t^N}{(1+t)^{3+\frac{2}{a}}}dt \\
&= \frac{4[(a+1)\lambda_1+\lambda_2]}{a\lambda_1}\left[\int_0^\infty \frac{t^N}{(1+t)^{3+\frac{2}{a}}}dt - \int_0^\infty \frac{t^{N+1}}{(1+t)^{3+\frac{2}{a}}}dt\right] \\
&= \frac{4[(a+1)\lambda_1+\lambda_2]}{a\lambda_1}\left[\frac{N!}{\prod_{k=2-N}^{2}\left(\frac{2}{a}+k\right)} - \frac{(N+1)!}{\prod_{k=1-N}^{2}\left(\frac{2}{a}+k\right)}\right] \\
&= \frac{4[(a+1)\lambda_1+\lambda_2]N!}{a\lambda_1\prod_{k=1-N}^{2}\left(\frac{2}{a}+k\right)}\left[\frac{2}{a}+1-N-(N+1)\right] \\
&= \frac{8[(a+1)\lambda_1+\lambda_2]N!(1-aN)}{a^2\lambda_1\prod_{k=1-N}^{2}\left(\frac{2}{a}+k\right)} = C_1. \qquad (2.20)
\end{aligned}
$$

The formula (1.8), on the other hand, follows from (2.18), observing $C_2 = aC_1$, since $\rho_1(r) = O(1)$ as $r \to \infty$. This completes the proof of Lemma 2.1. □

Now we compute the linearized operator of P. By direct computation we have

$$
\lim_{\varepsilon\to 0}\left.\frac{\partial g^I_{\delta,\varepsilon}(z)}{\partial\delta_1}\right|_{\delta=0} = -\frac{4}{a}\rho_1\varphi_+, \quad \lim_{\varepsilon\to 0}\left.\frac{\partial g^I_{\delta,\varepsilon}(z)}{\partial\delta_2}\right|_{\delta=0} = -\frac{4}{a}\rho_1\varphi_-,
$$

$$
\lim_{\varepsilon\to 0}\left.\frac{\partial g^{II}_{\delta,\varepsilon}(z)}{\partial\delta_1}\right|_{\delta=0} = -4\rho_2\varphi_+, \quad \lim_{\varepsilon\to 0}\left.\frac{\partial g^{II}_{\delta,\varepsilon}(z)}{\partial\delta_2}\right|_{\delta=0} = -4\rho_2\varphi_-.
$$

Let us set $P'_{u_1,u_2,\delta}(0,0,0,0) = \mathcal{A}$. Then, using the above preliminary computations and 2.13, we obtain

$$
\begin{aligned}
\mathcal{A}_1[v_1,v_2,\beta] = {} & \Delta v_1 + \lambda_1\rho_2 v_2 \\
& +4\left[(1+\frac{1}{a})(\lambda_1+\lambda_2)\rho_1\rho_2 - \lambda_1\rho_2 w_2\right](\varphi_+\beta_1+\varphi_-\beta_2),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{A}_2[v_1,v_2,\beta] = {} & \Delta v_2 + a\lambda_1\rho_2 v_2 \\
& +4\left[(1+a)(\lambda_1+\lambda_2)\rho_1\rho_2 - a\lambda_1\rho_2 w_2\right](\varphi_+\beta_1+\varphi_-\beta_2) \\
& -4\Delta[\rho_1(\varphi_+\beta_1+\varphi_-\beta_2)].
\end{aligned}
$$

For the linearized operator $\mathcal{A}[\cdot]$ we will establish the following key lemma.

Lemma 2.2 *The operator $\mathcal{A} : Y_\alpha^2 \times \mathbb{R}^2 \to X_\alpha^2$ defined above is onto. Moreover, the kernel of $\mathcal{A}$ is given by*

$$Ker\mathcal{A} = Span\{(1,0); (\frac{\varphi_\pm}{a}, \varphi_\pm); (\frac{\varphi_0}{a}, \varphi_0)\} \times \{(0,0)\}. \tag{2.21}$$

Thus, if we decompose $Y_\alpha^2 \times \mathbb{R}^2 = U_\alpha \oplus Ker\mathcal{A}$, where we set $U_\alpha = (Ker\mathcal{A})^\perp$, then $\mathcal{A}$ is an isomorphism from U_α onto X_α^2.

In order to prove the above lemma we need the following:

Proposition 2.3 *Let $w_2 \in Y_\alpha$ solve (2.14)-(2.15), then*

$$I_\pm = \int_{\mathbb{R}^2} [(1+a)(\lambda_1 + \lambda_2)\rho_1\rho_2 - a\lambda_1\rho_2 w_2]\, \varphi_+^2 dx \\ - \int_{\mathbb{R}^2} \Delta(\rho_1\varphi_+)\varphi_\pm dx \neq 0. \tag{2.22}$$

Proof: Integrating by part, we obtain

$$\begin{aligned} I_\pm &= \int_{\mathbb{R}^2} \left\{[(a+1)(\lambda_1+\lambda_2)\rho_1\rho_2 - a\lambda_1 w_2\rho_2]\varphi_\pm^2 - \rho_1\varphi_\pm\Delta\varphi_\pm\right\} dx \\ &= \int_{\mathbb{R}^2} [((2a+1)\lambda_1 + (a+1)\lambda_2)\rho_1\rho_2 - a\lambda_1 w_2\rho_2]\varphi_\pm^2 dx, \end{aligned} \tag{2.23}$$

where we used (2.21 $\Delta\varphi_\pm = -a\lambda_1\rho_2\varphi_\pm$. (Note that $L = \Delta + a\lambda_1\rho_2$.) Below we list useful formulas, which can be checked by elementary computations.

$$\varphi_\pm^2\rho_2 = \frac{1}{16}L_2\rho_2 \left\{ \begin{array}{c} \cos^2\theta \\ \sin^2\theta \end{array} \right\}, \tag{2.24}$$

$$\varphi_\pm^2 = \frac{a\lambda_1}{8} r^2\rho_2 \left\{ \begin{array}{c} \cos^2\theta \\ \sin^2\theta \end{array} \right\}, \tag{2.25}$$

$$\Delta\rho_2 = a\lambda_1(2r^2-1)\rho_2^2, \tag{2.26}$$

Also, from (2.15), we have

$$Lw_2 = a(\lambda_1+\lambda_2)\rho_1\rho_2 - a\Delta\rho_1. \tag{2.27}$$

Using (2.24)-(2.27), and integrating by parts, we transform the integral successively as follows.

$$\begin{aligned}
I_\pm &= \int_{\mathbb{R}^2}[(2a+1)\lambda_1+(a+1)\lambda_2]\rho_1\rho_2\varphi_\pm^2dx-\frac{a}{16}\int_0^\infty\int_0^{2\pi}w_2(L_2\rho_2)\left\{\begin{array}{c}\cos^2\theta\\ \sin^2\theta\end{array}\right\}d\theta rdr\\
&= \int_0^\infty\int_0^{2\pi}\left\{\frac{a\lambda_1[(2a+1)\lambda_1+(a+1)\lambda_2]}{8}r^2\rho_1\rho_2^2-\frac{a\lambda_1}{16}(Lw_2)\rho_2\right\}\left\{\begin{array}{c}\cos^2\theta\\ \sin^2\theta\end{array}\right\}d\theta rdr\\
&= \pi\int_0^\infty\left\{\frac{a\lambda_1[(2a+1)\lambda_1+(a+1)\lambda_2]}{8}r^2\rho_1\rho_2^2-\frac{a\lambda_1}{16}[a(\lambda_1+\lambda_2)\rho_1\rho_2-a\Delta\rho_1]\rho_2\right\}rdr\\
&= \pi\int_0^\infty\left\{\frac{a\lambda_1[(2a+1)\lambda_1+(a+1)\lambda_2]}{8}r^2\rho_1\rho_2^2-\frac{a^2\lambda_1(\lambda_1+\lambda_2)}{16}\rho_1\rho_2^2+\frac{a^2\lambda_1}{16}\rho_1\Delta\rho_2\right\}rdr\\
&= \pi\int_0^\infty\left\{\frac{a\lambda_1[(2a+1)\lambda_1+(a+1)\lambda_2]}{8}r^2\rho_1\rho_2^2-\frac{a^2\lambda_1(\lambda_1+\lambda_2)}{16}\rho_1\rho_2^2\right.\\
&\qquad\qquad\left.+\frac{a^3\lambda_1^2}{16}(2r^2-1)\rho_1\rho_2^2\right\}rdr\\
&= \frac{\lambda_1a[\lambda_1(a+1)+\lambda_2]\pi}{16}\int_0^\infty[2(a+1)r^2-a]\rho_1\rho_2^2rdr\\
&= \frac{4[\lambda_1(a+1)+\lambda_2]\pi}{a\lambda_1}\int_0^\infty\frac{[2(a+1)r^2-a]r^{2N+1}}{(1+r^2)^{\frac{2}{a}+4}}dr\quad(\text{Setting } r^2=t)\\
&= \frac{2[\lambda_1(a+1)+\lambda_2]\pi}{a\lambda_1}\int_0^\infty\left[\frac{2(a+1)t^{N+1}}{(1+t)^{\frac{2}{a}+4}}-\frac{at^N}{(1+t)^{\frac{2}{a}+4}}\right]dt
\end{aligned}$$

$$\begin{aligned}
&= \frac{2[\lambda_1(a+1)+\lambda_2]\pi}{a\lambda_1}\left[\frac{2(a+1)(N+1)!}{\prod_{k=2-N}^3\left(\frac{2}{a}+k\right)}-\frac{aN!}{\prod_{k=3-N}^3\left(\frac{2}{a}+k\right)}\right]\\
&= \frac{2[\lambda_1(a+1)+\lambda_2]\pi N!}{a\lambda_1\prod_{k=2-N}^3\left(\frac{2}{a}+k\right)}\left[2(a+1)(N+1)-a\left(\frac{2}{a}+2-N\right)\right]\\
&= \frac{2[\lambda_1(a+1)+\lambda_2]\pi(3a+2)N\cdot N!}{a\lambda_1\prod_{k=2-N}^3\left(\frac{2}{a}+k\right)}>0. \qquad (2.28)
\end{aligned}$$

This completes the proof of Proposition 2.3. □.

We are now ready to prove Lemma 3.1.

Proof of Lemma 2.2: Given $(f_1,f_2)\in X_\alpha^2$, we want first to show that there exists $(v,\eta)\in Y_\alpha^2$, $\beta=(\beta_1,\beta_2)\in\mathbb{R}^2$ such that

$$\mathcal{A}(v_1,v_2,\beta)=(f_1,f_2), \qquad (2.29)$$

which can be rewritten as

$$\begin{aligned}
&\Delta v_1+\lambda_1\rho_2v_2\\
&\qquad+4\left[(1+\frac{1}{a})(\lambda_1+\lambda_2)\rho_1\rho_2-\lambda_1\rho_2w_2\right](\varphi_+\beta_1+\varphi_-\beta_2)=f_1, \quad (2.30)
\end{aligned}$$

and

$$\Delta v_2 + a\lambda_1\rho_2 v_2 + 4\left[(1+a)(\lambda_1+\lambda_2)\rho_1\rho_2 - a\lambda_1\rho_2 w_2\right](\varphi_+\beta_1 + \varphi_-\beta_2)$$
$$-4\Delta[\rho_1(\varphi_+\beta_1 + \varphi_-\beta_2)] = f_2. \quad (2.31)$$

Let us set

$$\beta_1 = \frac{1}{4I_+}\int_{\mathbb{R}^2} f_2\varphi_+ dx, \qquad \beta_2 = \frac{1}{4I_-}\int_{\mathbb{R}^2} f_2\varphi_- dx, \qquad (2.32)$$

where $I_\pm > 0$ is defined in (2.22). We introduce $\tilde{f}$ by

$$\tilde{f}_2 = f_2 - \beta_1\varphi_+ - \beta_2\varphi_-. \qquad (2.33)$$

Using the fact $\int_0^{2\pi}\varphi_+\varphi_- d\theta = 0$, we find easily

$$\int_{\mathbb{R}^2} \tilde{f}_2\varphi_\pm dx = 0. \qquad (2.34)$$

Hence, by (2.12) there exists $v_2 \in Y_\alpha$ such that $\Delta v_2 + a\lambda_1\rho_2 v_2 = \tilde{f}_2$. Thus we have found $(v_2, \beta_1, \beta_2) \in Y_\alpha \times \mathbb{R}^2$ satisfying (2.31). Given such (v_2, β_1, β_2), in order to construct $v_1 \in Y_\alpha$ satisfying (2.30), we consider the following equation, obtained by (2.30) $\times\, a -$ (2.31),

$$\Delta(av_1 - v_2 + 4\rho_1\varphi_+\beta_1 + 4\rho_1\varphi_-\beta_2) = af_1 - f_2. \qquad (2.35)$$

For any harmonic function $h_1(x)$ the function

$$v_1(x) = \frac{1}{2\pi a}\int_{\mathbb{R}^2} \ln(|x-y|)(af_1(y) - f_2(y))dy$$
$$+\frac{1}{a}(v_2 - 4\rho_1\varphi_+\beta_1 - 4\rho_1\varphi_-\beta_2) + h_1(x) \qquad (2.36)$$

satisfies (2.30). The requirement $v_1 \in Y_\alpha$ implies $h_1(x)(x) \equiv$Constant thanks to Proposition 2.1(i).. We have just finished the proof that $\mathcal{A} : Y_\alpha^2 \times \mathbb{R}^2 \to X_\alpha^2$ is onto.
Now it is easy to check that the restricted operator(denoted by the same symbol), $\mathcal{A} : U_\alpha \to X_\alpha^2$ is one to one. We omit the details.

This completes the proof of the lemma.□

We are now ready to prove our main theorem.
Proof of Theorem 1.1: Lemma 2.2 shows that $P'_{(v,\xi,\beta)}(0,0,0,0) : U_\alpha \to X_\alpha \times X_\alpha$ is an isomorphism for $\alpha \in (0, \frac{1}{2})$. Then, the standard implicit function theorem(See e.g. [24]), applied to the functional $P : U_\alpha \times (-\varepsilon_0, \varepsilon_0) \to X_\alpha \times X_\alpha$, implies that there exists a constant $\varepsilon_1 \in (0, \varepsilon_0)$ and a continuous function $\varepsilon \mapsto \psi_\varepsilon^* := (v_{1,\varepsilon}^*, v_{2,\varepsilon}^*, \delta_\varepsilon^*)$ from $(0, \varepsilon_1)$ into a neighborhood of 0 in U_α such that

$$P(u_{1,\varepsilon}^*, u_{2,\varepsilon}^*, \delta_\varepsilon^*, \varepsilon) = (0,0), \quad \text{for all } \varepsilon \in (0, \varepsilon_1).$$

This completes the proof of Theorem 1.1. The representation of solutions u_1, u_2, and the explicit form of $\rho^I_{\varepsilon,\delta^*_\varepsilon}(z)$, $\rho^{II}_{\varepsilon,\delta^*_\varepsilon}(z)$, , together with the asymptotic behaviors of w_1, w_2 described in Lemma 2.1, the fact that $u^*_{1,\varepsilon}, u^*_{2,\varepsilon} \in Y_\alpha$, combined with Proposition 2.1, implies that the solutions satisfy the boundary condition in (P). Now, from Proposition 2.1 we obtain that for each $j = 1, 2$,

$$|u^*_{j,\varepsilon}(x)| \leq C\|u^*_{j,\varepsilon}\|_{Y_\alpha}(\ln^+|x| + 1) \leq C\|\psi_\varepsilon\|_{U_\alpha}(\ln^+|x| + 1). \tag{2.37}$$

This implies then

$$|u^*_{j,\varepsilon}(\varepsilon x)| \leq C\|\psi_\varepsilon\|_{U_\alpha}(\ln^+|\varepsilon x| + 1) \leq C\|\psi_\varepsilon\|_{U_\alpha}(\ln^+|x| + 1).$$

From the continuity of the function $\varepsilon \mapsto \psi_\varepsilon$ from $(0, \varepsilon_0)$ into U_α and the fact $\psi^*_0 = 0$ we have

$$\|\psi_\varepsilon\|_{U_\alpha} \to 0 \qquad \text{as } \varepsilon \to 0. \tag{2.38}$$

The proof of (1.11) follows from (2.37) combined with (2.38). This completes the proof of Theorem 1.1□

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