On the Doi Model for the suspensions of rod-like molecules in compressible fluids

Hantaek Bae* and Konstantina Trivisa†

Key words: Doi model, suspensions of rod-like molecules, fluid-particle interaction model, compressible Navier-Stokes equations, Fokker-Planck-type equation.

Abstract

Polymeric fluids arise in many practical applications in biotechnology, medicine, chemistry, industrial processes and atmospheric sciences. In this article, the Doi model for the suspensions of rod-like molecules in a compressible fluid is investigated. The model under consideration couples a Fokker-Planck-type equation on the sphere for the orientation distribution of the rods to the Navier-Stokes equations for compressible fluids, which are now enhanced by additional stresses reflecting the orientation of the rods on the molecular level. The coupled problem is 5-dimensional (three-dimensions in physical space and two degrees of freedom on the sphere) and it describes the interaction between the orientation of rod-like polymer molecules on the microscopic scale and the macroscopic properties of the fluid in which these molecules are contained. Prescribing arbitrarily the initial density of the fluid, the initial velocity, and the initial orientation distribution in suitable spaces we establish the global-in-time existence of a weak solution to our model defined on a bounded domain in the three dimensional space. The proof relies on the construction of a sequence of approximate problems by introducing appropriate regularization and the establishment of compactness.

Contents

*Center for Scientific Computation and Mathematical Modeling, University of Maryland, College Park, MD 20742. hbae@cscamm.umd.edu.

†Department of Mathematics & Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742. trivisa@math.umd.edu

1 Introduction

The evolution of rod-like molecules in both compressible and incompressible fluids is of great scientific interest with a variety of applications in science and engineering. The present article deals with the Doi model for the suspensions of rod-like molecules in a dilute regime. The model under consideration couples a Fokker-Planck-type equation on the sphere for the orientation distribution of the rods to the Navier-Stokes equations for compressible fluids, which are now enhanced by additional stresses reflecting the orientation of the rods on the molecular level. The coupled problem is 5-dimensional (three-dimensions in physical space and two degrees of freedom on the sphere) and it describes the interaction between the orientation of rod-like polymer molecules on the microscopic scale and the macroscopic properties of the fluid in which these molecules are contained. The macroscopic flow leads to a change of the orientation and, in the case of flexible particles, to a change in shape of the suspended microstructure. This process, in turn yields the production of a fluid stress. In this paper, we consider the Doi model for a compressible fluid in a bounded domain. The derivation of the system under consideration is described below.

A smooth motion of a body in continuum mechanics is described by a family of one-to-one mappings

$$X(t, \cdot) : \Omega \to \Omega, \quad t \in I.$$

The curve $X(t, x)$ represents the trajectory of a particle occupying at time t a spatial position x and this curve is completely determined by a velocity field $u : I \times \Omega \to \mathbb{R}^3$ through

$$\frac{\partial}{\partial t} X(t, x) = u(t, X(t, x)), \quad X(0, a) = a.$$

Then, the conservation of mass can be formulated as follows:

$$\frac{d}{dt} \int_{X(t,B)} \rho(t, x) dx = 0, \quad B \subset \Omega,$$

where ρ is a nonnegative function that corresponds to the density of the fluid. This equation is equivalent to

$$\frac{d}{dt}\int_B \rho(t,x)dx + \int_{\partial B} \rho(t,x)[u(t,x)\cdot \hat{n}]dS = 0,$$

where $\hat{n}$ is the unit outer normal vector on $\partial\Omega$. If ρ is smooth, one can use Green's theorem to deduce the following continuity equation:

$$\rho_t + \nabla\cdot(u\rho) = 0. \tag{1.1}$$

We next obtain equation of motion by applying Newton's second law of motion as follows:

$$\frac{d}{dt}\int_{X(t,B)} \rho(t,x)u(t,x)dx = \int_{X(t,B)} \rho(t,x)F(t,x)dx + \int_{\partial X(t,B)} \mathbf{t}(t,x,\hat{n})dS.$$

Then, we have

$$\frac{d}{dt}\int_B \rho(t,x)u(u,x)dx + \int_{\partial B}(\rho u)(t,x)[u(t,x)\cdot\hat{n}]dS = \int_B \rho(t,x)F(t,x)dx + \int_{\partial B}\mathbf{t}(t,x,\hat{n})dS. \tag{1.2}$$

For the simplicity, we take $F = 0$. The stress principle in continuum mechanics can be addressed through the fundamental laws of Cauchy: there is a symmetric stress tensor $\mathbb{T}(t,x)$ such that

$$\mathbf{t}(t,x,\hat{n}) = \mathbb{T}(t,x)\hat{n}.$$

Therefore, (1.2) becomes

$$\frac{d}{dt}\int_B \rho(t,x)u(u,x)dx + \int_{\partial B}(\rho u)(t,x)[u(t,x)\cdot\hat{n}]dS = \int_{\partial B}\mathbb{T}(t,x)\hat{n}dS. \tag{1.3}$$

By applying Green's lemma to (1.3), we finally have

$$(\rho u)_t + \nabla\cdot(\rho u\otimes u) = \nabla\cdot\mathbb{T}, \quad (\nabla\cdot\mathbb{T})_i = \sum_{j=1}^{3}\frac{\partial \mathbb{T}_{ij}}{\partial x_j}. \tag{1.4}$$

The stress tensor $\mathbb{T}$ of a general fluid obeys Stokes' law:

$$\mathbb{T} = \mathbb{S} - p\mathbb{I}_{3\times 3},$$

where p is the pressure and $\mathbb{S}$ is the stress tensor. Let us determine $\mathbb{S}$ and p in our model. The pressure p is of the form

$$p = a\rho^\gamma, \quad \gamma > \frac{3}{2}. \tag{1.5}$$

$\mathbb{S}$ consists of two parts:

$$\mathbb{S} = \mathbb{S}_1 + \mathbb{S}_2,$$

where $\mathbb{S}_1$ is the viscous stress tensor generated by the fluid

$$\mathbb{S}_1 = \mu\big(\nabla u + (\nabla u)^t\big) + \lambda(\nabla\cdot u)\mathbb{I}_{3\times 3},$$

and $\mathbb{S}_2$ is the macroscopic symmetric stress tensor derived from the orientation of the rods at the molecular level. The microscopic insertions at time t and macroscopic place x are described by the probability $f(t,x,\tau)d\tau$. The suspension stress tensor $\mathbb{S}_2$ is given by an expansion

$$\mathbb{S}_2(x,t) = \sigma^{(1)}(x,t) + \sigma^{(2)}(x,t) + \sigma^{(3)}(x,t),$$

where

$$\sigma^{(1)}(t,x) = \int_{S^2} (3\tau \otimes \tau - \mathbb{I}_{3\times 3}) f(t,x,\tau)d\tau,$$

$$\sigma^{(2)}(t,x) = -\sigma^{(2)}_{ij}(t,x)\mathbb{I}_{3\times 3}, \text{ with } \sigma^{(2)}_{ij}(t,x) = \int_{S^2} \gamma^{(2)}_{ij}(\tau) f(t,x,\tau)d\tau,$$

and

$$\sigma^{(3)}(t,x) = -\sigma^{(3)}_{ij}(t,x)\mathbb{I}_{3\times 3}, \text{ with } \sigma^{(3)}_{ij}(t,x) = \int_{S^2}\int_{S^2} \gamma^{(3)}_{ij}(\tau_1,\tau_2) f(t,x,\tau_1) f(t,x,\tau_2)d\tau_1 d\tau_2.$$

This, and more general expansions for $\mathbb{S}_2$ are encountered in the polymer literature (cf. Doi and Edwards [8]). We refer the reader to the articles by Constantin et al [5], [6], where a general class of stress tensors is presented in the context of incompressible fluids.

The structure coefficients in the expansion $\gamma^{(2)}_{ij}, \gamma^{(3)}_{ij}$ are in general smooth, time independent, x independent, and do not depend on f. Assuming for simplicity that

$$\gamma^{(2)}_{ij}(\tau) = \gamma^{(3)}_{ij}(\tau_1,\tau_2) = 1$$

and denoting

$$\eta(t,x) = \int_{S^2} f(t,x,\tau)d\tau$$

the suspension stress tensor $\mathbb{S}_2$ takes the form

$$\mathbb{S}_2(x,t) = \sigma^{(1)}(x,t) - \eta\mathbb{I}_{3\times 3} - \eta^2\mathbb{I}_{3\times 3}. \tag{1.6}$$

In this setting, f describes the time-dependent orientation distribution that a rod with a center mass at x has an axis τ in the area element $d\tau$ and it is described by a compressible Fokker-Plank type equation,

$$f_t + \nabla \cdot (uf) + \nabla_\tau \cdot (P_{\tau^\perp}\nabla u \tau f) - D_\tau \Delta_\tau f - D\Delta f = 0, \tag{1.7}$$

where $P_{\tau^\perp}(\nabla_x u\tau) = \nabla_x u\tau - (\tau \cdot \nabla_x u\tau)\tau$ is the projection of $\nabla u\tau$ on the tangent space of S^2 at $\tau \in S^2$. With ∇_τ and Δ_τ we denote the gradient and the Laplace operator on the unit sphere, while ∇ and Δ represent the gradient and the Laplacian operator in $\mathbb{R}^3$.

The second term $\nabla\cdot(uf)$ in (1.7) describes the change of f due to the displacement of the center of mass of the rods due to macroscopic advection. The term $\nabla_\tau \cdot (P_{\tau^\perp}\nabla u\tau f)$ is a drift-term on the sphere, which represents the shear-forces acting on the rods. The term $D_\tau\Delta_\tau f$ represents the rotational diffusion due to Brownian motion. This effect causes the rods to change their orientation spontaneously, whereas the term $D\Delta f$ is the translational diffusion due to Brownian effects.

By integrating (1.7) over S^2, we can obtain the equation of η:

$$\eta_t + \nabla \cdot (u\eta) - D\Delta\eta = 0. \tag{1.8}$$

By substituting (1.6) and (1.5) to (1.4), the equation of motion becomes

$$(\rho u)_t + \nabla \cdot (\rho u \otimes u) - \mu \Delta u - \lambda \nabla(\nabla \cdot u) + a \nabla \rho^\gamma + \nabla \eta^2 = \nabla \cdot \sigma - \nabla \eta. \tag{1.9}$$

In sum, after normalizing all the constants by 1 for the sake of simplicity, we have the following system of equations:

$$\rho_t + \nabla \cdot (\rho u) = 0 \quad \text{in} \quad (0,T) \times \Omega, \tag{1.10a}$$

$$(\rho u)_t + \nabla \cdot (\rho u \otimes u) - \Delta u - \nabla(\nabla \cdot u) + \nabla \rho^\gamma + \nabla \eta^2 = \nabla \cdot \sigma - \nabla \eta \quad \text{in} \quad (0,T) \times \Omega, \tag{1.10b}$$

$$f_t + \nabla \cdot (uf) + \nabla_\tau \cdot (P_{\tau^\perp}(\nabla_x u \tau) f) - \Delta_\tau f - \Delta_x f = 0 \quad \text{in} \quad (0,T) \times \Omega \times S^2, \tag{1.10c}$$

where Ω is a bounded domain and we impose Dirichlet boundary conditions to u, f, and η:

$$u = 0, \;\; f = 0, \text{ and } \eta = 0, \;\; \text{on } \partial\Omega.$$

In the sequel, we construct a sequence of approximating problems by regularizing the equations by extending functions to be zero outside Ω. Prescribing arbitrarily the initial fluid density, the initial velocity, and the initial orientation distribution in suitable spaces, we establish long-time and large data existence of a weak solution. Since the definition of a weak solution and the main result are rather complicated, they are stated in Section 2.

Related results on the Doi model for the suspensions of rod-like molecules in *incompressible* fluids have been studied by many authors. We refer the reader to Constantin [5, 6, 7], Lions and Masmoudi [14, 15], Masmoudi [16] and Otto and Tzavaras [19] for results on related models on the whole space. In [1] the authors treat the Doi model for an incompressible fluid within a bounded domain in the 3-dimensional space and establish results on the global existence of solutions. For *compressible* models, related results have been presented in a series of articles. We refer the reader to Carrillo et al [2, 3, 4], Goudon et al [11, 11, 12], and Mellet and Vasseur [17, 18], where asymptotic, analytical and numerical results on related fluid-particle interaction models are discussed. These articles deal with models coupling the Stokes, Navier Stokes or Euler system with either the Smolukowski equation or Fokker-Planck equation. What distinguishes the model presented in this article, besides the general type of the stress tensor under consideration, is the fact that, unlike other models, the Fokker-Planck-type equation presented here takes into consideration in addition to the Brownian effects the presence of the *shear forces* acting on the rods. This new element yields a new equation for the entropy induced by the probability density function f in the microscopic level and therefore new apriori estimates. We refer the reader to Appendix 1 for the derivation of equation of the entropy ψ.

The paper is organized as follows. In Section 2, we introduce the notion of a weak solution of the system (1.10), and we state the main results; compactness (Proposition 2.3) and existence (Theorem 2.2). In Section 3, we prove various convergence results and we provide a formal proof of the strong convergence of ρ under better assumptions. This formal convergence argument is recasted in Section 4, where the strong convergence of ρ is proved by using suitable cut-off functions in the renormalized equation of ρ. In Section 5, we generate an approximate sequence of weak solutions in three steps. (i) We first regularize the equation of ρ, which corresponds to the regularization of $\rho \frac{d}{dt} + \rho u \cdot \nabla$ in (1.10b). We also regularize u in the equation of f which requires to regularize η and σ in the right-hand side of (1.10b). Before regularization, we extend equations to zero outside Ω. (ii) Next we add nonlinear damping terms to the equation of ρ and η to increase integrability of ρ and η. (iii) We finally truncate ρ^γ and η^2 to increase regularity of $\{\rho, u, \eta\}$. By passing to the limits in sequences, we can prove Theorem 2.2.

Notations: • $L^p(0,T;X)$ denotes the Banach set of Bochner measurable functions f from $(0,T)$ to X endowed with either the norm $\left(\int_0^T \|g(\cdot,t)\|_X^p dt\right)^{\frac{1}{p}}$ for $1 \le p < \infty$ or $\sup_{t>\infty} \|g(\cdot,t)\|_X$ for $p=\infty$. In particular, $f \in L^r(0,T;XY)$ denotes $\left(\int_0^T \left\|\left(\|f(t)\|_{Y_\tau}\right)\right\|_X^p dt\right)^{\frac{1}{p}}$ or $\sup_{t>\infty} \left\|\left(\|f(t)\|_{Y_\tau}\right)\right\|_X$ for $p=\infty$.
• $A \lesssim B$ means there is a constant C such that $A \le CB$.
• $X \subset_{comp} Y$ means that X is compactly embedded in Y.
• $\mathbb{I}_X$ is the indicator function which is 1 for $x \in X$ and 0 otherwise.
• $C(T)$ is a function only depending on initial data and T.
• $\rightharpoonup$ and $\to$ denote weak limit and strong limit, respectively.

2 Definition of weak solution and main results

2.1 A priori estimate

Before introducing the concept of a weak solution of the system (1.10), let us present the energy estimate. Multiplying (1.10b) by u and integrating over Ω we get

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \Big[\frac{\rho|u|^2}{2} + \frac{\rho^\gamma}{\gamma-1} + \eta^2\Big]dx + \int_\Omega \Big[|\nabla u|^2 + |\nabla\cdot u|^2 + 2|\nabla\eta|^2\Big]dx \\ &= -\int_\Omega \nabla u : \sigma dx + \int_\Omega (\nabla\cdot u)\eta dx. \end{aligned} \tag{2.1}$$

Next we introduce an entropy induced by f in the microscopic level. Let

$$\psi(t,x) = \int_{S^2} (f\ln f)(t,x,\tau)d\tau.$$

Then, ψ satisfies

$$\psi_t + \nabla\cdot(u\psi) - \Delta\psi + 4\int_{S^2} |\nabla_\tau \sqrt{f}|^2 d\tau + 4\int_{S^2} |\nabla\sqrt{f}|^2 d\tau = \nabla u : \sigma - (\nabla\cdot u)\eta. \tag{2.2}$$

For the derivation of (2.2), we refer the reader to Section 6.1. Integrating (2.2) over Ω, we obtain

$$\frac{d}{dt}\int_\Omega \psi dx + 4\int_\Omega\int_{S^2} |\nabla_\tau\sqrt{f}|^2 d\tau dx + 4\int_\Omega\int_{S^2} |\nabla\sqrt{f}|^2 d\tau dx = \int_\Omega \nabla u : \sigma dx - \int_\Omega (\nabla\cdot u)\eta dx. \tag{2.3}$$

By adding (2.3) to (2.1), we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \Big[\frac{\rho|u|^2}{2} + \frac{\rho^\gamma}{\gamma-1} + \eta^2 + \psi\Big]dx + 4\int_\Omega\int_{S^2} |\nabla_\tau\sqrt{f}|^2 d\tau dx + 4\int_\Omega\int_{S^2} |\nabla\sqrt{f}|^2 d\tau dx \\ &+ \int_\Omega \Big[|\nabla u|^2 + |\nabla\cdot u|^2 + 2|\nabla\eta|^2\Big]dx = 0. \end{aligned} \tag{2.4}$$

In particular, η is bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$, which cannot be obtained derived from (1.8) in the three dimensions. From (2.4), we can obtain various estimates of $\{\rho, u, f, \eta, \sigma\}$. First,

$$\begin{aligned} &\rho|u|^2 \in L^\infty(0,T;L^1(\Omega)), \quad \rho \in L^\infty(0,T;L^\gamma(\Omega)), \quad \nabla u \in L^2(0,T;L^2(\Omega)), \\ &\psi \in L^\infty(0,T;L^1(\Omega)), \quad \eta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \end{aligned} \tag{2.5}$$

By expressing ρu as $\sqrt{\rho}\cdot\sqrt{\rho}u$, we have

$$\rho u \in L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega)). \tag{2.6}$$

From the entropy dissipation,

$$\sqrt{f}\in L^2\big(0,T;L^2(\Omega)H^1(S^2)\cap H^1(\Omega)L^2(S^2)\big)\subset L^2\big(0,T;L^2(\Omega)L^6(S^2)\cap L^6(\Omega)L^2(S^2)\big),$$

which implies that

$$f\in L^1\big(0,T;L^1(\Omega)L^3(S^2)\cap L^3(\Omega)L^1(S^2)\big)\subset L^1(0,T;L^2(\Omega\times S^2)). \tag{2.7}$$

Since $|\sigma(t,x)|\le 3\int_{S^2} f(t,x,\tau)d\tau = 3\eta(t,x)$,

$$\sigma\in L^1(0,T;L^3(\Omega))\cap L^\infty(0,T;L^2(\Omega)). \tag{2.8}$$

We next estimate the derivative of σ by using the entropy dissipation.

$$|\nabla\sigma(t,x)|\le 3\int_{S^2}|\nabla f(t,x,\tau)|d\tau \lesssim \Big[\int_{S^2}|\nabla\sqrt{f}|^2d\tau\Big]^{\frac12}\Big[\int_{S^2}(\sqrt{f})^2d\tau\Big]^{\frac12}=\Big[\int_{S^2}|\nabla\sqrt{f}|^2d\tau\Big]^{\frac12}\eta^{\frac12}.$$

Since $\eta^{\frac12}\in L^\infty(0,T;L^4(\Omega))\cap L^2(0,T;L^6(\Omega))$,

$$\nabla\sigma\in L^1(0,T;L^{\frac32}(\Omega))\cap L^2(0,T;L^{\frac43}(\Omega)). \tag{2.9}$$

2.2 Definition of weak solution, main result

We now define a weak solution of the system (1.10). By a notational abuse, we include η and σ in the definition of weak solution.

Definition 2.1 *We say that $\{\rho,u,f,\eta,\sigma\}$ is a weak solution of the system (1.10) if*
(i) (1.10a) holds in the sense of renormalized solutions, i.e.,

$$b(\rho)_t+\nabla\cdot(b(\rho)u)+(b^{'}(\rho)\rho-b(\rho))\nabla\cdot u=0 \tag{2.10}$$

for any $b\in C^1$ such that $|b^{'}(z)z|+|b(z)|\le C$ for all $z\in R$.
(ii) (1.10b) and (1.10c) hold in the sense of distributions.
(iii) Moreover, $\{\rho,u,f,\eta,\sigma\}$ satisfies the following energy inequality:

$$\begin{aligned}&\int_\Omega\Big[\frac{\rho|u|^2}{2}+\frac{\rho^\gamma}{\gamma-1}+\eta^2+\psi\Big](t)dx+4\int_0^t\int_\Omega\int_{S^2}|\nabla_\tau\sqrt{f}|^2d\tau dxdt\\&+4\int_0^t\int_\Omega\int_{S^2}|\nabla\sqrt{f}|^2d\tau dxdt+\int_0^t\int_\Omega\Big[|\nabla u|^2+|\nabla\cdot u|^2+2|\nabla\eta|^2\Big]dxdt\\&\le\int_\Omega\Big[\frac{\rho_0|u_0|^2}{2}+\frac{\rho_0^\gamma}{\gamma-1}+\eta_0^2+\psi_0\Big]dx\end{aligned} \tag{2.11}$$

Remark 1 *(1) The central difficulty in showing the existence of a weak solution in the theory of compressible fluids is typically the dependence of the pressure on nonlinear terms, for instance* ρ^γ. *From the a priori estimate, we have* $\rho \in L^\infty(0,T;L^\gamma(\Omega))$, *which is not enough to pass to the limit to* $\nabla\rho^\gamma$ *in the sense of distributions. The issue is resolved by showing that* ρ *satisfies a better integrability condition by choosing appropriate cut-off functions in the renormalized form (2.10) in the spirit of Feireisl [9] (see also Lions [13]). Note that in the present context the suspension stress tensor depends on the density of the particles in a nonlinear way as well. In this case, the regularity of* η, $\eta \in L^2(0,T;H^1(\Omega))$ *enables us to handle the nonlinearity.*

(2) The main additional difficulties in the present context involve the presence of two nonlinear terms in the equation of f. *In fact, letting* $\chi \in C_c^\infty(\Omega\times S^2)$ *we obtain from the advection term* $\nabla\cdot(uf)$,

$$\int_\Omega\int_{S^2}\nabla\cdot(u^{(n)}f^{(n)})\chi d\tau dx = -\int_\Omega u_i^{(n)}\Big[\int_{S^2}\partial_{x_i}\chi f^{(n)}d\tau\Big]dx, \tag{2.12}$$

whereas from the shear term $\nabla_\tau\cdot(P_{\tau^\perp}(\nabla_x u\tau)f)$, *we get*

$$\int_\Omega\int_{S^2}\nabla_\tau\cdot(P_{\tau^\perp}(\nabla_x u^{(n)}\tau)f^{(n)})\chi d\tau dx = -\int_\Omega \frac{\partial u_i^{(n)}}{\partial x_j}\Big[\int_{S^2}\tau_j f^{(n)}\frac{\partial\chi}{\partial\tau_i}d\tau\Big]dx. \tag{2.13}$$

To pass to the limit in (2.12) and (2.13), we need to show that

$$\int_{S^2}\partial_{x_i}\chi f^{(n)}d\tau, \quad \int_{S^2}\tau_j f^{(n)}\frac{\partial\chi}{\partial\tau_i}d\tau$$

converge strongly in $L^2(0,T;L^2(\Omega))$. *This is proved in Section 3.*

(3) In order to pass to the limit to linear terms in the equation of f, *we only need* $f \in L^p(0,T;L^q(\Omega\times S^2))$ *for some* $p>1$ *and* $q>1$. *Since* $f\in L^\infty(0,T;L^1(\Omega\times S^2))\cap L^1(0,T;L^2(\Omega\times S^2))$, *we can choose, for example,* $f\in L^2(0,T;L^{\frac{6}{5}}(\Omega\times S^2))$.

We now state the main result of the paper.

Theorem 2.2 *Let* $\gamma > \frac{3}{2}$ *and* Ω *be a* C^1 *bounded domain. Assume that a sequence* $\{\rho_0,u_0,f_0,\eta_0\}$ *satisfies*

$$\begin{aligned}&\rho_0\in L^1\cap L^\gamma(\Omega),\quad \rho_0u_0=m_0\in L^{\frac{2\gamma}{\gamma+1}}(\Omega),\quad f_0\in L^1(\Omega\times S^2),\quad \eta_0\in L^2(\Omega),\\ &\frac{m_0^2}{\rho_0}\in L^1(\Omega)\quad \text{for}\quad \rho_0\neq 0,\quad \frac{m_0^2}{\rho_0}=0\quad\text{for}\quad \rho_0=0,\end{aligned} \tag{2.14}$$

Then, there exists a weak solution $\{\rho,u,f,\eta,\sigma\}$ *of the system (1.10) satisfying (2.14) at* $t=0$. *Moreover,*

$$\rho\in L^p(\Omega\times(0,T)),\quad p=\frac{5}{3}\gamma-1. \tag{2.15}$$

The proof of Theorem 2.2 consists of two parts. First, we prove the compactness of an approximate sequence $\{\rho^n,u^n,f^n,\eta^n,\sigma^n\}_{n\geq1}$ under the assumption $\gamma>\frac{3}{2}$ in Section 3 and 4. We state the detailed statement in Proposition 2.3 below. Secondly, we construct an approximate sequence of solutions through regularizing equations in Section 5. The reader should contrast the approximating scheme presented here with the schemes presented in [4], [10] and [13] for different models.

We begin with the compactness result. Suppose there is a an approximate sequence of solutions $\{\rho^n, u^n, f^n, \eta^n, \sigma^n\}_{n\geq 1}$ such that

$$
\begin{aligned}
&\{\rho^n\} \text{ is bounded in } L^\infty(0,T;L^1\cap L^\gamma(\Omega)), \quad \{\rho^n|u^n|^2\} \text{ is bounded in } L^\infty(0,T;L^1(\Omega)),\\
&\{u^n\} \text{ is bounded in } L^2(0,T;H^1(\Omega)),\\
&\{f^n\} \text{ is bounded in } L^2(0,T;L^{\frac{6}{5}}(\Omega\times S^2)),\\
&\{\eta^n\} \text{ is bounded in } L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)),\\
&\{\sigma^n\} \text{ is bounded in } L^\infty(0,T;L^2(\Omega))\cap L^1(0,T;L^3(\Omega)),\\
&\{\nabla\sigma^n\} \text{ is bounded in } L^1(0,T;L^{\frac{3}{2}}(\Omega))\cap L^2(0,T;L^{\frac{4}{3}}(\Omega)).
\end{aligned}
\tag{2.16}
$$

Then, we can extract a subsequence, using the same notation, $\{\rho^n, u^n, f^n, \eta^n, \sigma^n\}_{n\geq 1}$ such that

$$
\begin{aligned}
&\rho^n \rightharpoonup \rho \text{ in } L^\gamma(\Omega\times(0,T)) \text{ and } \rho\in L^\infty(0,T;L^1\cap L^\gamma(\Omega)),\\
&\sqrt{\rho^n}u^n \rightharpoonup v \text{ in } L^2(0,T;L^2(\Omega)) \text{ and } v\in L^\infty(0,T;L^2(\Omega)),\\
&u^n \rightharpoonup u \text{ in } L^2(0,T;H^1(\Omega)), \quad \sqrt{\rho^n} \rightharpoonup \overline{\sqrt{\rho}} \text{ in } L^{2\gamma}(\Omega\times(0,T)),\\
&\rho^n u^n \rightharpoonup m \text{ in } L^{\frac{2\gamma}{\gamma+1}}(\Omega\times(0,T)) \text{ and } m\in L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega))\\
&\rho^n u_i^n u_j^n \rightharpoonup e_{ij} \text{ in the sense of measures and } e_{ij} \text{ is a bounded measure},\\
&f^n \rightharpoonup f \text{ in } L^2(0,T;L^{\frac{6}{5}}(\Omega\times S^2)),\\
&\eta^n \rightharpoonup \eta \text{ in } L^2(0,T;H^1(\Omega)) \text{ and } \eta\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)),\\
&\sigma^n \rightharpoonup \sigma \text{ in } L^2(0,T;L^2(\Omega)) \text{ and } \sigma\in L^\infty(0,T;L^2(\Omega))\cap L^1(0,T;L^3(\Omega)),\\
&\nabla\sigma^n \rightharpoonup \nabla\sigma \text{ in } L^2(0,T;L^{\frac{4}{3}}(\Omega)) \text{ and } \nabla\sigma\in L^1(0,T;L^{\frac{3}{2}}(\Omega))\cap L^2(0,T;L^{\frac{4}{3}}(\Omega)).
\end{aligned}
\tag{2.17}
$$

Proposition 2.3 (Compactness) *Let $\gamma > \frac{3}{2}$ and Ω be a C^1 bounded domain. Assume that the energy inequality (2.11) holds for a sequence $\{\rho^n, u^n, f^n, \eta^n, \sigma^n\}_{n\geq 1}$. Then, limit functions in (2.17) satisfy the followings.*
(i) $v = \sqrt{\rho}u$, $m = \rho u$, $e_{ij} = \rho u_i u_j$.
(ii) η^n converges strongly to η in $L^2(\Omega\times(0,T))$, and σ^n converges strongly to σ in $L^2(\Omega\times(0,T))$.
(iii) $\rho^n(\eta^n)^2$ converges to $\rho\eta^2$ in the sense of distributions.
(iv) ρ and u solve (1.10a) in the sense of renormalized solutions.
(v) If in addition we assume that ρ_0^n converges to ρ_0 in $L^1(\Omega)$, then $\{\rho, u, f, \eta, \sigma\}$ is a weak solution of (1.10) such that

$$
\rho^n \to \rho \;\; in \;\; L^1(\Omega\times(0,T))\cap C([0,T];L^p(\Omega)) \;\; for\ all \;\; 1\leq p<\gamma. \tag{2.18}
$$

(vi) Finally, we have the following strong convergence:

$$
\begin{aligned}
&\rho_n u^n \to \rho u \;\; in \;\; L^p(0,T;L^r(\Omega)) \;\; for\ all \;\; 1\leq p<\infty, \quad 1\leq r<\frac{2\gamma}{\gamma+1},\\
&u^n \to u \;\; in \;\; L^p(\Omega\times(0,T))\cap\{\rho>0\} \;\; for\ all \;\; 1\leq p<2,\\
&u^n \to u \;\; in \;\; L^2(\Omega\times(0,T))\cap\{\rho\geq\delta\} \;\; for\ all \;\; \delta>0,\\
&\rho^n u_i^n u_j^n \to \rho u_i u_j \;\; in \;\; L^p(0,T;L^1(\Omega)) \;\; for\ all \;\; 1\leq p<\infty.
\end{aligned}
\tag{2.19}
$$

We will prove this proposition in Section 3 and 4.

3 Proof of Proposition 2.3 (i), (ii), (iii), formal proof of (v), and (vi)

3.1 Proof of Proposition 2.3. (i)

We begin with the proof of Proposition 2.3 (i). To this end, we need the following lemma.

Lemma 3.1 *[13] Let g^n and h^n converge weakly to g and h respectively in $L^{p_1}(0,T;L^{p_2}(\Omega))$ and $L^{q_1}(0,T;L^{q_2}(\Omega))$, where $1 \le p_1, p_2 \le \infty$, $\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{p_2}+\frac{1}{q_2}=1$. Suppose*

$$\begin{aligned}&\frac{d}{dt}g^n \ \text{ is bounded in } \ L^1(0,T;W^{-m,1}(\Omega)) \ \text{ for some } \ m \ge 0 \ \text{ independent of } n,\\ &\|h^n - h^n(\cdot+\xi,t)\|_{L^{q_1}(0,T;L^{q_2}(\Omega))} \to 0 \ \text{ as } \ |\xi| \to 0, \ \text{ uniformly in } \ n.\end{aligned} \tag{3.1}$$

Then, $g^n h^n$ converges to gh in the sense of distributions.

We would like to apply Lemma 3.1 to $h^n = u^n$ with $q_1 = 2$ and $q_2 \in [2,6)$, and $g^n = \rho^n$, $\rho^n u^n$, or $g^n = \sqrt{\rho^n}$.

• First, we need to show that $\{\rho^n\}$ is bounded in $L^2(0,T;L^p)$, $p > \frac{6}{5}$. But, this is clear because $\rho^n \in L^\infty(0,T;L^\gamma(\Omega))$, $\gamma > \frac{3}{2} > \frac{6}{5}$. Next, from the equation of ρ,

$$(\rho^n)_t = -\nabla\cdot(\rho^n u^n) \ \text{ is bounded in } \ L^\infty(0,T;W^{-1,\frac{2\gamma}{\gamma+1}}(\Omega)) \subset L^1(0,T;W^{-1,1}(\Omega)).$$

Therefore, $m = \rho u$.

• Since $u^n \in L^2(0,T;L^6(\Omega))$,

$$\rho^n u^n \in L^2(0,T;L^p(\Omega)), \quad \frac{1}{p} = \frac{1}{\gamma}+\frac{1}{6} < \frac{2}{3}+\frac{1}{6} = \frac{5}{6}.$$

Next, from the equation of motion,

$$\begin{aligned}(\rho^n u^n)_t &= -\nabla\cdot(\rho^n u^n \otimes u^n) + \Delta u^n + \nabla\nabla\cdot u^n - \nabla(\rho^n)^\gamma - \nabla(\eta^n)^2 + \nabla\cdot\sigma^n - \nabla\eta^n\\ &\subset L^\infty(0,T;W^{-1,1}(\Omega)) + L^2(0,T;H^{-1}(\Omega)) + L^\infty(0,T;W^{-1,1}(\Omega))\\ &\text{is bounded in } \ L^1(0,T;W^{-1,1}(\Omega)).\end{aligned}$$

Here, we use the fact that $(\eta^n)^2$, σ^n, and η^n are bounded in $L^\infty(0,T;L^1(\Omega))$. Therefore, $e = m\otimes u = \rho u \otimes u$.

• $\sqrt{\rho^n}$ satisfies that

$$\frac{d}{dt}\sqrt{\rho^n} + \nabla\cdot(u^n\sqrt{\rho^n}) = -\frac{1}{2}(\nabla\cdot u^n)\sqrt{\rho^n},$$

from which $\frac{d}{dt}\sqrt{\rho^n}$ is bounded in $L^\infty(0,T;W^{-1,2}(\Omega)) + L^2(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega)) \subset L^1(0,T;W^{-1,1}(\Omega))$. Since $\sqrt{\rho^n} \in L^\infty(0,T;L^{2\gamma}(\Omega))$, $2\gamma > \frac{6}{5}$, we conclude as above that $v = \sqrt{\rho}u$.

3.2 Proof of Proposition 2.3. (ii)

To show the strong convergence of η^n and σ^n, we need the following lemma.

Lemma 3.2 *[20] Let X, B, and Y be Banach spaces such that X is compactly embedded in B and B is a subset of Y. Then, for $1 \le p < \infty$, $\{v; v \in L^p(0,T;X), v_t \in L^1(0,T;Y)\}$ is compactly embedded in $L^p(0,T;B)$.*

• **Strong convergence of η^n:** First,

$$\eta_t^n = -\nabla\cdot(u^n\eta^n) + \Delta\eta^n \in L^1(0,T;W^{-1,1}(\Omega)) + L^2(0,T;W^{-1,2}(\Omega)) \subset L^1(0,T;W^{-1,1}(\Omega)).$$

Since $H^1(\Omega) \subset_{comp} L^2(\Omega) \subset W^{-1,1}(\Omega)$,

$$\eta^n \to \eta \ \text{ in } \ L^2(0,T;L^2(\Omega)). \tag{3.2}$$

• **Strong convergence of σ^n:** First,

$$\begin{aligned}
\sigma_t &= \int_{S^2}(3\tau\otimes\tau - \mathbb{I}_{3\times3})f_t(t,x,\tau)d\tau \\
&= \int_{S^2}(3\tau\otimes\tau - \mathbb{I}_{3\times3})\Big[-\nabla\cdot(uf) - \nabla_\tau\cdot(P_{\tau^\perp}(\nabla_x u\tau)f) + \Delta_\tau f + \Delta f\Big]d\tau \\
&= -\nabla\cdot(u\sigma) - \int_{S^2}[\nabla_\tau\cdot(3\tau\otimes\tau - \mathbb{I}_{3\times3})]\cdot[(\nabla u\tau)f]d\tau + \int_{S^2}[\nabla_\tau\cdot(3\tau\otimes\tau - \mathbb{I}_{3\times3})]\cdot[\nabla_\tau f]d\tau + \Delta\sigma \\
&\lesssim |\nabla\cdot(u\sigma)| + |\nabla u|\eta + \int_{S^2}|\nabla_\tau f|d\tau + |\Delta\sigma| \\
&\in L^1(0,T;W^{-1,1}(\Omega)) + L^1(0,T;L^1(\Omega)) + L^1(0,T;L^{\frac{3}{2}}(\Omega)) + L^1(0,T;W^{-1,\frac{3}{2}}(\Omega)) \\
&\subset L^1(0,T;W^{-1,1}(\Omega)).
\end{aligned}$$

From $\nabla\sigma \in L^1(0,T;L^3(\Omega)) \cap L^2(0,T;L^{\frac{4}{3}}(\Omega))$, we have, for example, $\nabla\sigma \in L^{\frac{3}{2}}(0,T;L^{\frac{18}{11}}(\Omega))$. From $W^{1,\frac{18}{11}}(\Omega) \subset_{comp} L^2(\Omega) \subset W^{-1,1}(\Omega)$, we have

$$\sigma^n \to \sigma \ \text{ in } \ L^{\frac{3}{2}}(0,T;L^2(\Omega)).$$

Since $\{\sigma^n\}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$, $\sigma^n \rightharpoonup \sigma$ in $L^p(0,T;L^2(\Omega))$ for all $p<\infty$. Therefore,

$$\sigma^n \to \sigma \ \text{ in } \ L^2(0,T;L^2(\Omega)). \tag{3.3}$$

• **Strong convergence of $\int_{S^2}\partial_{x_i}\chi f^{(n)}d\tau$, $\int_{S^2}\tau_j f^{(n)}\frac{\partial\chi}{\partial\tau_i}d\tau$:** We note that these two terms are of the form $\int_{S^2}\varsigma f^n d\tau$, where $\varsigma \in C_c^\infty(\Omega\times S^2)$, and $\int_{S^2}\varsigma f^n d\tau$ and their time and spatial derivatives satisfy the same bound of σ. Therefore, these two terms converge strongly in $L^2(0,T;L^2(\Omega))$ as well.

3.3 Proof of Proposition 2.3. (iii)

To show (iii), we need to show $(\eta^n)^2$ converges strongly in $L^p(0,T;L^q(\Omega))$ for some $p>1$ and q such that $\frac{1}{q}+\frac{1}{\gamma}\le 1$. Given $\gamma > \frac{3}{2}$, we take $\epsilon>0$ such that $\frac{1}{\gamma} = \frac{2}{3}-\epsilon$. Then, for $\theta = 1-\epsilon$,

$$\|\eta^n - \eta\|_{L^{\bar p}(0,T;L^{\bar q}(\Omega))} \le \|\eta^n-\eta\|^{1-\epsilon}_{L^2(0,T;L^{6-\delta}(\Omega))}\|\eta^n-\eta\|^{\epsilon}_{L^\infty(0,T;L^2(\Omega))},$$

where

$$\frac{1}{\bar p} = \frac{1-\epsilon}{2}, \quad \frac{1}{\bar q} = \frac{1-\epsilon}{6-2\delta} + \frac{\epsilon}{2}.$$

Since $H^1(\Omega) \subset_{comp} L^{6-2\delta}(\Omega)$ for any $\delta > 0$,

$$\eta^n \to \eta \ \text{ in } \ L^{\bar{p}}(0,T;L^{\bar{q}}(\Omega)).$$

Let $p = \frac{\bar{p}}{2}$ and $q = \frac{\bar{q}}{2}$. Then,

$$(\eta^n)^2 \to \eta^2 \ \text{ in } \ L^p(0,T;L^q(\Omega)), \tag{3.4}$$

where $\frac{1}{q} + \frac{1}{\gamma} \le 1$ by taking $\delta < 3\epsilon$. Therefore, $\rho^n(\eta^n)^2$ converges to $\rho\eta^2$ in the sense of distributions.

3.4 Proof of Proposition 2.3 (v): formal proof on the whole spaces

Before proving Proposition 2.3 (iv) and (v), we provide a formal proof of the convergence of ρ^n on the whole spaces under a stronger assumption:

$$\{\rho^n\} \ \text{ is bounded in } \ L^{\gamma+1}((0,T)\times R^3) \cap L^\infty(0,T;L^s(R^3)), \quad s > 3. \tag{3.5}$$

For details of the proof, see Section 6.2. From (1.10a),

$$(\rho\log\rho)_t + \nabla\cdot(u\rho\log\rho) + (\nabla\cdot u)\rho = 0. \tag{3.6}$$

Next, we take $(-\Delta)^{-1}\nabla\cdot$ to (1.10b). Then,

$$\frac{d}{dt}\Big[(-\Delta)^{-1}\nabla\cdot(\rho u)\Big] + (-\Delta)^{-1}\partial_i\partial_j(\rho u_i u_j) + 2\nabla\cdot u - \rho^\gamma - \eta^2 = (-\Delta)^{-1}\nabla\cdot(\nabla\cdot\sigma - \nabla\eta),$$

from which we have

$$2\nabla\cdot u = -\frac{d}{dt}\Big[(-\Delta)^{-1}\nabla\cdot(\rho u)\Big] - (-\Delta)^{-1}\partial_i\partial_j(\rho u_i u_j) + \rho^\gamma + \eta^2 + (-\Delta)^{-1}\nabla\cdot(\nabla\cdot\sigma - \nabla\eta). \tag{3.7}$$

By (3.6) and (3.7),

$$\begin{aligned} &2\Big[(\rho\log\rho)_t + \nabla\cdot(u\rho\log\rho)\Big] + \rho^{\gamma+1} = -\rho\Big[(-\Delta)^{-1}\nabla\cdot(\nabla\cdot\sigma - \nabla\eta)\Big] - \rho\eta^2 \\ &+ \rho\frac{d}{dt}\Big[(-\Delta)^{-1}\nabla\cdot(\rho u)\Big] + \nabla\cdot(\rho u)(-\Delta)^{-1}\nabla\cdot(\rho u) + \rho(-\Delta)^{-1}\partial_i\partial_j(\rho u_i u_j). \end{aligned} \tag{3.8}$$

Since

$$\begin{aligned} \rho\frac{d}{dt}\Big[(-\Delta)^{-1}\nabla\cdot(\rho u)\Big] &= \frac{d}{dt}\Big[\rho(-\Delta)^{-1}\nabla\cdot(\rho u)\Big] - \rho_t\Big[(-\Delta)^{-1}\nabla\cdot(\rho u)\Big] \\ &= \frac{d}{dt}\Big[\rho(-\Delta)^{-1}\nabla\cdot(\rho u)\Big] + \nabla\cdot\rho u\Big[(-\Delta)^{-1}\nabla\cdot(\rho u)\Big] \\ &= \frac{d}{dt}\Big[\rho(-\Delta)^{-1}\nabla\cdot(\rho u)\Big] + \nabla\cdot\Big[\rho u(-\Delta)^{-1}\nabla\cdot(\rho u)\Big] - \rho u\cdot\nabla\Big[(-\Delta)^{-1}\nabla\cdot(\rho u)\Big], \end{aligned}$$

we rewrite (3.8) as follows.

$$\begin{aligned} &2\Big[(\rho\log\rho)_t + \nabla\cdot(u\rho\log\rho)\Big] + \rho^{\gamma+1} \\ &= -\rho\eta^2 - \rho\Big[(-\Delta)^{-1}\nabla\cdot(\nabla\cdot\sigma - \nabla\eta)\Big] + \frac{d}{dt}\Big[\rho(-\Delta)^{-1}\nabla\cdot(\rho u)\Big] \\ &+ \nabla\cdot\Big[\rho u(-\Delta)^{-1}\nabla\cdot(\rho u)\Big] + \rho\Big[(-\Delta)^{-1}\partial_i\partial_j(\rho u_i u_j) - u\cdot\nabla(-\Delta)^{-1}\nabla\cdot(\rho u)\Big]. \end{aligned} \tag{3.9}$$

Suppose that (3.9) also holds for $\{\rho^n, u^n, f^n, \eta^n, \sigma^n\}_{n\geq 1}$.

$$
\begin{aligned}
&2\Big[(\rho^n \log \rho^n)_t + \nabla\cdot(u^n\rho^n\log\rho^n)\Big] + (\rho^n)^{\gamma+1}\\
&= -\rho^n(\eta^n)^2 - \rho^n\Big[(-\Delta)^{-1}\nabla\cdot(\nabla\cdot\sigma^n - \nabla\eta^n)\Big] + \frac{d}{dt}\Big[\rho^n(-\Delta)^{-1}\nabla\cdot(\rho^n u^n)\Big]\\
&+\nabla\cdot\Big[\rho^n u^n(-\Delta)^{-1}\nabla\cdot(\rho^n u^n)\Big] + \rho^n\Big[(-\Delta)^{-1}\partial_i\partial_j(\rho^n u_i^n u_j^n) - u^n\cdot\nabla(-\Delta)^{-1}\nabla\cdot(\rho^n u^n)\Big].
\end{aligned}
\tag{3.10}
$$

Let $\bar{s}$ be a weak limit of $\rho^n\log\rho^n$. By taking the limit of (3.10) for $n\to\infty$,

$$
\begin{aligned}
&2\Big[\bar{s}_t + \nabla\cdot(u\bar{s})\Big] + \overline{\rho^{\gamma+1}}\\
&= -\rho\eta^2 - \rho\Big[(-\Delta)^{-1}\nabla\cdot(\nabla\cdot\sigma - \nabla\eta)\Big] + \frac{d}{dt}\Big[\rho(-\Delta)^{-1}\nabla\cdot(\rho u)\Big]\\
&+\nabla\cdot\Big[\rho u(-\Delta)^{-1}\nabla\cdot(\rho u)\Big] + \rho\Big[(-\Delta)^{-1}\partial_i\partial_j(\rho u_i u_j) - u\cdot\nabla(-\Delta)^{-1}\nabla\cdot(\rho u)\Big],
\end{aligned}
\tag{3.11}
$$

where we use Proposition 2.3 (iii). Next, we take the limit to (1.10b).

$$
(\rho u)_t + \nabla\cdot(\rho u\otimes u) - \Delta u - \nabla(\nabla\cdot u) + \nabla\overline{\rho^\gamma} + \nabla\eta^2 = \nabla\cdot\sigma - \nabla\eta,
\tag{3.12}
$$

where we use Proposition 2.3 (ii). Let $s = \rho\log\rho$. By following the same calculations above, we have from which we obtain that

$$
\begin{aligned}
&2\Big[s_t + \nabla\cdot(us)\Big] + \rho\overline{\rho^\gamma}\\
&= -\rho\eta^2 - \rho\Big[(-\Delta)^{-1}\nabla\cdot(\nabla\cdot\sigma - \nabla\eta)\Big] + \frac{d}{dt}\Big[\rho(-\Delta)^{-1}\nabla\cdot(\rho u)\Big]\\
&+\nabla\cdot\Big[\rho u(-\Delta)^{-1}\nabla\cdot(\rho u)\Big] + \rho\Big[(-\Delta)^{-1}\partial_i\partial_j(\rho u_i u_j) - u\cdot\nabla(-\Delta)^{-1}\nabla\cdot(\rho u)\Big].
\end{aligned}
\tag{3.13}
$$

Comparing (3.11) and (3.13), we have

$$
(\bar{s}-s)_t + \nabla\cdot(u(\bar{s}-s)) + \frac{1}{2}\Big[\overline{\rho^{\gamma+1}} - \rho\overline{\rho^\gamma}\Big] = 0.
\tag{3.14}
$$

Since

$$
\begin{gathered}
\bar{s}\leq s,\;\; \overline{\rho^{\gamma+1}} \geq \rho\overline{\rho^\gamma}\;\; a.e.,\\
\frac{d}{dt}\int_\Omega(\bar{s}-s)dx \leq 0,\;\; \text{while}\;\; \int_\Omega(\bar{s}_0 - s_0)dx = 0.
\end{gathered}
$$

Therefore, $\bar{s} = s$ almost everywhere, and ρ^n converges strongly to ρ in $C([0,T];L^1(\Omega))$.

Remark 2 *In Section 4, we show the strong convergence of ρ^n in $C([0,T];L^1(\Omega))$ using (2.10) with appropriate cut-off functions approximating $\rho\log\rho$.*

3.5 Proof of Proposition 2.3 (vi)

We now prove Proposition 2.3 (vi) assuming that we have already resolved Proposition 2.3 (v). We will prove Proposition 2.3 (iv) and (v) in the next section. Let us begin with the convergence of $\rho^n u^n$. First, we show that $(\rho^n u^n)_\epsilon$ converges to $\rho^n u^n$ in $L^2(0,T;L^1(R^3))$. Here, we extended functions to zero outside Ω, and $g_\epsilon = g \star k_\epsilon$ and k_ϵ is the usual mollifier. Since

$$\Big|\big((\rho^n u^n)_\epsilon - \rho^n u^n\big)(x)\Big| = \Big|\int_\Omega \big[\rho^n(y,t) - \rho^n(x,t)\big]u^n(y,t)k_\epsilon(x-y)dy + \rho^n(x,t)\big[u^n_\epsilon(x,t) - u^n(x,t)\big]\Big|,$$

we have

$$\begin{aligned}
&\int_\Omega \big|\big((\rho^n u^n)_\epsilon - \rho^n u^n\big)(x)\big|dx \\
&\le \Big[\int_\Omega dx \int_\Omega \big|\rho^n(y,t) - \rho^n(x,t)\big|^p k_\epsilon(x-y)\Big]^{\frac{1}{p}} \Big\|\Big(|u^n|^{\frac{p}{p-1}}\Big)_\epsilon\Big\|^{\frac{p-1}{p}}_{L^1(\Omega)} + \|\rho^n\|_{L^p(\Omega)}\|u^n_\epsilon - u^n\|_{L^{\frac{p}{p-1}}(\Omega)} \\
&\le \Big[\sup_{|z|\le\epsilon} \|\rho^n(\cdot+z) - \rho^n\|_{L^p(\Omega)}\Big]\|u^n\|_{L^{\frac{p}{p-1}}(\Omega)} + \|\rho^n\|_{L^p(\Omega)}\|u^n_\epsilon - u^n\|_{L^{\frac{p}{p-1}}(\Omega)}.
\end{aligned}$$

Now, we choose $p > \frac{6}{5}$ such that $\frac{p}{p-1} < 6$. Then, $\|u^n_\epsilon - u^n\|_{L^{\frac{p}{p-1}}(\Omega)}$ converges to 0 as ϵ goes to 0. Moreover, (2.18) implies that $\sup_{|z|\le\epsilon} \|\rho^n(\cdot+z) - \rho^n\|_{L^p(\Omega)}$ converges to 0 as ϵ goes to 0. Therefore, $(\rho^n u^n)_\epsilon$ converges to $\rho^n u^n$ in $L^2(0,T;L^1(\Omega))$ as ϵ goes to 0, uniformly in n. Next, we deduce that $(\rho^n u^n)_\epsilon$ converges to $(\rho u)_\epsilon$ as n goes to ∞. Since $(\rho^n u^n)_\epsilon$ is smooth in x and $\frac{d}{dt}(\rho^n u^n)_\epsilon$ is bounded in $L^2(0,T;H^m(R^3))$ for any $m \ge 0$, $(\rho^n u^n)_\epsilon$ converges to $(\rho u)_\epsilon$ as n goes to ∞ in $L^1(R^3 \times (0,T))$ for each $\epsilon > 0$. Since $\rho^n u^n$ is uniformly bounded in $L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega))$ and

$$\begin{aligned}
\|\rho^n u^n - \rho u\|_{L^1(\Omega\times(0,T))} \le{}& \|\rho^n u^n - (\rho^n u^n)_\epsilon\|_{L^1(\Omega\times(0,T))} + \|(\rho^n u^n)_\epsilon - (\rho u)_\epsilon\|_{L^1(\Omega\times(0,T))} \\
&+ \|(\rho u)_\epsilon - \rho u\|_{L^1(\Omega\times(0,T))},
\end{aligned}$$

$\rho^n u^n$ converges to ρu in $L^1(\Omega \times (0,T))$.

The second statement in (2.19) is immediate consequence of the uniform bound of u^n in $L^2(\Omega \times (0,T))$.

Since

$$\rho^n|u - u^n|^2 = \rho^n|u^n|^2 - 2\rho^n u^n \cdot u + \rho^n|u|^2$$

converges to 0 in $L^1(\Omega)$, the third statement is proved.

Finally, three previous results implies $\rho^n u^n_i u^n_j$ converges to $\rho u_i u_j$ almost everywhere, and $\rho^n u^n_i u^n_j$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega)) \cap L^1(0,T;L^p(\Omega))$, $\frac{1}{p} = \frac{1}{\gamma} + \frac{1}{3} < 1$. Therefore, the last statement in (2.19) holds.

4 Proof of Proposition 2.3 (iv) and (v)

The strong convergence of $\{\rho^{(n)}\}_{n\ge1}$ can be proved by introducing a family of cut-off functions approximating $\rho\log\rho$ in the renormalized solution setting. First, we show that ρ satisfies higher integrability condition.

4.1 Higher integrability of ρ

We first define the inverse of the divergence operator. We denote the solution v of

$$\nabla\cdot v = g \ \text{ in } \ \Omega, \quad v = 0 \ \text{ on } \ \partial\Omega.$$

by $v = \mathscr{T} g$. This operator $\mathscr{T} = (\mathscr{T}1, \mathscr{T}_2, \mathscr{T}_3)$ is the inverse of the divergence operator such that

$$\mathscr{T} : \Big\{ g \in L^p; \int_\Omega g dx = 0 \Big\} \to W_0^{1,p}(\Omega),$$

with the following boundedness property:

$$\| \mathscr{T}(g) \|_{W^{1,p}(\Omega)} \leq C \| g \|_{L^p(\Omega)}.$$

If in addition g can be written as $g = \nabla \cdot h$ for a certain $h \in L^r$ with $h \cdot \hat{n} = 0$ on $\partial\Omega$, then

$$\| \mathscr{T}(g) \|_{L^r(\Omega)} \leq C \| h \|_{L^r(\Omega)}.$$

We will use this operator to obtain higher integrability of ρ. By extending (2.10) to zero outside Ω and regularizing it, we have,

$$\partial_t b(\rho)_\epsilon + \nabla \cdot (b(\rho)_\epsilon u) + \Big(\big[b^{'}(\rho)\rho - b(\rho) \big] \nabla \cdot u \Big)_\epsilon = r_\epsilon, \tag{4.1}$$

where $b(\rho)_\epsilon = b(\rho) \star g_\epsilon$. As proved in [13], we have

$$r_\epsilon \to 0 \ \text{ in } \ L^2(R^3 \times (0,T)). \tag{4.2}$$

We are now ready to prove the following result.

Lemma 4.1 *Let $\gamma > \frac{3}{2}$ and $\{\rho, u, f, \eta, \sigma\}$ be a weak solution of the system (1.10). Then, there exists $\theta > 0$, depending only γ, such that*

$$\| \rho \|_{L^{\gamma+\theta}(\Omega \times (0,T))} \leq C(T).$$

Proof: We take a test function of the form

$$\phi_i = \chi(t) \mathscr{T}_i \Big[b(\rho)_\epsilon - \oint_\Omega b(\rho)_\epsilon dy \Big], \quad \oint_\Omega b(\rho)_\epsilon dy = \frac{1}{|\Omega|} \int_\Omega b(\rho)_\epsilon dy, \quad \chi \in \mathscr{D}(0,T)$$

and test it against (1.10b). Then, with the aid of (4.1),

$$\begin{aligned}
&\int_0^T \int_\Omega \chi \rho^\gamma b(\rho)_\epsilon dx dt \\
&= \int_0^T \int_\Omega \chi \rho^\gamma \Big[\oint_\Omega b(\rho)_\epsilon dy \Big] dx dt - \int_0^T \int_\Omega \chi_t \rho u \cdot \mathscr{T} \Big[b(\rho)_\epsilon - \oint_\Omega b(\rho)_\epsilon dy \Big] dx dt \\
&+ \int_0^T \int_\Omega \chi \rho u \cdot \mathscr{T} \Big[\big((b^{'}(\rho)\rho - b(\rho)) \nabla \cdot u \big)_\epsilon - \oint_\Omega \big((b^{'}(\rho)\rho - b(\rho)) \nabla \cdot u \big)_\epsilon dy \Big] dx dt \\
&- \int_0^T \int_\Omega \chi \rho u \cdot \mathscr{T} \Big[r_\epsilon - \oint_\Omega r_\epsilon dy \Big] dx dt + \int_0^T \int_\Omega \chi \rho u \cdot \mathscr{T} \Big[\nabla \cdot \big(b(\rho)_\epsilon u \big) \Big] dx dt \\
&- \int_0^T \int_\Omega \chi \rho u_i u_j \partial_i \mathscr{T}_j \Big[b(\rho)_\epsilon - \oint_\Omega b(\rho)_\epsilon dy \Big] dx dt + \int_0^T \int_\Omega \chi \partial_i u_j \partial_i \mathscr{T}_j \Big[b(\rho)_\epsilon - \oint_\Omega b(\rho)_\epsilon dy \Big] dx dt \\
&+ \int_0^T \int_\Omega \chi \nabla \cdot u \Big[b(\rho)_\epsilon - \oint_\Omega b(\rho)_\epsilon dy \Big] dx dt - \int_0^T \int_\Omega \chi \eta^2 \Big[b(\rho)_\epsilon - \oint_\Omega b(\rho)_\epsilon dy \Big] dx dt \\
&+ \int_0^T \int_\Omega \chi \sigma_{ij} \partial_i \mathscr{T}_j \Big[b(\rho)_\epsilon - \oint_\Omega b(\rho)_\epsilon dy \Big] dx dt - \int_0^T \int_\Omega \chi \eta \Big[b(\rho)_\epsilon - \oint_\Omega b(\rho)_\epsilon dy \Big] dx dt \\
&= I_1 + \cdots + I_{11}.
\end{aligned}$$

We now estimate $I_1, \cdots, I_{11}$. For details, see [10].

- $I_1 \lesssim C(T)$.

- $I_2 \lesssim \|\rho u\|_{L^\infty(0,T);L^{\frac{2\gamma}{\gamma+1}}(\Omega)} \|b(\rho)_\epsilon\|_{L^\infty(0,T;L^{\frac{6\gamma}{5\gamma-3}}(\Omega))} \le C(T)\|b(\rho)_\epsilon\|_{L^\infty(0,T;L^{\frac{6\gamma}{5\gamma-3}}(\Omega))}$.

- $I_3 \lesssim \|\rho\|_{L^\infty(0,T;L^\gamma(\Omega))}\|\nabla u\|^2_{L^2(\Omega\times(0,T))}\|b(\rho)_\epsilon\|_{L^\infty(0,T;L^{\frac{3\gamma}{2\gamma-3}}(\Omega))} \le C(T)\|b(\rho)_\epsilon\|_{L^\infty(0,T;L^{\frac{3\gamma}{2\gamma-3}}(\Omega))}$.

- $I_4 \lesssim \|\rho u\|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega))}\|r_\epsilon\|_{L^2(\Omega\times(0,T))} \le C(T)\|r_\epsilon\|_{L^2(\Omega\times(0,T))}$.

- $I_5 + I_6 \lesssim \|\rho\|_{L^\infty(0,T;L^\gamma(\Omega))}\|\nabla u\|^2_{L^2(\Omega\times(0,T))}\|b(\rho)_\epsilon\|_{L^\infty(0,T;L^{\frac{3\gamma}{2\gamma-3}}(\Omega))} \le C(T)\|b(\rho)_\epsilon\|_{L^\infty(0,T;L^{\frac{3\gamma}{2\gamma-3}}(\Omega))}$.

- $I_7 + I_8 \lesssim \|\nabla u\|_{L^2(\Omega\times(0,T))}\|b(\rho)_\epsilon\|_{L^2(\Omega\times(0,T))} \le C(T)\|b(\rho)_\epsilon\|_{L^2(\Omega\times(0,T))}$.

- $I_9 + I_{10} + I_{11} \lesssim \Big(\|\eta\|^2_{L^2(0,T;L^6(\Omega))} + \|\sigma\|_{L^1(0,T;L^3(\Omega))} + \|\eta\|_{L^1(0,T;L^3(\Omega))}\Big)\|b(\rho)_\epsilon\|_{L^\infty(0,T;L^{\frac{3}{2}}(\Omega))}$
$\le C(T)\|b(\rho)_\epsilon\|_{L^\infty(0,T;L^{\frac{3}{2}}(\Omega))}$.

In sum,

$$\begin{aligned}\int_0^T\int_\Omega \chi\rho^\gamma(b(\rho))_\epsilon dxdt \le C(T) &+ \|b(\rho)_\epsilon\|_{L^\infty(0,T;L^{\frac{6\gamma}{5\gamma-3}}(\Omega))} + \|b(\rho)_\epsilon\|_{L^\infty(0,T;L^{\frac{3\gamma}{2\gamma-3}}(\Omega))} \\ &+ \|b(\rho)_\epsilon\|_{L^\infty(0,T;L^{\frac{3}{2}}(\Omega))} + \|b(\rho)_\epsilon\|_{L^2(\Omega\times(0,T))} + \|r_\epsilon\|_{L^2(\Omega\times(0,T))}.\end{aligned} \tag{4.3}$$

By taking the limit in $\epsilon \to 0$,

$$\begin{aligned}\int_0^T\int_\Omega \chi\rho^\gamma b(\rho) dxdt \le C(T) &+ \|b(\rho)\|_{L^\infty(0,T;L^{\frac{6\gamma}{5\gamma-3}}(\Omega))} + \|b(\rho)\|_{L^\infty(0,T;L^{\frac{3\gamma}{2\gamma-3}}(\Omega))} \\ &+ \|b(\rho)\|_{L^\infty(0,T;L^{\frac{3}{2}}(\Omega))} + \|b(\rho)\|_{L^2(\Omega\times(0,T))}.\end{aligned} \tag{4.4}$$

We approximate $z \mapsto z^\theta$ by a sequence of $\{b_n\}$ in (2.10), and approximate χ to the identity function of $(0,T)$. Then,

$$\begin{aligned}\int_0^T\int_\Omega \rho^{\gamma+\theta} dxdt \le C(T) &+ \|\rho^\theta\|_{L^\infty(0,T;L^{\frac{6\gamma}{5\gamma-3}}(\Omega))} + \|\rho^\theta\|_{L^\infty(0,T;L^{\frac{3\gamma}{2\gamma-3}}(\Omega))} \\ &+ \|\rho^\theta\|_{L^\infty(0,T;L^{\frac{3}{2}}(\Omega))} + \|\rho^\theta\|_{L^2(\Omega\times(0,T))}.\end{aligned} \tag{4.5}$$

We note that $\frac{3}{2}, \frac{6\gamma}{5\gamma-3} < \frac{3\gamma}{2\gamma-3}$. The relation between $\frac{3\gamma}{2\gamma-3}$ and 2 depends on the range of γ: $\frac{3\gamma}{2\gamma-3} > 2$ for $\gamma < 6$, $\frac{3\gamma}{2\gamma-3} \le 2$ for $\gamma \ge 6$. In either cases, we take θ such that

$$\frac{3\gamma}{2\gamma-3}\theta \le \gamma. \tag{4.6}$$

Then,

$$\int_0^T\int_\Omega \rho^{\gamma+\theta} dxdt \le C(T) \tag{4.7}$$

which completes the proof. ■

Remark 3 *From (4.6), the best possible θ is $\frac{2}{3}\gamma - 1$, and higher integrability of ρ can be obtained by choosing appropriate cut-off functions in (2.10) in the spirit of Feireisl [10]. If $\gamma \geq \frac{9}{5}$, then $\gamma + \theta \geq 2$. In [13], Lions used this idea in order to show higher integrability of ρ by multiplying (3.7) by ρ^θ.*

4.2 Limit of the effective viscous flux

We now study the limit of the so called the effective viscous flux, $\rho^\gamma - 2\nabla \cdot u$. In the formal proof, we take $(-\Delta)^{-1}\nabla$ to (1.10b) and multiply by ρ. In this section, we instead take $\chi(t)\phi(x)\Delta^{-1}\nabla T_k(\rho^n)$ as test functions to (2.10) to obtain a better convergence result on the effective viscous flux. Here, the cut-off function T_k is defined as

$$T_k(z) = kT(\frac{z}{k}), \tag{4.8}$$

where $T \in C^\infty(\mathbb{R})$ satisfies

$$T(z) = z \ \text{ for } \ |z| \leq 1, \ \ T(z) = 2 \ \text{ for } \ z \geq 3, \ \ T \ \text{ is concave on } \ [0, \infty),$$

and $T(-z) = -T(z)$. From (2.10),

$$\partial_t T_k(\rho^n) + \nabla \cdot (T_k(\rho^n)u^n) + \big(T_k'(\rho^n)\rho^n - T_k(\rho^n)\big)\nabla \cdot u^n = 0 \tag{4.9}$$

holds in the sense of the distributions. Passing to the limit in (4.9), we have

$$\partial_t \overline{T_k(\rho)} + \nabla \cdot (\overline{T_k(\rho)}u) + \overline{(T_k'(\rho)\rho - T_k(\rho))\nabla \cdot u} = 0 \tag{4.10}$$

in the sense of distributions.

To take the limit to the effective viscous flux, we need the following lemma. For the proof, we refer the reader to Feireisl [10].

Lemma 4.2 *Suppose*

$$v_n \rightharpoonup v \ \text{ in } \ L^p(\Omega), \ \ w_n \rightharpoonup w \ \text{ in } \ L^q(\Omega),$$

with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1$. Then,

$$v_n R_{ij}(w_n) - w_n R_{ij}(v_n) \rightharpoonup vR_{ij}(w) - wR_{ij}(v) \ \text{ in } \ L^r(\Omega),$$

where $R_{ij} = \partial_i\partial_j\Delta^{-1}$.

Lemma 4.3 *Under the condition in Proposition 2.3, we have*

$$\lim_{n\to\infty} \int_0^T \int_\Omega \chi\phi\Big[(\rho^n)^\gamma - 2\nabla \cdot u^n\Big]T_k(\rho^n)dxdt = \int_0^T \int_\Omega \chi\phi\Big[\overline{\rho^\gamma} - 2\nabla \cdot u\Big]\overline{T_k(\rho)}dxdt$$

for $\chi \in \mathscr{D}(0,T)$ and $\phi \in \mathscr{D}(\Omega)$.

Proof: We take $\chi(t)\phi(x)\Delta^{-1}\nabla T_k(\rho^n)$ as test functions to (1.10b). Then,

$$\begin{aligned}
&\int_0^T\int_\Omega \chi\phi\Big[(\rho^n)^\gamma - 2\nabla\cdot u^n\Big]T_k(\rho^n)dxdt\\
&=\int_0^T\int_\Omega \chi\Big[\nabla\cdot u^n-(\rho^n)^\gamma\Big]\nabla\phi\cdot\Delta^{-1}\nabla T_k(\rho^n)dxdt\\
&+\int_0^T\int_\Omega \chi\Big[\nabla\phi\cdot\nabla u^n\cdot\Delta^{-1}\nabla T_k(\rho^n)-u_i^n\partial_j\phi\partial_j\Delta^{-1}\partial_i T_k(\rho^n)\Big]dxdt\\
&+\int_0^T\int_\Omega \chi u^n\cdot\nabla\phi T_k(\rho^n)dxdt-\int_0^T\int_\Omega \chi\rho^n u_i^n u_j^n\partial_j\phi\Delta^{-1}\partial_i T_k(\rho^n)dxdt\\
&-\int_0^T\int_\Omega \phi\rho^n u^n\cdot\Big[\chi_t\Delta^{-1}\nabla T_k(\rho^n)+\chi\Delta^{-1}\nabla\big[\big(T_k(\rho^n)-T_k^{'}(\rho^n)\rho^n\big)\nabla\cdot u^n\big]\Big]dxdt\\
&-\int_0^T\int_\Omega \chi\phi u_i^n\Big[T_k(\rho^n)R_{ij}(\rho^n u_j^n)-\rho^n u_j^n R_{ij}T_k(\rho^n)\Big]dxdt\\
&-\int_0^T\int_\Omega \chi(\eta^n)^2\nabla\phi\cdot\Delta^{-1}\nabla T_k(\rho^n)dxdt-\int_0^T\int_\Omega \chi\phi(\eta^n)^2T_k(\rho^n)dxdt\\
&+\int_0^T\int_\Omega \chi\sigma_{ij}\partial_i\phi\Delta^{-1}\partial_j T_k(\rho^n)dxdt+\int_0^T\int_\Omega \chi\phi\sigma_{ij}\partial_i\Delta^{-1}\partial_j T_k(\rho^n)dxdt\\
&-\int_0^T\int_\Omega \chi\eta\nabla\phi\cdot\Delta^{-1}\nabla T_k(\rho^n)dxdt-\int_0^T\int_\Omega \chi\phi\eta T_k(\rho^n)dxdt
\end{aligned}\tag{4.11}$$

We now take the limit of (4.11) for $n\to\infty$.

$$\begin{aligned}
&\int_0^T\int_\Omega \chi\phi\Big[\overline{\rho^\gamma}-2\nabla\cdot u\Big]\overline{T_k(\rho)}dxdt\\
&=\int_0^T\int_\Omega \chi\Big[\nabla\cdot u-\overline{\rho^\gamma}\Big]\nabla\phi\cdot\Delta^{-1}\nabla\overline{T_k(\rho)}dxdt\\
&+\int_0^T\int_\Omega \chi\Big[\nabla\phi\cdot\nabla u\cdot\Delta^{-1}\nabla\overline{T_k(\rho)}-u_i\partial_j\phi\partial_j\Delta^{-1}\partial_i\overline{T_k(\rho)}\Big]dxdt\\
&+\int_0^T\int_\Omega \chi u\cdot\nabla\phi\overline{T_k(\rho)}dxdt-\int_0^T\int_\Omega \chi\rho u_i u_j\partial_j\phi\Delta^{-1}\partial_i\overline{T_k(\rho)}dxdt\\
&-\int_0^T\int_\Omega \phi\rho u\cdot\Big[\chi_t\Delta^{-1}\nabla\overline{T_k(\rho)}+\chi\Delta^{-1}\nabla\overline{\big(T_k(\rho)-T_k^{'}(\rho)\rho\big)\nabla\cdot u}\Big]dxdt\\
&-\int_0^T\int_\Omega \chi\phi u_i\Big[\overline{T_k(\rho)}R_{ij}(\rho u_j)-\rho^n u_j R_{ij}\overline{T_k(\rho)}\Big]dxdt.\\
&-\int_0^T\int_\Omega \chi\eta^2\nabla\phi\cdot\Delta^{-1}\nabla\overline{T_k(\rho)}dxdt-\int_0^T\int_\Omega \chi\phi\eta^2\overline{T_k(\rho)}dxdt\\
&+\int_0^T\int_\Omega \chi\sigma_{ij}\partial_i\phi\Delta^{-1}\partial_j\overline{T_k(\rho)}dxdt+\int_0^T\int_\Omega \chi\phi\sigma_{ij}\partial_i\Delta^{-1}\partial_j\overline{T_k(\rho)}dxdt\\
&-\int_0^T\int_\Omega \chi\eta\nabla\phi\cdot\Delta^{-1}\nabla\overline{T_k(\rho)}dxdt-\int_0^T\int_\Omega \chi\phi\eta\overline{T_k(\rho)}dxdt,
\end{aligned}\tag{4.12}$$

where we use lemma 4.2 to show

$$
\begin{aligned}
&\lim_{n\to\infty}\int_0^T\int_\Omega \chi\phi u_i^n\Big[T_k(\rho^n)R_{ij}(\rho^n u_j^n)-\rho^n u_j^n R_{ij}T_k(\rho^n)\Big]dxdt\\
&=\int_0^T\int_\Omega \chi\phi u_i\Big[\overline{T_k(\rho)}R_{i,j}(\rho u_j)-\rho^n u_j R_{i,j}\overline{T_k(\rho)}\Big]dxdt.
\end{aligned}
\tag{4.13}
$$

We now take $\chi\phi\Delta^{-1}\nabla\overline{T_k(\rho)}$ as test functions to

$$
(\rho u)_t+\nabla\cdot(\rho u\otimes u)-\Delta u-\nabla(\nabla\cdot u)+\nabla\overline{\rho^\gamma}+\nabla\eta^2=\nabla\cdot\sigma-\nabla\eta
$$

and do the same calculation using (4.10). Then,

$$
\begin{aligned}
&\int_0^T\int_\Omega \chi\phi\Big[\overline{\rho^\gamma}-2\nabla\cdot u\Big]\overline{T_k(\rho)}dxdt\\
&=\int_0^T\int_\Omega \chi\Big[\nabla\cdot u-\overline{\rho^\gamma}\Big]\nabla\phi\cdot\Delta^{-1}\nabla\overline{T_k(\rho)}dxdt\\
&+\int_0^T\int_\Omega \chi\Big[\nabla\phi\cdot\nabla u\cdot\Delta^{-1}\nabla\overline{T_k(\rho)}-u_i\partial_j\phi\partial_j\Delta^{-1}\partial_i\overline{T_k(\rho)}\Big]dxdt\\
&+\int_0^T\int_\Omega \chi u\cdot\nabla\phi\overline{T_k(\rho)}dxdt-\int_0^T\int_\Omega \chi\rho u_i u_j\partial_j\phi\Delta^{-1}\partial_i\overline{T_k(\rho)}dxdt\\
&-\int_0^T\int_\Omega \phi\rho u\cdot\Big[\chi_t\Delta^{-1}\nabla\overline{T_k(\rho)}+\chi\Delta^{-1}\nabla\overline{\big(T_k(\rho)-T_k'(\rho)\rho\big)\nabla\cdot u}\Big]dxdt\\
&-\int_0^T\int_\Omega \chi\phi u_i\Big[\overline{T_k(\rho)}R_{ij}(\rho u_j)-\rho^n u_j R_{ij}\overline{T_k(\rho)}\Big]dxdt\\
&-\int_0^T\int_\Omega \chi\eta^2\nabla\phi\cdot\Delta^{-1}\nabla\overline{T_k(\rho)}dxdt-\int_0^T\int_\Omega \chi\phi\eta^2\overline{T_k(\rho)}dxdt\\
&+\int_0^T\int_\Omega \chi\sigma_{ij}\partial_i\phi\Delta^{-1}\partial_j\overline{T_k(\rho)}dxdt+\int_0^T\int_\Omega \chi\phi\sigma_{ij}\partial_i\Delta^{-1}\partial_j\overline{T_k(\rho)}dxdt\\
&-\int_0^T\int_\Omega \chi\eta\nabla\phi\cdot\Delta^{-1}\nabla\overline{T_k(\rho)}dxdt-\int_0^T\int_\Omega \chi\phi\eta\overline{T_k(\rho)}dxdt.
\end{aligned}
\tag{4.14}
$$

Therefore, by comparing (4.12) and (4.14), we complete the proof. ■

Corollary 4.4 *Let ρ be a weak limit of the sequence $\{\rho^n\}$. Then,*

$$
\limsup_{n\to\infty}\big\|T_k(\rho^n)-T_k(\rho)\big\|_{L^{\gamma+1}(\Omega\times(0,T))}\le C(T).
\tag{4.15}
$$

Proof: As $z\mapsto z^\gamma$ is convex, T_k is concave, and

$$
(z^\gamma-y^\gamma)(T_k(z)-T_k(y))\ge|T_k(z)-T_k(y)|^{\gamma+1},
$$

we have

$$
\begin{aligned}
&\limsup_{n\to\infty}\int_0^T\int_\Omega |T_k(\rho^n)-T_k(\rho)|^{\gamma+1}dxdt\\
&\leq \lim_{n\to\infty}\int_0^T\int_\Omega ((\rho^n)^\gamma-\rho^\gamma)(T_k(\rho^n)T_k(\rho))dxdt\\
&\leq \lim_{n\to\infty}\int_0^T\int_\Omega ((\rho^n)^\gamma-\rho^\gamma)(T_k(\rho^n)-T_k(\rho))dxdt+\int_0^T\int_\Omega(\overline{\rho^\gamma}-\rho^\gamma)(T_k(\rho)-\overline{T_k(\rho)})dxdt\\
&= \lim_{n\to\infty}\int_0^T\int_\Omega (\rho^n)^\gamma T_k(\rho^n)-\overline{\rho^\gamma}\overline{T_k(\rho)}dxdt
\end{aligned}
\tag{4.16}
$$

By lemma 4.3, the last term in (4.16) can be estimated as follows.

$$
\begin{aligned}
&\lim_{n\to\infty}\int_0^T\int_\Omega (\rho^n)^\gamma T_k(\rho^n)-\overline{\rho^\gamma}\overline{T_k(\rho)}dxdt\\
&=\lim_{n\to\infty}\int_0^T\int_\Omega (\nabla\cdot u^n)T_k(\rho^n)-(\nabla\cdot u)\overline{T_k(\rho)}dxdt\\
&=\lim_{n\to\infty}\int_0^T\int_\Omega \Big[T_k(\rho^n)-T_k(\rho)+T_k(\rho)-\overline{T_k(\rho)}\Big](\nabla\cdot u^n)dxdx\\
&\leq 2\sup_n\|\nabla\cdot u^n\|_{L^2(\Omega\times(0,T))}\limsup_{n\to\infty}\|T_k(\rho^n)-T_k(\rho)\|_{L^2(\Omega\times(0,T))}.
\end{aligned}
\tag{4.17}
$$

Since $\gamma+1>2$, (4.16) and (4.17) implies (4.15). ■

4.3 Proof of Proposition 2.3. (iv)

To show that ρ and u solve (1.10a) in the sense of renormalized solutions, we regularize (4.10) to get

$$
\partial_t\Big[\overline{T_k(\rho)}\Big]_\epsilon+\nabla\cdot\Big[\big(\overline{T_k(\rho)}u\big)_\epsilon\Big]+\Big[\overline{(T_k'(\rho)\rho-T_k(\rho))\nabla\cdot u}\Big]_\epsilon=r_\epsilon,
\tag{4.18}
$$

where $r_\epsilon\to 0$ in $L^2(R^3\times(0,T))$. We multiply (4.18) by $b'\Big[\Big(\overline{T_k(\rho)}\Big)_\epsilon\Big]$ and take $\epsilon\to 0$. Then,

$$
\begin{aligned}
&\partial_t b\Big(\overline{T_k(\rho)}\Big)+\nabla\cdot\Big[b\Big(\overline{T_k(\rho)}\Big)u\Big]+\Big[b'\Big(\overline{T_k(\rho)}\Big)\overline{T_k(\rho)}-b\Big(\overline{T_k(\rho)}\Big)\Big](\nabla\cdot u)\\
&=b'\Big(\overline{T_k(\rho)}\Big)\overline{(T_k(\rho)-T_k'(\rho)\rho)\nabla\cdot u}
\end{aligned}
\tag{4.19}
$$

in the sense of distributions. By the interpolation,

$$
\|T_k(\rho^n)-T_k'(\rho^n)\rho^n\|_{L^2(\Omega\times(0,T))}\leq\|T_k(\rho^n)-T_k'(\rho^n)\rho^n\|^\alpha_{L^1(\Omega\times(0,T))}\|T_k(\rho^n)-T_k'(\rho^n)\rho^n\|^{1-\alpha}_{L^{\gamma+1}(\Omega\times(0,T))},
$$

where $\alpha=\frac{\gamma-1}{\gamma}$. Since

$$
\|T_k(\rho^n)-T_k'(\rho^n)\rho^n\|_{L^1(\Omega\times(0,T))}\lesssim k^{1-\gamma}\sup_n\|\rho^n\|^\gamma_{L^\gamma(\Omega\times(0,T))}
$$

and

$$
\limsup_{n\to\infty}\|T_k(\rho^n)-T_k'(\rho^n)\rho^n\|_{L^{\gamma+1}(\Omega\times(0,T))}\leq C(T)
$$

from corollary 4.4, we have

$$b'\left(\overline{T_k(\rho)}\right)\overline{(T_k(\rho)-T_k'(\rho)\rho)\nabla\cdot u}\to 0 \ \text{ in } \ L^1(\Omega\times(0,T))$$

as $k\to\infty$. This completes Proposition 2.3 (iv).

4.4 Proof of Proposition 2.3. (v)

We introduce a family of functions L_k such that

$$L_k(z)=\begin{cases} z\log z \ \text{ for } \ 0\le z<k,\\ z\log k+z\displaystyle\int_k^z \frac{T_k(s)}{s^2}ds \ \text{ for } \ z\ge k.\end{cases}$$

Setting

$$L_k(z)=\beta_k(z)+b_k(z),\ \ |b_k(z)|\le C(k),\ \ b_k'(z)z-b_k(z)=T_k(z),$$

we have

$$\partial_t L_k(\rho^n)+\nabla\cdot(L_k(\rho^n)u^n)+T_k(\rho^n)\nabla\cdot u^n=0. \tag{4.20}$$

On the other hand, by (1.10a),

$$\partial_t L_k(\rho)+\nabla\cdot(L_k(\rho)u)+T_k(\rho)\nabla\cdot u=0. \tag{4.21}$$

By (4.20) and (4.21), for $\phi\in C_c^\infty(\Omega)$

$$\begin{aligned}&\int_\Omega\left[L_k(\rho^n)-L_k(\rho)\right](t)\phi dx\\ &=\int_\Omega\left[L_k(\rho_0^n)-L_k(\rho_0)\right]\phi dx\\ &+\int_0^t\int_\Omega\left[L_k(\rho^n)u^n-L_k(\rho)u\right]\cdot\nabla\phi+\left[T_k(\rho)\nabla\cdot u-T_k(\rho^n)\nabla\cdot u^n\right]\phi dxds\end{aligned} \tag{4.22}$$

Passing to the limit in (4.22) for $n\to\infty$,

$$\begin{aligned}&\int_\Omega\left[\overline{L_k(\rho)}-L_k(\rho)\right](t)\phi dx\\ &=\int_0^t\int_\Omega\left[\overline{L_k(\rho)}u-L_k(\rho)u\right]\cdot\nabla\phi dxds+\lim_{n\to\infty}\int_0^t\int_\Omega\left[T_k(\rho)\nabla\cdot u-T_k(\rho^n)\nabla\cdot u^n\right]\phi dxds\end{aligned} \tag{4.23}$$

By approximating ϕ to the identity function of Ω,

$$\begin{aligned}&\int_\Omega\left[\overline{L_k(\rho)}-L_k(\rho)\right](t)dx\\ &=\int_0^t\int_\Omega T_k(\rho)(\nabla\cdot u)dxdt-\lim_{n\to\infty}\int_0^t\int_\Omega T_k(\rho^n)(\nabla\cdot u^n)dxds\\ &\le\int_0^t\int_\Omega\left[T_k(\rho)-\overline{T_k(\rho)}\right](\nabla\cdot u)dxds.\end{aligned} \tag{4.24}$$

We approximate $z\log z$ by $L_k(z)$. By corollary 4.4, with $\gamma+1>2$, the right-hand side of (4.24) tends to 0 as $k\to\infty$. Therefore,

$$\overline{\rho\log\rho}(t)=(\rho\log\rho)(t),\quad \text{for all}\quad t\in[0,T] \tag{4.25}$$

which implies the strong convergence of $\{\rho^n\}$ in $L^1(\Omega\times(0,T))$. This completes the proof of Proposition 2.3 (v). ■

5 Construction of approximate sequences

We now construct an approximate sequence of solutions to the system (1.10) so that we can apply the compactness argument (Proposition 2.3) to obtain a weak solution. This section consists of three parts: the regularization of $\rho\frac{d}{dt} + \rho u\cdot\nabla$ in (1.10b), nonlinear damping to the equation of ρ and η, and the truncation of ρ^γ and η^2. Collecting all these steps, we can prove Theorem 2.2.

5.1 Smoothing $\rho\frac{d}{dt} + \rho u\cdot\nabla$

We consider the following system of equations:

$$\rho_t + \nabla\cdot(\rho u) = 0, \tag{5.1a}$$

$$(\rho_\epsilon u)_t + \nabla\cdot((\rho u)_\epsilon \otimes u) - \Delta u - \nabla(\nabla\cdot u) + \nabla\rho^\gamma + \nabla\eta^2 = \nabla\cdot\sigma_\epsilon - \nabla\eta_\epsilon, \tag{5.1b}$$

$$f_t + \nabla\cdot(u_\epsilon f) + \nabla_\tau\cdot(P_{\tau^\perp}(\nabla_x u_\epsilon \tau)f) - \Delta_\tau f - \Delta f = 0. \tag{5.1c}$$

In Section 5.2 will be shown that there exists a solution $\{\rho, u, f, \eta, \sigma\}$ to (5.1) such that $\rho \in L^p(\Omega\times(0,T))$, $p = \frac{5}{3}\gamma$. By the energy identity

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\Big[\frac{\rho_\epsilon|u|^2}{2} + \frac{\rho^\gamma}{\gamma-1} + \eta^2 + \psi\Big]dx + 4\int_\Omega\int_{S^2}|\nabla_\tau\sqrt{f}|^2 d\tau dx + 4\int_\Omega\int_{S^2}|\nabla\sqrt{f}|^2 d\tau dx\\ &+\int_\Omega\Big[|\nabla u|^2 + |\nabla\cdot u|^2 + 2|\nabla\eta|^2\Big]dx = 0,\end{aligned} \tag{5.2}$$

we can obtain the following a priori estimates:

$$\begin{aligned}&\rho\in C([0,T];L^1(\Omega))\cap L^\infty(0,T;L^\gamma(\Omega)),\quad \rho_\epsilon|u|^2\in L^\infty(0,T;L^1(\Omega)),\quad u\in L^2(0,T;H^1(\Omega))\\ &\rho_\epsilon u\in L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega))\cap L^2(0,T;L^r(\Omega)),\quad \frac{1}{r} = \frac{1}{6}+\frac{1}{\gamma}\\ &f\in L^2(0,T;L^{\frac{6}{5}}(\Omega\times S^2)),\quad \eta\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)),\\ &\sigma\in L^1(0,T;L^3(\Omega))\cap L^\infty(0,T;L^2(\Omega)),\quad \nabla\sigma\in L^1(0,T;L^{\frac{3}{2}}(\Omega))\cap L^2(0,T;L^{\frac{4}{3}}(\Omega)).\end{aligned} \tag{5.3}$$

If $u\in L^2(0,T;H^1(\Omega))$, (5.1c) has a smooth solution, and the entropy ψ and its equation make sense. Thus, we only focus on the existence of a solution $\{\rho,u,\eta\}$ under the assumption that f is smooth. By taking $\theta = \frac{2}{3}\gamma - 1$ in Lemma 4.1, we have

$$\rho\in L^p(\Omega\times(0,T)),\quad p = \frac{5}{3}\gamma - 1. \tag{5.4}$$

5.2 Nonlinear damping

We wish to build solutions of (5.1) as a limit of the following system of equations:

$$\rho_t + \nabla\cdot(\rho u) + \delta\rho^q = 0, \tag{5.5a}$$

$$(\rho_\epsilon u)_t + \nabla\cdot((\rho u)_\epsilon\otimes u) - \Delta u - \nabla(\nabla\cdot u) + \nabla[\rho^\gamma + \eta^2] + \delta[(\rho^q)_\epsilon + (\eta^m)_\epsilon]u = \nabla\cdot\sigma_\epsilon - \nabla\eta_\epsilon, \tag{5.5b}$$

$$f_t + \nabla\cdot(u_\epsilon f) + \nabla_\tau\cdot(P_{\tau^\perp}(\nabla_x u_\epsilon\tau)f) - \Delta_\tau f - \Delta f = 0, \tag{5.5c}$$

$$\eta_t + \nabla\cdot(\eta u) - \Delta\eta + \delta\eta^m = 0. \tag{5.5d}$$

where $q > \gamma+1$ and $m > 3$, with $m \geq q$, will be determined later. We note that we add a nonlinear damping to the equation of η, not the equation of f. We will prove the existence of a solution $\{\rho, u, f, \eta, \sigma\}$ to (5.5) satisfying

$$
\begin{aligned}
&u \in L^s(0,T;W^{1,s}(\Omega)), \quad u_t \in L^s(0,T;W^{-1,s}(\Omega)), \quad \rho,\eta \in C([0,T];L^s(\Omega)), \quad 1 \leq s < \infty \\
&\rho_\epsilon \in C^1([0,T];C^k(\bar{\Omega})), \quad (\rho u)_\epsilon \in C([0,T];C^k(\bar{\Omega})), \quad k \geq 0
\end{aligned}
\tag{5.6}
$$

in Section 5.3. From (5.5), we have the following energy identity:

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big[\frac{\rho_\epsilon |u|^2}{2} + \frac{\rho^\gamma}{\gamma-1} + \eta^2 + \psi\Big]dx + 4\int_\Omega\int_{S^2} |\nabla_\tau \sqrt{f}|^2 d\tau dx + 4\int_\Omega\int_{S^2}|\nabla\sqrt{f}|^2 d\tau dx \\
&+ \delta\int_0^T\int_\Omega (\rho^q)_\epsilon |u|^2 dxdt + \delta\int_0^T\int_\Omega \rho^{q+\gamma-1}dxdt + \delta\int_0^T\int_\Omega (\eta^m)_\epsilon |u|^2 dxdt \\
&+ \delta\int_0^T\int_\Omega \eta^{m+1}dxdt + \int_\Omega \Big[|\nabla u|^2 + |\nabla\cdot u|^2 + 2|\nabla\eta|^2\Big]dx = 0.
\end{aligned}
\tag{5.7}
$$

Therefore, we have

$$
\begin{aligned}
&\rho \in C([0,T];L^1(\Omega)) \cap L^\infty(0,T;L^\gamma(\Omega)), \quad \rho_\epsilon |u|^2 \in L^\infty(0,T;L^1(\Omega)), \quad u \in L^2(0,T;H^1(\Omega)) \\
&\delta\int_0^T\int_\Omega (\rho^q)_\epsilon |u|^2 dxdt \leq C, \quad \delta\int_0^T\int_\Omega \rho^{q+\gamma-1}dxdt \leq C, \\
&\rho_\epsilon u \in L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega)) \cap L^2(0,T;L^r(\Omega)), \quad \frac{1}{r} = \frac{1}{6} + \frac{1}{\gamma} \\
&f \in L^2(0,T;L^{\frac{6}{5}}(\Omega\times S^2)), \quad \eta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \\
&\delta\int_0^T\int_\Omega (\eta^m)_\epsilon |u|^2 dxdt \leq C, \quad \delta\int_0^T\int_\Omega \eta^{m+1}dxdt \leq C, \\
&\sigma \in L^1(0,T;L^3(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \quad \nabla\sigma \in L^1(0,T;L^{\frac{3}{2}}(\Omega)) \cap L^2(0,T;L^{\frac{4}{3}}(\Omega)).
\end{aligned}
\tag{5.8}
$$

Moreover, for a sufficiently small δ,

$$
\rho_\epsilon \geq \frac{1}{C} \ \text{ on } \ \bar{\Omega}\times[0,T]
\tag{5.9}
$$

holds. To prove this, we integrate (5.5a) over Ω.

$$
\begin{aligned}
\int_\Omega \rho(t)dx &= \int_\Omega \rho_{0,\epsilon}dx - \delta\int_0^t\int_\Omega \rho^q(s)dxds \geq \int_\Omega \rho_0 dx - \delta\int_0^T\int_\Omega \rho^q(t)dxdt \\
&\geq \int_\Omega \rho_0 dx - \delta\int_0^T\int_\Omega \rho^{q+\alpha}\mathbb{I}_{\{\rho>1\}}dxdt - \delta\int_0^T\int_\Omega \rho\mathbb{I}_{\{\rho<1\}}dxdt \\
&\geq \int_\Omega \rho_0 dx - \delta T^{1-\frac{q+\alpha}{q+\gamma-1}}\Big[\int_0^T\int_\Omega \rho^{q+\gamma-1}dxdt\Big]^{\frac{q+\alpha}{q+\gamma-1}} - \delta T\int_\Omega \rho_0 dx \\
&\geq \int_\Omega \rho_0 dx - C(\delta T)^{\frac{q+\alpha}{q+\gamma-1}} - \delta T\int_\Omega \rho_0 dx
\end{aligned}
$$

where $0 < \alpha < \gamma-1$. By taking a sufficiently small δ, we have

$$
\int_\Omega \rho(t)dx \geq \frac{1}{2}\int_\Omega \rho_0 dx
$$

from which we have

$$\rho_\epsilon(x,t) = \int_\Omega \rho(y,t)k_\epsilon(x-y)dy \geq \inf_{z\in 2\overline{\Omega}} k_\epsilon(z)\int_\Omega \rho(y)dy \geq \frac{1}{C}. \tag{5.10}$$

Since $\rho_\epsilon|u|^2 \in L^\infty(0,T;L^1(\Omega))$, $u \in L^\infty(0,T;L^2(\Omega))$. Moreover, from $\rho \in L^\infty(0,T;L^1(\Omega))$, $\rho_\epsilon \in L^\infty(\Omega\times(0,T))$.

Next, we obtain higher integrability of ρ by using (3.9), with replacing the multiplication of ρ by ρ^θ to (3.8). We also have additional terms from the damping terms.

$$\begin{aligned}
\rho^{\gamma+\theta} &= \frac{d}{dt}\Big[\rho^\theta(-\Delta)^{-1}\nabla\cdot(\rho_\epsilon u)\Big] + \nabla\cdot\Big[u\rho^\theta(-\Delta)^{-1}\nabla\cdot(\rho_\epsilon u)\Big] + 2(\nabla\cdot u)\rho^\theta \\
&\quad + (\theta-1)(\nabla\cdot u)\rho^\theta(-\Delta)^{-1}\nabla\cdot(\rho_\epsilon u) - \rho^\theta(-\Delta)^{-1}\nabla\cdot(\nabla\cdot\sigma_\epsilon - \nabla\eta_\epsilon) \\
&\quad + \rho^\theta\Big[R_iR_j((\rho u_i)_\epsilon u_j) - u_jR_iR_j(\rho_\epsilon u_i)\Big] + \rho^\theta\eta^2 \\
&\quad + \delta\rho^\theta(-\Delta)^{-1}\nabla\cdot((\rho^q)_\epsilon u) + \theta\delta\rho^{q+\theta-1}(-\Delta)^{-1}\nabla\cdot(\rho_\epsilon u) + \delta\rho^\theta(-\Delta)^{-1}\nabla\cdot((\eta^m)_\epsilon u).
\end{aligned} \tag{5.11}$$

Integrating (5.11) over $\Omega\times(0,T)$, we have

$$\begin{aligned}
\int_0^T\int_\Omega \rho^{\gamma+\theta}dxdt &\lesssim C(T) + \int_0^T\int_\Omega \big|(\nabla\cdot u)\rho^\theta\big|dxdt + \int_0^T\int_\Omega \big|(\nabla\cdot u)\rho^\theta(-\Delta)^{-1}\nabla\cdot(\rho_\epsilon u)\big|dxdt \\
&\quad + \int_0^T\int_\Omega |\rho^\theta(-\Delta)^{-1}\nabla\cdot(\nabla\cdot\sigma_\epsilon - \nabla\eta_\epsilon)|dxdt + \int_0^T\int_\Omega |\rho^\theta\eta^2\big|dxdt \\
&\quad + \int_0^T\int_\Omega \Big|\rho^\theta\Big[R_iR_j((\rho u_i)_\epsilon u_j) - u_jR_iR_j(\rho_\epsilon u_i)\Big]\Big|dxdt \\
&\quad + \delta\int_0^T\int_\Omega \big|\rho^\theta(-\Delta)^{-1}\nabla\cdot((\rho^q)_\epsilon u)\big|dxdt \\
&\quad + \delta\int_0^T\int_\Omega \big|\rho^\theta(-\Delta)^{-1}\nabla\cdot((\eta^m)_\epsilon u)\big|dxdt \\
&\quad + \delta\int_0^T\int_\Omega \big|\rho^{q+\theta-1}(-\Delta)^{-1}\nabla\cdot(\rho_\epsilon u)\big|dxdt.
\end{aligned} \tag{5.12}$$

Let $\theta = \frac{2}{3}\gamma$. We only provide estimations involving δ. The rest of them are easily by the regularization. First,

$$\begin{aligned}
&\delta\|\rho^\theta(-\Delta)^{-1}\nabla\cdot((\rho^q)_\epsilon u)\|_{L^1(\Omega\times(0,T))} \\
&\lesssim \delta\|\rho\|^\theta_{L^\infty(0,T;L^\gamma(\Omega))}\|\sqrt{(\rho^q)_\epsilon}\|_{L^{\frac{2(q+\gamma-1)}{q}}(0,T;L^\infty(\Omega))}\|\sqrt{\rho_\epsilon}u\|_{L^2(0,T;L^2(\Omega))} \\
&\lesssim \delta\|\rho\|^\theta_{L^\infty(0,T;L^\gamma(\Omega))}\epsilon^{-\alpha}\|\sqrt{\rho^q}\|_{L^{\frac{2(q+\gamma-1)}{q}}(\Omega\times(0,T))}\|\sqrt{\rho_\epsilon}u\|_{L^2(0,T;L^2(\Omega))} \\
&\lesssim \epsilon^{-\alpha}\delta\delta^{-\frac{2}{2(q+\gamma-1)}}\delta^{-\frac{1}{2}} \to 0 \quad \text{as} \quad \delta\to 0,
\end{aligned} \tag{5.13}$$

where $0 < \frac{1}{2} + \frac{q}{2(q+\gamma-1)} < 1$. Similarly,

$$\delta\|\rho^\theta(-\Delta)^{-1}\nabla\cdot((\eta^m)_\epsilon u)\|_{L^1(\Omega\times(0,T))} \lesssim \epsilon^{-\beta}\delta\delta^{-\frac{1}{m}}\delta^{-\frac{1}{2}} \to 0 \ \text{ as } \ \delta\to 0, \tag{5.14}$$

where $0 < \frac{1}{2} + \frac{1}{m} < 1$. Next, we want to estimate $\delta\rho^{p+\theta-1}(-\Delta)^{-1}\nabla\cdot(\rho_\epsilon u)$. Since $\rho_\epsilon u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;L^6(\Omega))$ and

$$\delta\|\rho^{q+\theta-1}\|_{L^{\frac{q+\gamma-1}{q+\theta-1}}(\Omega\times(0,T))} \lesssim \delta\delta^{-\frac{q+\theta-1}{q+\gamma-1}},$$

we have

$$\delta\|\rho^{q+\theta-1}(-\Delta)^{-1}\nabla\cdot(\rho_\epsilon u)\|_{L^1(\Omega\times(0,T))} \to 0 \text{ as } \delta \to 0 \tag{5.15}$$

by taking $q > \gamma+1$, which is close to $\gamma+1$. Therefore, as $\delta \to 0$,

$$\begin{aligned} &\rho^\delta \text{ is bounded in } L^\infty(0,T;L^\gamma(\Omega)) \cap L^p(\Omega\times(0,T)), \quad p = \frac{5}{3}\gamma \\ &u^\delta \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \end{aligned} \tag{5.16}$$

so we can pass to the limit to (5.5) to obtain a solution of (5.1) by the compactness argument in Section 4. Indeed,

$$(\rho^\delta)_\epsilon \to \rho_\epsilon \in L^r(\Omega\times(0,T)), \quad u^\delta \to u \in L^r(0,T;L^2(\Omega)) \quad 1 \le r < \infty$$

so the analysis is easier than the case without ϵ.

5.3 Truncation of the pressure

In this section, we construct a solution of (5.5) by

$$\rho_t + \nabla\cdot(\rho u) + \delta\rho^q = 0, \tag{5.17a}$$

$$(\rho_\epsilon u)_t + \nabla\cdot((\rho u)_\epsilon \otimes u) - \Delta u - \nabla(\nabla\cdot u) + \nabla T(R) + \delta[(\rho^q)_\epsilon + (\eta^m)_\epsilon]u = \nabla\cdot\sigma_\epsilon - \nabla\eta_\epsilon, \tag{5.17b}$$

$$f_t + \nabla\cdot(u_\epsilon f) + \nabla_\tau\cdot(P_{\tau^\perp}(\nabla_x u_\epsilon \tau)f) - \Delta_\tau f - \Delta f = 0, \tag{5.17c}$$

$$\eta_t + \nabla(\eta u) - \Delta\eta + \delta\eta^m = 0. \tag{5.17d}$$

where $T(R) = (\rho\wedge R)^\gamma + (\eta\wedge R)^2$ and $f \wedge R = \min\{f, R\}$. We will show at the end of this section that there exists a smooth solution to (5.17) such that

$$\rho,\eta \in C([0,T];W^{1,s}(\Omega)), \quad u \in L^s(0,T;W^{2,s}(\Omega)), \quad u_t \in L^s(\Omega\times(0,T)), \quad 1 \le s < \infty. \tag{5.18}$$

We begin with the energy identity of (5.17).

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \Big[\frac{\rho_\epsilon|u|^2}{2} + A_R(\rho) + B_R(\eta) + \psi + \eta\Big]dx + 4\int_\Omega\int_{S^2}|\nabla_\tau\sqrt{f}|^2 d\tau dx \\ &+ 4\int_\Omega\int_{S^2}|\nabla\sqrt{f}|^2 d\tau dx + \delta\int_0^T\int_\Omega(\rho^q)_\epsilon|u|^2 dxdt + \delta\int_0^T\int_\Omega \rho^q A_R'(\rho)dxdt \\ &+ \delta\int_0^T\int_\Omega(\eta^m)_\epsilon|u|^2 dxdt + \delta\int_0^T\int_\Omega \rho^m B_R'(\eta)dxdt \\ &+ \int_\Omega\Big[|\nabla u|^2 + |\nabla\cdot u|^2 + 2|\nabla(\eta\wedge R)|^2\Big]dx \le 0, \end{aligned} \tag{5.19}$$

where

$$A_R(\rho) = \rho\int_0^\rho \frac{(t\wedge R)^\gamma}{t^2}dt, \;\; A_R'(\rho) = \frac{\gamma(\rho\wedge R)^{\gamma-1}}{\gamma-1}, \;\; B_R(\eta) = \eta\int_0^\eta \frac{(t\wedge R)^2}{t^2}dt, \;\; B_R'(\eta) = 2(\eta\wedge R).$$

From (5.19), we have

$$
\begin{aligned}
&\rho \in C([0,T];L^1(\Omega)), \quad \rho_\epsilon |u|^2 \in L^\infty(0,T;L^1(\Omega)), \quad u \in L^2(0,T;H^1(\Omega)) \\
&A_R(\rho) \in L^\infty(0,T;L^1(\Omega)), \quad A_R'(\rho)\rho^q \in L^1(\Omega \times (0,T)), \\
&\delta \int_0^T \int_\Omega (\rho^q)_\epsilon |u|^2 dxdt \le C, \quad \delta \int_0^T \int_\Omega \rho^{q+\gamma-1} dxdt \le C, \\
&\rho_\epsilon u \in L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega)) \cap L^2(0,T;L^r(\Omega)), \quad \frac{1}{r} = \frac{1}{6} + \frac{1}{\gamma} \\
&f \in L^2(0,T;L^{\frac{6}{5}}(\Omega \times S^2)), \quad B_R(\eta) \in L^\infty(0,T;L^2(\Omega)), \quad \eta \wedge R \in L^2(0,T;H^1(\Omega)), \\
&\sigma \in L^1(0,T;L^3(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \quad \nabla\sigma \in L^1(0,T;L^{\frac{3}{2}}(\Omega)) \cap L^2(0,T;L^{\frac{4}{3}}(\Omega)).
\end{aligned}
\tag{5.20}
$$

We still have $\rho_\epsilon \ge \frac{1}{C}$ uniformly in $R \ge 1$ as for sufficiently small δ so that $u \in L^\infty(0,T;L^2(\Omega))$ and $(\rho u)_\epsilon \in L^p(0,T;C^k(\bar{\Omega}))$. We are going to prove

$$
\rho \in L^\infty(0,T;L^s(\Omega)), \quad u \in L^s(0,t;W^{1,s}(\Omega)), \quad u_t \in L^s(0,T;W^{-1,s}(\Omega)), \quad 1 < s < \infty. \tag{5.21}
$$

For any $1 < r < \infty$,

$$
\frac{d}{dt}\rho^r + \nabla \cdot (u\rho^r) + \delta r \rho^{q+r-1} = (r-1)(\nabla \cdot u)\rho^r.
$$

By taking $s = \frac{q+r-1}{q-1}$,

$$
\|\rho\|^{\frac{r}{s}}_{L^\infty(0,T;L^r(\Omega))} + \|\rho^{q-1}\|_{L^s(\Omega\times(0,T))} \lesssim \|\nabla u\|_{L^s(\Omega\times(0,T))}. \tag{5.22}
$$

We can estimate η by using the same method. We set $a = \frac{m+r-1}{m-1}$. For $m \ge q$, $s > a$. Therefore,

$$
\|\eta\|^{\frac{r}{a}}_{L^\infty(0,T;L^r(\Omega))} + \|\eta^{m-1}\|_{L^a(\Omega\times(0,T))} \lesssim \|\nabla u\|_{L^s(\Omega\times(0,T))}. \tag{5.23}
$$

We now estimate u by using the estimation of the heat equation ([13]). By writing $(\rho u)_\epsilon \cdot \nabla u = \nabla \cdot ((\rho u)_\epsilon \otimes u) - (\nabla \cdot (\rho u)_\epsilon)u$,

$$
\begin{aligned}
&\|u_t\|_{L^s(0,T;W^{-1,s}(\Omega))} + \|u\|_{L^s(0,T;W^{1,s}(\Omega))} \\
&\lesssim C(T) + \|(\rho \wedge R)^\gamma\|_{L^s(\Omega\times(0,T))} + \|(\rho u)_\epsilon \otimes u\|_{L^s(\Omega\times(0,T))} + \|(\nabla\cdot(\rho u)_\epsilon)u\|_{L^s(\Omega\times(0,T))} \\
&+ \|\sigma_\epsilon\|_{L^s(\Omega\times(0,T))} + \|\eta_\epsilon\|_{L^s(\Omega\times(0,T))} + \|\eta\|^2_{L^{2s}(\Omega\times(0,T))} \\
&\lesssim C(T) + \|\rho\|^\gamma_{L^{\gamma s}(\Omega\times(0,T))} + \|\rho u\|_{L^{\frac{ls}{l-s}}(0,T;L^1(\Omega))} \|u\|_{L^l(0,T;L^s(\Omega))} + \|\eta\|^2_{L^{2s}(\Omega\times(0,T))}
\end{aligned}
\tag{5.24}
$$

for any $s \le l \le \infty$. Since $q > \gamma+1$ and $m > 3$, $(m-1)a > 2s$ and $(q-1)s > \gamma s$. Therefore, we can obtain (5.21) by (5.22), (5.23), and (5.24) provided that we can bound $\|\rho u\|_{L^{\frac{ls}{l-s}}(0,T;L^1(\Omega))}\|u\|_{L^l(0,T;L^s(\Omega))}$ in (5.24). This can be done by a bootstrap argument. First, we take $s = 2$, $l = \infty$. Then,

$$
\rho \in L^\infty(0,T;L^{q-1}(\Omega)), \quad \text{thus} \quad \rho \in L^\infty(0,T;L^\gamma(\Omega)). \tag{5.25}
$$

If $\gamma \ge 2$, then $\rho u \in L^\infty(0,T;L^1(\Omega))$ so that

$$
\begin{aligned}
\|u_t\|_{L^s(0,T;W^{-1,s}(\Omega))} + \|u\|_{L^s(0,T;W^{1,s}(\Omega))} \lesssim C(T) + \|\rho\|^\gamma_{L^{\gamma s}(\Omega\times(0,T))} + \|u\|_{L^s(\Omega\times(0,T))} \\
+ \|\eta\|^2_{L^{2s}(\Omega\times(0,T))}.
\end{aligned}
\tag{5.26}
$$

We note that we do not have $\nabla\cdot\sigma_\epsilon-\nabla\eta_\epsilon$ in the right-hand side of (5.26) due to the regularization. If $\gamma<2$, then we take $s=\frac{10q}{3q+4}$, $k=\frac{10q}{3q-6}$ so that $q=\frac{ks}{k-s}$. Then,

$$\rho\in L^\infty(0,T;L^w(\Omega)),\quad w=\frac{s-1}{q-1}. \tag{5.27}$$

Since $w>2$, $\rho u\in L^\infty(0,T;L^1(\Omega))$. Therefore, we can conclude as the previous case.

Now, we want to pass to the limit in $R\to\infty$. Since

$$\|\rho^\theta(\rho\wedge R)^\gamma-\rho^{\theta+\gamma}\|_{L^1(\Omega\times(0,T))}\le\int_0^T\int_\Omega\rho^{\theta+\gamma}\mathbb{I}_{\{\rho>R\}}\le\frac{1}{R}\int_0^T\int_\Omega\rho^{\theta+\gamma-1}dxdt\lesssim\frac{1}{R},$$

we have $\|\rho^\theta(\rho\wedge R)^\gamma-\rho^{\theta+\gamma}\|_{L^1(\Omega\times(0,T))}\to0$. Similarly, $\|\rho^\theta(\eta\wedge R)^2-\rho^\theta\eta^2\|_{L^1(\Omega\times(0,T))}\to0$. Therefore, we can recover a solution of (5.5) satisfying (5.6) in Section 5.2.

It remains to show (5.18). First, we show that $\rho\in L^\infty(\Omega\times(0,T))$. From the equation of ρ,

$$\frac{d}{dt}(\log\rho)+u\cdot\nabla\log\rho+\nabla\cdot u+\delta\rho^{p-1}=0 \tag{5.28}$$

Moreover,

$$\begin{aligned}\nabla\cdot u=&\frac{1}{2}(\rho\wedge R)^\gamma-\frac{1}{2}\frac{1}{|\Omega|}\int_\Omega(\rho\wedge R)^\gamma dx+\frac{1}{2}(\eta\wedge R)^2-\frac{1}{2}\frac{1}{|\Omega|}\int_\Omega(\eta\wedge R)^2dx\\&+(-\Delta)^{-1}\nabla\cdot(\nabla\cdot\sigma_\epsilon-\nabla\eta_\epsilon)-(-\Delta)^{-1}\nabla\cdot(\delta(\rho^q)_\epsilon u)-(-\Delta)^{-1}\nabla\cdot(\delta(\eta^m)_\epsilon u)\\&-R_iR_j((\rho u_i)_\epsilon u_j)-\frac{d}{dt}(-\Delta)^{-1}\nabla\cdot(\rho_\epsilon u)\end{aligned} \tag{5.29}$$

Let $\Phi=(-\Delta)^{-1}\nabla\cdot(\rho_\epsilon u)$. By (5.28) and (5.29),

$$\frac{d}{dt}(\log\rho+\Phi)+u\cdot\nabla(\log\rho+\Phi)+\delta\rho^{p-1}=\Psi\in L^\infty(\Omega\times(0,T)). \tag{5.30}$$

By the maximum principle, $\log\rho+\Phi\le C$. Therefore, $\rho\in L^\infty(\Omega\times(0,T))$. Next, we estimate $\nabla\rho$. For any $1\le r<\infty$,

$$\frac{d}{dt}|\nabla\rho|^r+\nabla\cdot(u|\nabla\rho|^r)\lesssim|\nabla u||\nabla\rho|^r+|\nabla^2u||\nabla\rho|^{r-1} \tag{5.31}$$

which implies that

$$\frac{d}{dt}\|\nabla\rho\|_{L^r(\Omega)}\lesssim\|\nabla u\|_{L^\infty(\Omega)}\|\nabla\rho\|_{L^r(\Omega)}+\|\nabla^2u\|_{L^r(\Omega)}. \tag{5.32}$$

Similarly,

$$\frac{d}{dt}\|\nabla\eta\|_{L^r(\Omega)}\lesssim\|\nabla u\|_{L^\infty(\Omega)}\|\nabla\eta\|_{L^r(\Omega)}+\|\nabla^2u\|_{L^r(\Omega)}. \tag{5.33}$$

We need to estimate derivatives of u. From standard parabolic estimates,

$$\begin{aligned}\|\nabla^2u\|_{L^r(\Omega\times(0,T))}&\lesssim C(T)+\|\nabla(\rho\wedge R)^\gamma\|_{L^r(\Omega\times(0,T))}+\|\nabla(\eta\wedge R)^2\|_{L^r(\Omega\times(0,T))}\\&\lesssim C(T)+\|\nabla\rho\|_{L^r(\Omega\times(0,T))}+\|\nabla\eta\|_{L^r(\Omega\times(0,T))}.\end{aligned} \tag{5.34}$$

By taking $r>3$,

$$\|\nabla u(t)\|_{L^\infty(\Omega)}\lesssim\log\Big[C(T)+\max_{0\le\tau\le t}\big(\|\nabla\rho(\tau)\|_{L^r(\Omega)}+\|\nabla\eta(\tau)\|_{L^r(\Omega)}\big)\|_{L^q(\Omega)}\Big]. \tag{5.35}$$

By the Gronwall's inequality, $\|\nabla\rho\|_{L^\infty(0,T;L^r(\Omega))}+\|\nabla\eta\|_{L^\infty(0,T;L^r(\Omega))}\le C(T)$, which implies that $\nabla u\in L^\infty(\Omega\times(0,T))$ and $\nabla^2u\in L^r(\Omega\times(0,T))$. Therefore, from the equation of u, $u_t\in L^r(\Omega\times(0,T))$. This completes the proof. ■

6 Appendix

6.1 Appendix 1: derivation of the equation of ψ

We take the time derivative to $\psi = \int_{S^2} f \ln f d\tau$. Then,

$$\psi_t = \int_{S^2} f_t \ln f + f_t d\tau = (I) + (II).$$

We obtain (I) and (II) by using the equation of f (1.10c).

$$(I) = -\int_{S^2} \Big[\nabla \cdot (uf) + \nabla_\tau \cdot (P_{\tau^\perp}(\nabla_x u\tau)f) - \Delta_\tau f - \Delta f\Big] \ln f d\tau = (a) + (b) + (c) + (d).$$

$$\begin{aligned}(a) &= -\int_{S^2} \nabla \cdot (uf) \ln f d\tau = -\int_{S^2} \nabla \cdot (uf \ln f) d\tau + \int_{S^2} u \cdot \nabla_x f d\tau \\ &= -\nabla \cdot (u\psi) + u \cdot \nabla \int_{S^2} f d\tau.\end{aligned} \tag{6.1}$$

$$(b) = -\int_{S^2} \nabla_\tau \cdot (P_{\tau^\perp}(\nabla_x u\tau)f) \ln f d\tau = \int_{S^2} \nabla u\tau f \cdot \frac{\nabla_\tau f}{f} = \nabla u : \int_{S^2} \tau \otimes \nabla_\tau f d\tau = \nabla u : \sigma. \tag{6.2}$$

The proof of last equality in relation (6.2) requires the expression of these quantities in spherical coordinates. We refer the reader to [19] for further details.

$$\begin{aligned}(c) &= \int_{S^2} \Delta_\tau f f d\tau = \int_{S^2} \Big[\Delta_\tau (f \ln f) - 2\nabla_\tau f \cdot \nabla_\tau \ln f - f\Delta(\ln f)\Big] d\tau \\ &= -2\int_{S^2} \frac{|\nabla_\tau f|^2}{f} - \int_{S^2} f\nabla_\tau \cdot \frac{\nabla_\tau f}{f} d\tau \\ &= -2\int_{S^2} \frac{|\nabla_\tau f|^2}{f} d\tau - \int_{S^2} \Delta_\tau f d\tau + \int_{S^2} \frac{|\nabla_\tau f|^2}{f} d\tau \\ &= -4\int_{S^2} |\nabla_\tau \sqrt{f}|^2 d\tau.\end{aligned} \tag{6.3}$$

$$\begin{aligned}(d) &= \int_{S^2} \Delta_x f f d\tau = \int_{S^2} \Big[\Delta_x (f \ln f) - 2\nabla_x f \cdot \nabla_x \ln f - f\Delta_x(\ln f)\Big] d\tau \\ &= \Delta_x \psi - 2\int_{S^2} \frac{|\nabla_x f|^2}{f} d\tau - \int_{S^2} \Delta_x f + \int_{S^2}\int_{S^2} \frac{|\nabla_x f|^2}{f} d\tau \\ &= \Delta_x \psi - 4\int_{S^2} |\nabla_\tau \sqrt{f}|^2 d\tau - \Delta_x \eta.\end{aligned} \tag{6.4}$$

Now, we calculate (II).

$$(II) = -\int_{S^2} \Big[\nabla \cdot (uf) + \nabla_\tau \cdot (P_{\tau^\perp}(\nabla_x u\tau)f) - \Delta_\tau f - \Delta_x f\Big] d\tau = -\nabla \cdot (u\eta) + \Delta_x \eta. \tag{6.5}$$

Collecting all terms in (6.1) - (6.5),

$$
\begin{aligned}
\psi_t &= -\nabla\cdot(u\psi) + u\cdot\nabla\eta + \nabla u : \sigma - 4\int_{S^2} |\nabla_\tau \sqrt{f}|^2 d\tau - \Delta_x \psi \\
&\quad - 4\int_{S^2} |\nabla_x \sqrt{f}|^2 d\tau - \Delta\eta - \nabla\cdot(u\eta) + \Delta\eta \\
&= -\nabla(u\psi) + \Delta\psi - 4\int_{S^2} |\nabla_\tau \sqrt{f}|^2 d\tau - 4\int_{S^2} |\nabla_x \sqrt{f}|^2 d\tau + \nabla u : \sigma - (\nabla\cdot u)\eta.
\end{aligned}
\tag{6.6}
$$

6.2 Appendix 2: verification of the formal proof in Section 3.4

In this section, we provide a rigorous proof of the formal proof in Section 3.4. First, we verify (3.6). From the equation of the density,

$$\rho_t + \nabla\cdot(\rho u) = 0,$$

we can obtain

$$\frac{d}{dt}\beta(\rho) + \nabla\cdot(u\beta(\rho)) + (\nabla\cdot u)\Big[\rho\beta'(\rho) - \beta(\rho)\Big] = 0, \tag{6.7}$$

where β is a C^1 function from $[0,\infty]$ to $\mathbb{R}$ such that

$$\beta'(t) \le C(1+t^\alpha), \quad \alpha = \frac{q-2}{2}.$$

We note that (6.7) makes sense because $u \in L^2(0,T;H^1_{loc}(\Omega))$. We approximate $\rho\log\rho$ by $\beta_\delta(\rho) = \rho\log(\rho+\delta)$. Then,

$$\rho\beta_\delta'(\rho) - \beta_\delta(\rho) \to 0 \quad \text{in} \quad L^q,$$

which verifies (3.6) in the sense of distributions.

Next, we want to verify (3.8) by showing that each term in the right-hand side of (3.7) can be multiplied by ρ to be in $L^1(\Omega\times(0,T))$.

- $\rho\times\rho^\gamma$ is from $\rho\in L^{\gamma+1}(\Omega)$.
- Since $\rho \in L^\infty(0,T;L^s)$, $s>3$ and $\eta^2 \in L^1(0,T;L^3)$, the product $\rho\eta^2$ is in $L^1(\Omega\times(0,T))$.
- $\rho(-\Delta)^{-1}\nabla\cdot(\nabla\cdot\sigma - \nabla\eta) \sim \rho(\sigma - \eta\mathbb{I}) \in L^\infty(0,T;L^s(\Omega))\times L^1(0,T;L^3(\Omega))$. Since $s>3$, $\frac{1}{s}+\frac{1}{3}\le 1$.
- $\rho(-\Delta)^{-1}\partial_i\partial_j(\rho u_i u_j) \sim \rho\rho u\otimes u$: Since $u\otimes u \in L^1(0,T;L^3(\Omega))$,

$$\rho u\otimes u \in L^1(0,T;L^r(\Omega))\cap L^p(0,T;L^q(\Omega)), \quad \frac{1}{r} = \frac{1}{s}+\frac{1}{3}\le 1, \quad \frac{1}{q} = \frac{1}{rp} + 1 - \frac{1}{p}.$$

Since $\rho\in L^\infty(0,T;L^s)$, $\rho\rho u\otimes u \in L^1(\Omega\times(0,T))$, with $\frac{1}{s}+\frac{1}{r} = \frac{2}{s}+\frac{1}{3} \le \frac{2}{3}+\frac{1}{3} = 1$.
- We rewrite $\rho\frac{d}{dt}\Big[(-\Delta)^{-1}\nabla\cdot(\rho u)\Big]$ as

$$
\begin{aligned}
\rho\frac{d}{dt}\Big[(-\Delta)^{-1}\nabla\cdot(\rho u)\Big] &= \frac{d}{dt}\Big[\rho(-\Delta)^{-1}\nabla\cdot(\rho u)\Big] - \rho_t(-\Delta)^{-1}\nabla\cdot(\rho u) \\
&= \frac{d}{dt}\Big[\rho(-\Delta)^{-1}\nabla\cdot(\rho u)\Big] + \nabla\cdot(\rho u)(-\Delta)^{-1}\nabla\cdot(\rho u) \\
&= \frac{d}{dt}\Big[\rho(-\Delta)^{-1}\nabla\cdot(\rho u)\Big] + \nabla\cdot\Big[(\rho u)(-\Delta)^{-1}\nabla\cdot(\rho u)\Big] - (\rho u)\nabla(-\Delta)^{-1}\nabla\cdot(\rho u).
\end{aligned}
$$

First, $(\rho u)\cdot\nabla(-\Delta)^{-1}\nabla\cdot(\rho u) \sim \rho\rho u\otimes u$ which is already done. Next, we estimate $\rho(-\Delta)^{-1}\nabla\cdot(\rho u)$. Since $\rho u \in L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega))$, $(-\Delta)^{-1}\nabla\cdot(\rho u) \in L^\infty(0,T;L^{\frac{6\gamma}{\gamma+3}}(\Omega))$. Therefore, $\rho(-\Delta)^{-1}\nabla\cdot(\rho u) \in$

$L^\infty(0,T;L^1(\Omega))$ because $\frac{1}{s}+\frac{\gamma+3}{6\gamma}<\frac{1}{3}+\frac{\gamma+3}{6\gamma}<\frac{3\gamma+3}{6\gamma}<1$. Finally, $(\rho u)(-\Delta)^{-1}\nabla\cdot(\rho u)$ makes sense because $(-\Delta)^{-1}\nabla\cdot(\rho u)\in L^\infty(0,T;L^{\frac{6\gamma}{\gamma+3}}(\Omega))$, $\rho\in L^\infty(0,T;L^s(\Omega))$, and $u\in L^2(0,T;L^6(\Omega))$, with $\frac{1}{s}+\frac{1}{6}+\frac{\gamma+3}{6\gamma}<\frac{1}{3}+\frac{1}{6}+\frac{\gamma+3}{6\gamma}<1$.

Finally, we verify (3.11) by passing to the limit in (3.10).

- $\rho^n\log\rho^n u^n$: We take $g^n=\rho^n\log\rho^n$ and $h^n=u^n$. To verify the condition of g^n, we can use (4.6).
- $\rho^n(-\Delta)^{-1}\nabla(\nabla\cdot\sigma^n-\nabla\eta^n)$: We take $g^n=\rho^n$, $h^n=(-\Delta)^{-1}\nabla\cdot(\nabla\cdot\sigma^n-\nabla\eta^n)$. This is possible because $\nabla\cdot\sigma^n,\nabla\eta^n\in L^1(0,T;L^3(\Omega))$ and $\rho^n\in L^\infty(0,T;L^s(\Omega))$, $s>3$.
- We already show that $\rho^n(\eta^n)^2$ converges to $\rho\eta^2$ in the sense of distributions.
- $\rho^n(-\Delta)^{-1}\nabla\cdot(\rho^n u^n)$ and $\rho^n u^n(-\Delta)^{-1}\nabla\cdot(\rho^n u^n)$: Since

$$\rho^n u^n\in L^\infty(0,T;L^q(\Omega))\cap L^2(0,T;L^r(\Omega)),\quad \frac{1}{q}=\frac{1}{2s}+\frac{1}{2},\quad \frac{1}{r}=\frac{1}{6}+\frac{1}{s}\le\frac{1}{2}.$$

we have

$$(-\Delta)^{-1}\nabla\cdot(\rho^n u^n)\in L^\infty(0,T;W^{1,q}_{loc}(\Omega))\cap L^2(0,T;W^{1,r}_{loc}(\Omega)).$$

Therefore, we can take $g^n=\rho^n,\rho^n u^n$ and $h^n=(-\Delta)^{-1}\nabla\cdot(\rho^n u^n)$.

- $\rho^n\Big[(-\Delta)^{-1}\partial_i\partial_j(\rho^n u_i^n u_j^n)-u^n\cdot\nabla(-\Delta)^{-1}\nabla\cdot(\rho^n u^n)\Big]=\rho^n[u_j^n,R_iR_j](\rho^n u^n)$: First, we control the commutator. Since $\nabla u^n\in L^2(0,T;L^2(\Omega))$ and $\rho^n u^n\in L^2(0,T;L^p(\Omega))$, with $\frac{1}{p}=\frac{1}{6}+\frac{1}{s}\le\frac{1}{2}$,

$$[u_j^n,R_iR_j](\rho^n u^n)\ \text{ is bounded in }\ L^1(0,T;W^{1,q}(\Omega)),\quad \frac{1}{q}=\frac{1}{2}+\frac{1}{p}.$$

Let $g^n=\rho^n u_i^n$ and $h^n=u_j^n$. Then,

$$U^n=[u_j^n,R_iR_j](\rho^n u_i^n)\to U=[u_j,R_iR_j](\rho u_i)\in L^1(0,T;L^{q_2}(\Omega)),\quad \frac{1}{q_2}<1+\frac{1}{s}-\frac{2}{3}.$$

in the sense of distributions. Now, let $g^n=\rho^n$ and $h^n=U^n$. Then, $\rho^n U^n$ converges to ρU in the sense of distributions. This completes the proof of the formal argument in Section 3.4. ■

Acknowledgments

H.B. gratefully acknowledges the support provided by the Center for Scientific Computation and Mathematical Modeling (CSCAMM) at the University of Maryland where this research was performed. H.B is partially supported by NSF grants DMS10-08397 and FRG07-57227. K.T. acknowledges the support by the National Science Foundation under the awards DMS 0807815 and DMS 1109397.

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