# Frames for Linear Reconstruction without Phase

Bernhard G. Bodmann
Department of Mathematics
University of Houston
Houston, TX 77204-3008

Peter G. Casazza
and Dan Edidin
Department of Mathematics
University of Missouri
Columbia, MO 65211-4100

Radu Balan
Mathematics Department
and CSCAMM
University of Maryland
College Park, MD 20742

***Abstract*— The objective of this paper is the linear reconstruction of a vector, up to a unimodular constant, when all phase information is lost, meaning only the magnitudes of frame coefficients are known. Reconstruction algorithms of this type are relevant for several areas of signal communications, including wireless and fiber-optical transmissions. The algorithms discussed here rely on suitable rank-one operator valued frames defined on finite-dimensional real or complex Hilbert spaces. Examples of such operator-valued frames are the rank-one Hermitian operators associated with vectors from maximal sets of equiangular lines or maximal sets of mutually unbiased bases. A more general type of examples is obtained by a tensor product construction. We also study erasures and show that in addition to loss of phase, a maximal set of mutually unbiased bases can correct for erased frame coefficients as long as no more than one erasure occurs among the coefficients belonging to each basis, and at least one basis remains without erasures.**

## I. Introduction

Maximizing bandwidth is a priority in the design of today's digital communication systems. Analog transmissions, whether wireless or via optical fibers, have to exhaust what is physically possible for the given medium. To this end, transmissions send parallel streams of data, e.g., from antenna arrays through a number of links to the receiver (see e.g. [35], [24], [5]), or through multiple electromagnetic modes in an optical fiber (as described in [40], [1], [38]). The benefit of using analog channels in parallel comes, however, at the cost of an increased susceptibility to oscillator instabilities and of a resulting loss of coherence in the transmission ([21], [27]).

The purpose of this paper is to investigate linear encoding and decoding strategies for analog signals that use redundancy to overcome the dependence on phase information. In other words, we are concerned with the question of reconstructing a vector in a finite-dimensional real or complex Hilbert space when only the magnitudes of the coefficients of the vector under a linear map are known. In a previous paper [3], part of the authors discussed this problem in the context of signal processing, in particular the analysis of speech. It was shown that the magnitudes of inner products with a generic set containing a sufficient number of (frame) vectors characterize each vector, up to a unimodular constant. However, at least in the complex case, reconstruction algorithms were difficult to implement.

To obtain the linear algorithms presented here, we use that characterizing a vector $x$ in a Hilbert space $\mathcal{H}$, up to a unimodular factor, is equivalent to reconstructing the rank-one Hermitian operator $x \otimes x^*$. After "squaring" the vector, we are able to provide linear reconstruction algorithms in terms of $\{|\langle x, f_j\rangle|^2\}_{j=1}^N$, the squares of frame coefficients of $x$ with respect to a suitable frame $\{f_1, f_2, \dots f_N\}$ for $\mathcal{H}$. The same strategy appears under the name of state tomography in quantum theory, see e.g. [36] or [37]. While the quantum literature emphasizes the design of minimal measurements (smallest number of frame vectors) to characterize an unknown operator $x \otimes x^*$ with $x$ of unit-norm (as in [18], [19]), we focus on types of frames which provide an efficient, linear reconstruction algorithm for $x \otimes x^*$. The frames we use for this purpose are maximal equiangular tight frames or maximal sets of mutually unbiased bases, which contain $N \geq d^2$ frame vectors in the complex case and $N \geq d(d+1)/2$ vectors in the real case. In the case of complex Hilbert spaces, a more general type of frames providing linear reconstruction without phase is obtained from tensor products of maximal equiangular tight frames or maximal sets of mutually unbiased bases.

In addition, we consider the situation when coefficients are lost, e.g. in the course of a data transmission ([22], [30], [11], [7], [6]). We investigate which of the encoding strategies provide a correction mechanism for erasures. Maximal sets of mutually unbiased bases can correct under certain conditions up to $d$ erasures in the complex case or $d/2$ in the real case, in addition to loss of phase information. Therefore, the encoding requires $N = d(d+1)$ or $N = d(d/2+1)$ frame vectors, respectively. The condition for correctibility is that for a given partition of the coefficients in subsets of size $d$ corresponding to each basis, at most one loss occurs within each subset.

The organization of this paper is as follows. Section II introduces the notion of operator-valued frames. Section III presents examples of frames which provide a linear reconstruction algorithm from the magnitudes of frame coefficients. This algorithm is described in Section IV. Finally, Section V discusses the correction of erasures.

## II. From Frames to Operator-Valued Frames

In this section we introduce the main idea in this paper: Reconstructing a vector $x$ in a Hilbert space $\mathcal{H}$ from the magnitudes of its frame coefficients, up to a unimodular constant, is equivalent to reconstructing the rank-one Hermitian operator $x \otimes x^*$, given by $x \otimes x^* y = \langle y, x\rangle x$, $y \in \mathcal{H}$, from its expansion with respect to an operator-valued frame. For a more exhaustive treatment of operator-valued frames, see [28].

*Definition 2.1:* Let $\mathcal{H}$ be a $d$-dimensional real or complex Hilbert space. A finite family of vectors $\mathcal{F} = \{f_1, f_2, \dots f_N\} \in \mathcal{H}^N$ is called a **frame**, if there are non-zero constants $A_1$ and $A_2$ such that for all $x \in \mathcal{H}$,

$$A_1\|x\|^2 \le \sum_{j=1}^{N} |\langle x, f_j\rangle|^2 \le A_2\|x\|^2 \,.$$

With each frame, we associate its **Grammian** $G = (\langle f_j, f_l\rangle)_{j,l=1}^N$, the matrix formed by the inner products of the frame vectors.

If we can choose $A_1 = A_2 = A$ in the above chain of inequalities, then the frame is called $A$-tight. If, in addition, there is $b > 0$ such that $\|f_j\| = b$ for all $j \in \{1, 2, \dots N\}$, then we call the family $\{f_j\}_{j=1}^N$ a **uniform** $A$-tight frame. Such frames are also called equal-norm tight frames.

A family of vectors $\mathcal{F}$ is a frame for a finite-dimensional Hilbert space $\mathcal{H}$ if and only if it spans $\mathcal{H}$, because then it contains a linearly independent, spanning subset, and thus it is straightforward to verify the norm inequalities in Definition 2.1.

Any given vector $x$ can be reconstructed from the linear combination of its frame coefficients $\{\langle x, f_j\rangle\}_{j=1}^N$ with a dual frame $\{g_j\}_{j=1}^N$. A canonical choice for this dual frame is to take the pseudo-inverse $M$ of the Grammian matrix, meaning the Hermitian $M$ which has the same range as $G$, satisfies $GM = MG$ and $GMG = G$, and define $g_j = \sum_{l=1}^N M_{j,l} f_l$, $j \in \{1, 2, \dots N\}$. It can then be verified that for any $x \in \mathcal{H}$,

$$x = \sum_{j=1}^{N} \langle x, f_j\rangle g_j \,.$$

If $\mathcal{F}$ is an $A$-tight frame, then the canonical dual is simply given by $g_j = f_j/A$, and thus any vector $x \in \mathcal{H}$ can be reconstructed according to

$$x = \frac{1}{A}\sum_{j=1}^{N} \langle x, f_j\rangle f_j \,.$$

This identity is equivalent to the matrix $\frac{1}{A}G$ being an orthogonal rank-$d$ projection. Thus, $d = \frac{1}{A}\operatorname{tr}[G] = \frac{1}{A}\sum_{j=1}^N \|f_j\|^2$ implies that all the frame vectors in an $A$-tight uniform frame have the same norm

$$b = \sqrt{\frac{Ad}{N}} \,.$$

With a frame $\mathcal{F} = \{f_j\}_{j=1}^N$, we associate the set of rank-one Hermitian operators $\mathcal{S} = \{f_j \otimes f_j^*\}_{j=1}^N$ on $\mathcal{H}$.

*Definition 2.2:* Let $\{f_j\}_{j=1}^N$ be a frame for a Hilbert space $\mathcal{H}$. Let $S_j = f_j \otimes f_j^*$ denote the rank-one Hermitian operator associated with each $f_j$. Let $\mathcal{X}$ be the span of the family $\mathcal{S} = \{S_j\}_{j=1}^N$, equipped with the Hilbert-Schmidt inner product. We say that $\{S_j\}_{j=1}^N$ is the **operator-valued frame** for $\mathcal{X}$ associated with $\{f_j\}_{j=1}^N$. The Grammian $H$ of $\mathcal{S}$ has entries $H_{j,k} = \operatorname{tr}[S_j S_k] = |\langle f_j, f_k\rangle|^2$.

Using the same argument for the operator-valued frame $\mathcal{S} = \{S_j\}_{j=1}^N$ as for $\{f_j\}_{j=1}^N$, we can define the canonical dual $\mathcal{R} = \{R_j\}_{j=1}^N$ of $\mathcal{S}$ with respect to the Hilbert-Schmidt inner product. If $X$ is an operator in the span of $\mathcal{S}$, then

$$X = \sum_{j=1}^{N} \operatorname{tr}[X S_j] R_j \,.$$

Therefore, reconstruction without phase requires that this identity holds for any rank-one Hermitian $X = x \otimes x^*$, $x \in \mathcal{H}$.

## III. Operator-Valued Frames with Maximal Span

In this section, we want to find conditions which guarantee that $\mathcal{X}$ contains all rank-one Hermitian operators. We then say that the operator-valued frame $\mathcal{S}$ has **maximal span**, and that the underlying frame $\mathcal{F}$ is **maximal**.

*Proposition 3.1:* Let $\{f_j\}_{j=1}^N$ be a frame for a real or complex Hilbert space $\mathcal{H}$ and $\mathcal{S}$ the associated operator-valued frame with span $\mathcal{X}$. The rank of the Grammian $H$ is at most $d(d+1)/2$ in the real case or $d^2$ if $\mathcal{H}$ is complex. Moreover, the rank of $H$ is maximal if and only if $\mathcal{X}$ contains all rank-one Hermitian operators on $\mathcal{H}$.

*Proof:* We note that the space $\mathcal{Q}$ spanned by all rank-one Hermitian operators has dimension $d(d+1)/2$ or $d^2$ in the real or complex case, respectively. Since $\mathcal{S}$ contains only such rank-one operators, $\mathcal{X} \subset \mathcal{Q}$ and the rank of $H$ as well as the dimension of $\mathcal{X}$ can be at most $d(d+1)/2$ or $d^2$, respectively. Moreover, if the rank of $H$, and thus the dimension of $\mathcal{X}$, is maximal, then $\mathcal{X} = \mathcal{Q}$. ■

*Corollary 3.2:* If $\mathcal{F} = \{f_j\}_{j=1}^N$ is a maximal frame for a real or complex Hilbert space $\mathcal{H}$, then $N \ge d(d+1)/2$ in the real case and $N \ge d^2$ in the complex case

*Proof:* By the preceding theorem, if the span of the operator-valued frame $\mathcal{S}$ associated with $\mathcal{F}$ contains all rank-one Hermitian operators, then the rank of the Grammian $H$ is $d(d+1)/2$ or $d^2$ in the real or complex case, respectively. This provides the desired lower bound for $N$, because the rank of the $N \times N$ matrix $H$ can be at most $N$. ■

Now we discuss specific types of maximal frames.

For this purpose, we consider uniform tight frames with the property that the magnitudes of the inner products between frame vectors form a small set. If this set has size one, we call the frame 2-uniform ([26], [7]) or (equal-norm) equiangular tight frame ([41], [33], [39], [42]). Another type of frame, for which this set has size two, is obtained from a number of bases for a Hilbert space which are chosen in such a way that, between basis vectors belonging to different bases, their inner products have a fixed magnitude. These frames are referred to as sets of mutually unbiased bases.

*Definition 3.3:* A family of vectors $\mathcal{F} = \{f_j\}_{j=1}^N$ is said to form a **2-uniform** or **equiangular** $A$-tight frame if it is uniform and if there is $c > 0$ such that for all pairs of frame vectors $f_j$ and $f_k$, $j \ne k$, we have $|\langle f_j, f_k\rangle| = c$.

Using that $G$ is a scaled projection, we obtain $d = \frac{1}{A}\operatorname{tr}[G] = \frac{1}{A^2}\operatorname{tr}[G^2] = \frac{1}{A^2}\sum_{j,k=1}^N |\langle f_j, f_k\rangle|^2$ which, together with the known value for the diagonal of $G$, determines the constant $c$ in Definition 3.3,

$$c = \frac{A}{N}\sqrt{\frac{d(N-d)}{N-1}}.$$

An observation of Lemmens and Seidel characterizes when the operator-valued frame associated with an equiangular tight frame has maximal span.

*Proposition 3.4 (Lemmens and Seidel [33]):* Let $\mathcal{H}$ be a real or complex Hilbert space, and $\mathcal{F} = \{f_1, f_2, \dots f_N\}$ be an equiangular tight frame, then $\mathcal{F}$ is maximal if and only if the frame consists of $N = d(d+1)/2$ or $N = d^2$ vectors in the real or complex case, respectively.

*Proof:* We observe that the Grammian $H$ of the operator-valued frame $\mathcal{S}$ associated with $\mathcal{F}$, with entries $H_{j,k} = |\langle f_j, f_k\rangle|^2$, is of rank $N$ because $H = (b^2 - c^2)I + c^2 J$, the matrix $J$ containing all 1's is non-negative and $b > c$. Thus, the span of $\mathcal{S}$ is maximal if and only if $N = d(d+1)/2$ or $N = d^2$ vectors, depending on whether $\mathcal{H}$ is real or complex. ■

For examples of such frames, see [43], [26], [7], [2], [20]. We describe a simple example for a two-dimensional real or complex Hilbert space.

*Example 3.5:* Let $\{e_1, e_2\}$ denote the canonical basis for either $\mathbb{R}^2$ or $\mathbb{C}^2$.

We first consider the Hilbert space $\mathbb{R}^2$. Let $R$ be the rotation matrix such that $R^3 = I$ and $R \neq I$. Choose $f_1 = e_1$, $f_2 = Re_1$ and $f_3 = R^2 e_1$. Then $\{f_1, f_2, f_3\}$ is a 2-uniform 3/2-tight frame with $|\langle f_i, f_j\rangle| = 1/2$ for $i \neq j$. This frame is sometimes called the Mercedes-Benz frame.

For the case of $\mathbb{C}^2$, we introduce the unitary Pauli matrices $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Let $f_1 = \alpha e_1 + \beta e_2$ where $\alpha = \sqrt{\frac{1}{2}(1 - \frac{1}{\sqrt{3}})}$ and $\beta = e^{(\frac{5\pi}{4})i}\sqrt{\frac{1}{2}(1 + \frac{1}{\sqrt{3}})}$, and let $f_2 = Xf_1$, $f_3 = Zf_1$, $f_4 = XZf_1$. Then $\{f_1, \dots, f_4\}$ is an equal-norm equiangular 2-tight frame with $|\langle f_i, f_j\rangle| = \frac{1}{\sqrt{3}}$ for all $i \neq j$.

Mutually unbiased bases form another type of frame which has an associated operator-valued frame with maximal span. This type of frame contains vectors from a number of orthonormal bases for a Hilbert space which are chosen in such a way that, between basis vectors belonging to different bases, their inner products have a fixed magnitude.

*Definition 3.6:* Let $\mathcal{H}$ be a real or complex Hilbert space of dimension $d$. A family of vectors $\{e_k^{(j)}\}$ in $\mathcal{H}$ indexed by $k \in \mathbb{K} = \{1, 2, \dots d\}$ and $j \in \mathbb{J} = \{1, 2, \dots m\}$ is said to form $\boldsymbol{m}$ **mutually unbiased bases** if for all $j, j' \in \mathbb{J}$ and $k, k' \in \mathbb{K}$ the magnitude of the inner product between $e_k^{(j)}$ and $e_{k'}^{(j')}$ is given by

$$|\langle e_k^{(j)}, e_k^{(j')}\rangle| = \delta_{k,k'}\delta_{j,j'} + \frac{1}{\sqrt{d}}(1 - \delta_{j,j'}),$$

where Kronecker's $\delta$ symbol is one when its indices are equal and zero otherwise.

*Proposition 3.7 (Delsarte, Goethals and Seidel [17]):* Let $\mathcal{H}$ be a real or complex Hilbert space, and $\mathcal{F} = \{f_1, f_2, \dots f_N\}$ a frame formed by a set of $m$ mutually unbiased bases. Then the operator-valued frame $\mathcal{S}$ associated with $\mathcal{F}$ has maximal span if and only if $m = d/2 + 1$ in the real case or $m = d + 1$ in the complex case.

*Proof:* The Grammian $H$ of the operator-valued frame associated with $m$ mutually unbiased bases has the form $H = I_m \otimes I_d + (J_m - I_m) \otimes J_d/d$, where $I_m$ and $I_d$ are the $m \times m$ and $d \times d$ identity matrices, respectively, and $J_m$ and $J_d$ denote the $m \times m$ and $d \times d$ matrices containing only 1's. The kernel of the Grammian matrix is the space of vectors $a \otimes b$ such that $J_d b = db$ and $J_m a = 0$, so it is $m - 1$-dimensional. Consequently, the rank of $H$ and thus the dimension of the span of $\mathcal{S}$ is $md - m + 1$. This shows that the maximal rank is achieved if and only if there are $m = d + 1$ mutually unbiased bases in a $d$-dimensional complex Hilbert space $\mathcal{H}$ and $m = \frac{d}{2} + 1$ in the real case. For an alternative proof, see [43]. ■

*Example 3.8:* The simplest example of a set of mutually unbiased bases in a complex Hilbert space is the standard basis, together with the basis of eigenvectors of the Pauli matrices $X$ and $Y = iXZ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

This example and others can be found in [4]. If $d$ is prime, then there exists a maximal set of mutually unbiased bases called **discrete chirps**, see [10], [25].

*Example 3.9:* Let $d$ be a prime number, and $\omega$ a primitive $d$-th root of unity. Denote the canonical basis of $\mathbb{C}^d$ by $\{e_k^{(1)}\}_{k=1}^d$, then for $j \in \{2, 3, \dots d+1\}$,

$$e_k^{(j)} = \frac{1}{\sqrt{d}}\sum_{l=1}^{d} \omega^{-(j-1)l^2 + kl} e_l^{(1)}$$

defines together with the canonical basis a family of $d+1$ mutually unbiased bases called the discrete chirps.

To see that these vectors form bases, we first consider inner products between vectors of same $j$. For $j = 1$, this is clear. If $j > 1$,

$$\langle e_{k'}^{(j)}, e_k^{(j)}\rangle = \frac{1}{d}\sum_{l=1}^{d} \omega^{k'l - kl} = \delta_{k,k'}\,.$$

The bases are mutually unbiased because if $j \neq j'$, and one of them is equal to one, then $|\langle e_{k'}^{(j')}, e_k^{(j)}\rangle| = 1/\sqrt{d}$. If neither basis index is equal to one, then

$$\langle e_{k'}^{(j')}, e_k^{(j)}\rangle = \frac{1}{d}\sum_{l=1}^{d} \omega^{-j'l^2 + jl^2 + k'l - kl}$$

and by completing the square and using cyclicity

$$|\langle e_{k'}^{(j')}, e_k^{(j)}\rangle| = \frac{1}{d}\left|\sum_{l=1}^{d} \omega^{l^2}\right|.$$

Now squaring this expression yields

$$\begin{aligned} |\langle e_{k'}^{(j')}, e_k^{(j)}\rangle|^2 &= \frac{1}{d^2}\sum_{l,l'=1}^{d} \omega^{l^2 - (l')^2} \\ &= \frac{1}{d^2}\sum_{l,l'=1}^{d} \omega^{(l+l')(l-l')} = \frac{1}{d}\,. \end{aligned}$$

*Remark 3.10:* A similar construction applies when $d$ is a power of a prime ([43]). If $d$ is not prime, then the maximal number of mutually unbiased bases is generally unknown ([20]). In the real case, even in the case of prime dimensions, the construction of maximal sets of mutually unbiased bases is more difficult ([8]), but at least for $d$ a power of 4 this is possible, see [16], [9].

Whenever the span of the operator-valued frame associated with an equiangular tight frame or of mutually unbiased bases is maximal, we refer to them as **maximal equiangular tight frames** or **maximal sets of mutually unbiased bases**, respectively. From these two types of examples, we can construct a large class of maximal frames in the complex case.

*Theorem 3.11:* Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ be a tensor product of two complex Hilbert spaces, $\mathcal{H}_1$ of dimension $d_1$ and $\mathcal{H}_2$ of dimension $d_2$. If $\mathcal{F}_1 = \{f_1, f_2, \dots f_{N_1}\}$ and $\mathcal{F}_2 = \{h_1, h_2, \dots h_{N_2}\}$ are frames for $\mathcal{H}_1$ and $\mathcal{H}_2$, then the **tensor product frame** $\mathcal{F} = \{f_j \otimes h_l : j \in \{1, 2, \dots N_1\}, l \in \{1, 2, \dots N_2\}\}$ for $\mathcal{H}$ is maximal if and only if both $\mathcal{F}_1$ and $\mathcal{F}_2$ are.

*Proof:* The Grammian $H$ of the operator-valued frame $\mathcal{S}$ associated with $\mathcal{F}$ is the Kronecker product of the Grammians of $\mathcal{S}_1$ and $\mathcal{S}_2$. Therefore, the rank of $H = H_1 \otimes H_2$ is the product of the ranks of $H_1$ and $H_2$. If the ranks of $H_1$ and of $H_2$ are both maximal, then they are $\operatorname{rk} H_1 = d_1^2$ and $\operatorname{rk} H_2 = d_2^2$, and the rank of $H$ is $d_1^2 d_2^2 = (\dim \mathcal{H})^2$. This shows that the rank of $H$ is maximal if and only if the ranks of $H_1$ and $H_2$ are. ■

Tensor products of maximal equiangular tight frames or mutually unbiased bases yield a more general type of operator-valued frame $\mathcal{S}$ with maximal span, because the entries in the Grammian $H = H_1 \otimes H_2$ assume a larger set of values than previously. From a practical point of view, such tensor product frames are appealing because they allow for distributed processing. This means, for each fixed $f \in \mathcal{F}_1$, the family $\{f \otimes h_j\}_{j=1}^{N_2}$ is a maximal frame for the subspace $f \otimes \mathcal{H}_2$ and for each $h \in \mathcal{F}_2$, $\{f_j \otimes h\}_{j=1}^{N_1}$ is a maximal frame for the subspace $\mathcal{H}_1 \otimes h$. Thus, reconstruction can proceed first in "local" subspaces, and then on the "global" level of the entire Hilbert space. Frames for subspaces ([12]), recently also called fusion frames ([13]), have been studied because of their relevance in distributed processing ([14]) and because of their efficient error-correction capabilities, see [6] and [15].

*Corollary 3.12:* Every complex Hilbert space can be endowed with a frame for which the associated operator-valued frame has maximal span.

*Proof:* Since the dimension of $\mathcal{H}$ factors into primes, we can interpret it as a tensor product $\mathcal{H} = \bigotimes_{j=1}^m \mathcal{H}_j$ where the dimension of each $\mathcal{H}_j$ is prime. Thus, we can construct a maximal set of mutually unbiased bases for each $\mathcal{H}_j$. The tensor product of these maximal frames provides the desired frame for $\mathcal{H}$. ■

## IV. Reconstructing a Vector from Absolute Values of its Frame Coefficients

In this section we present the reconstruction formula which determines any vector $x$ in a finite-dimensional Hilbert space $\mathcal{H}$, up to a unimodular constant, from the magnitudes of its frame coefficients $\{\langle x, f_j\rangle\}_{j=1}^N$. More precisely, the reconstruction formula characterizes the self-adjoint rank-one operator $x \otimes x^*$, from which the vector $x$ can be determined, up to a unimodular constant, by any non-vanishing row of the matrix with entries $(x \otimes x^*)_{i,l} = x_i \overline{x_l}$, $i, l \in \{1, 2, \dots d\}$, representing $x \otimes x^*$ with respect to an orthonormal basis.

To obtain the reconstruction, we require a frame $\{f_j\}_{j=1}^N$ for which the associated operator-valued frame has maximal span.

*Theorem 4.1:* Let $\mathcal{H}$ be a $d$-dimensional real or complex Hilbert space and $F = \{f_1, f_2, \dots f_N\}$ a maximal $N/d$-tight frame, such that the associated operator-valued frame $\mathcal{S}$ has maximal span. Let $M$ be the pseudo-inverse of the Grammian $H$, so $M$ is self-adjoint, $HM = MH$, and $HMH = H$, and denote the canonical dual of $\mathcal{S}$ as $\mathcal{R}$, containing operators $R_j = \sum_{k=1}^N M_{j,k} S_k$. Given a vector $x \in \mathcal{H}$, then

$$x \otimes x^* = \sum_{j=1}^N |\langle x, f_j\rangle|^2 R_j \,.$$

*Proof:* Instead of deriving the claimed identity directly, we show that both sides coincide after taking their Hilbert-Schmidt inner product with any operator $y \otimes y^*$, $y \in \mathcal{H}$. Inserting the expression for $R_j$ means we have to prove the identity

$$|\langle x, y\rangle|^2 = \sum_{j,k=1}^N |\langle x, f_j\rangle|^2 M_{j,k} |\langle f_k, y\rangle|^2$$

for all $y \in \mathcal{H}$. Using that $x \otimes x^* = \sum_{l=1}^N c_l f_l \otimes f_l^*$ with some coefficients $\{c_l\}$ by the maximality of the span of $\mathcal{S}$ and similarly $y \otimes y^* = \sum_{l'=1}^N c'_{l'} f_{l'} \otimes f_{l'}^*$, the matrix identity $HMH = H$ yields that both sides are equal to $\sum_{l,l'=1}^N c_l c'_{l'} H_{l,l'}$. ■

After this general result, we consider the examples of maximal equiangular frames and of maximal sets of mutually unbiased bases.

*Corollary 4.2:* Let $\mathcal{H}$ be a complex Hilbert space. If $\mathcal{F}$ is a maximal equiangular $N/d$-tight frame or a tight frame formed by a maximal set of mutually unbiased bases, then the reconstruction identity becomes

$$x \otimes x^* = \frac{d(d+1)}{N} \sum_{j=1}^N |\langle x, f_j\rangle|^2 (f_j \otimes f_j^* - I/(d+1)) \,.$$

*Proof:* This result follows from the preceding theorem by proving that the canonical (Hilbert-Schmidt) dual of $\{S_j\}$ is $\{R_j\}$ with $R_j = d(d+1)S_j/N - dI/N$. Let $0 < c = |\langle f_j, f_k\rangle|$ for any $j \neq k$ if the frame is equiangular and a non-orthogonal pair if it consists of mutually unbiased bases. Since the frame vectors are normalized in either case, it is straightforward to verify that $\operatorname{tr}[S_j R_j] = d^2/N$. If $j \neq k$,

$$\operatorname{tr}[S_j R_k] = \frac{d(d+1)}{N} |\langle f_j, f_k\rangle|^2 - \frac{d}{N} \,.$$

If $\mathcal{F}$ is equiangular, then $|\langle f_j, f_k\rangle|^2 = c^2 = (N-d)/d(N-1) = 1/(d+1)$ and consequently $\operatorname{tr}[S_j R_k] = 0$.

In the case of mutually unbiased bases, we have either $\mathrm{tr}[S_j R_k] = -d/N$ if the indices $j$ and $k$ belong to two vectors from the same basis, or $|\langle f_j, f_k\rangle|^2 = 1/d$ and $\mathrm{tr}[S_j R_k] = 1/N$. Thus, the matrix $K$ with entries $K_{j,k} = \mathrm{tr}[S_j R_k]$ has the form $K = \frac{d(d+1)}{N} I_{d+1} \otimes I_d - \frac{1}{N}((d+1)I_{d+1} - J_{d+1}) \otimes J_d$, where the first component specifies the basis and the second refers to the index within each basis. Since $N = d(d+1)$, $K$ can be identified as a rank-$d^2$ orthogonal projection with the same range as $H$. This shows that in both cases, whether equiangular tight frame or mutually unbiased bases, $\{R_j\}_{j=1}^N$ is the canonical dual of the operator-valued frame $\{S_j\}_{j=1}^N$. ■

It is at first surprising that the reconstruction formula for maximal equiangular tight frames and for maximal mutually unbiased bases is identical. The reason for this can be traced to the fact that these two frame families are both projective 2-designs, see [23], [31], [32] and [29]. The structure of projective 2-designs and their use for reconstruction without phase will be left to a future investigation.

## V. Loss of Phase and Erasures

In this section, we show that a maximal set of mutually unbiases bases admits the reconstruction from magnitudes of frame coefficients even if some of these coefficients are lost. Since we want to reconstruct linearly from the remaining coefficients, the operator-valued frame must retain maximal span after removing elements corresponding to erased coefficients.

*Definition 5.1:* Let $\{e_j\}_{j=1}^N$ be a frame for a real or complex Hilbert space $\mathcal{H}$ and $\mathcal{S} = \{S_j\}_{j=1}^N$ the associated operator-valued frame. We call an erasure of coefficients indexed with $\mathbb{L} \subset \mathbb{J} = \{1, 2, \dots N\}$ **correctible** if the set $\{S_j\}_{j \in \mathbb{J}\setminus\mathbb{L}}$ is a frame for the span of all rank-one Hermitian operators.

By rank considerations, it is clear that an equiangular tight frame cannot admit erased coefficients. However, since the number of frame vectors coming from a maximal family of mutually unbiased bases is larger than the dimension of the space spanned by all Hermitian rank-one operators, we expect that possibly, lost coefficients are correctible. This is indeed the case for at most one lost coefficient in each basis, as long as a set of coefficients belonging to at least one basis contains no losses.

*Theorem 5.2:* Let $\mathcal{H}$ be a real or complex Hilbert space of dimension $d$. Let $\{e_k^{(j)} : k \in \mathbb{K}, j \in \mathbb{J}\}$, $\mathbb{K} = \{1, 2, \dots d\}$, $\mathbb{J} = \{1, 2, \dots m\}$, be a maximal family of $m$ mutually unbiased bases for $\mathcal{H}$ such that the associated operator-valued frame $\{S_k^{(j)}\}_{j \in \mathbb{J}}$ has maximal span. If for each $j \in \mathbb{J}$, $\mathbb{L}_j \subset \mathbb{K}$ is of size at most one, and for at least one $j$, $\mathbb{L}_j = \emptyset$, then $\mathcal{S} = \{S_k^{(j)} : j \in \mathbb{J}, k \in \{1, 2, \dots d\} \setminus \mathbb{L}_j\}$ has maximal span. Conversely, if one set $\mathbb{L}_j$ is of size larger than one, or at least $m$ coefficients are erased, then the erasure is not correctible.

*Proof:* We recall that the span of the operators $\{S_k^{(j)} : j \in \mathbb{J}, k \in \mathbb{K} \setminus \mathbb{L}_j\}$ is maximal if and only if the rank of the Grammian $H$ is.

We first consider at most $m-1$ erased coefficients. To determine the rank of $H$, we view the equation $Ha = 0$ as a block matrix equation with blocks labeled by the basis indices such that $H^{(j,j)} = I_{d_j}$ for diagonal blocks and $H^{(j,l)} = J_{d_j,d_l}/d$ for the off-diagonal blocks.

Collecting the entries of the vector $a$ belonging to one basis index $j$ in $a^{(j)}$, we deduce from $a^{(j)} = \sum_l J_{d_j,d_l} a^{(l)}/d$ that all of its entries are identical, $a_k^{(j)} = \alpha_j$, for each $j \in \{1, 2, \dots m\}$.

This means for each solution $a$ of $Ha = 0$, there is a corresponding solution $H'\alpha = 0$, where each diagonal block $H^{(j,j)}$ in the matrix $H$ has been replaced by the eigenvalue 1 of $a^{(j)}$ and each off-diagonal block $H^{(j,l)}$ by $\delta_l = 1 - |\mathbb{L}_l|/d$ to obtain $H'$.

We assume that we have ordered the blocks in such a way that $\{\delta_l\}_{l=1}^m$ is increasing. If at least one $\mathbb{L}_j$ is empty, then there is $r \geq 0$ such that $\delta_j = 1$ for all $j > r$. Moreover, if the sets $\mathbb{L}_j$ are at most of size one, then for $r$ erasures there are $m-r$ bases without erasures, meaning the last $m-r$ columns of $H'$ contain 1's. Now taking the difference between consecutive rows of $H'$ gives the equation $H''\alpha = 0$ with

$$H'' = \begin{pmatrix} 1 & \delta_2 & \delta_3 & \dots & \delta_r & 1 & \dots & 1 \\ \delta_1 - 1 & 1-\delta_2 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \delta_2 - 1 & 1-\delta_3 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & & \dots & \\ 0 & \dots & \dots & 0 & \delta_r - 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 & 0 & \dots & \\ 0 & \dots & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Since we assumed $\delta_r \neq 1$, we conclude $\alpha_r = 0$. Now using the identity $(\delta_{j-1}-1)\alpha_{j-1} = (\delta_j - 1)\alpha_j$ from rows $2 \leq j \leq r$, it can be verified that $\alpha_1 = \alpha_2 = \dots = \alpha_r = 0$. This means, $H''\alpha = 0$ if and only if the first $r$ entries of $\alpha$ vanish and $\sum_{j>r} \alpha_j = 0$, which is a space of solutions of dimension $m - r - 1$. Consequently, if $r$ erasures occur and each set $\mathbb{L}_j$ contains at most one erasure, then $m - r \geq 1$ sets among the family $\{\mathbb{L}_j\}$ are empty, then the Grammian $H$ is an $(md-r) \times (md-r)$ matrix of rank $md - r - (m-r-1) = m(d-1)+1$. In the real case, the rank is $(d/2+1)(d-1)+1 = d(d+1)/2$, and in the complex case, $(d+1)(d-1)+1 = d^2$. This means, the rank of $H$ is maximal.

The remaining case to be examined is that of $m$ or more erasures, or at least two coefficients belonging to one basis being erased.

If $m$ or more coefficients are erased, then the Grammian $H$ of the operator-valued frame $\mathcal{S}$ has at most $N = m(d-1)$ rows and columns, which means its rank is at most $(d-1)(d+1) = d^2 - 1$ in the complex case and $(d-1)(d/2+1) = d(d+1)/2 - 1$ in the real case, which is not maximal.

To cover the case of at least two coefficients belonging to one basis are erased, say $|\mathbb{L}_1| > 1$, it is enough to consider $|\mathbb{L}_1| = 2$ and $\mathbb{L}_j = \emptyset$ for all $j > 1$, because erasing more coefficients only reduces the rank of $H$ further.

In this case, the matrix $H''$ has the entries $H'_{j,1} = \delta_1 < 1$ for all $j > 1$ and all other entries are 1's. Therefore, the space of solutions to $H'\alpha = 0$ is $m-2$-dimensional. However, $H$ is an $(md-2) \times (md-2)$ matrix, which means its rank is $md - 2 - (m-2) = m(d-1)$. Now using the same argument as in the case of $m$ or more erasures, we see that the rank of $H$ is not maximal. ■

*Corollary 5.3:* The preceding theorem implies that if $\{e_k^{(j)} : k \in \mathbb{K}, j \in \mathbb{J}\}$, $\mathbb{K} = \{1, 2, \ldots d\}$, $\mathbb{J} = \{1, 2, \ldots m\}$, is a set of mutually unbiased bases for a real or complex Hilbert space of dimension $d$, with an operator-valued frame of maximal span, then erasures of up to $d$ coefficients in the complex case or $d/2$ coefficients in the real case are correctible, as long as no more than one coefficient is erased in each basis.

## VI. Conclusion

Maximal equal-norm equiangular tight frames and maximal sets of mutually unbiased bases provide simple reconstruction algorithms that only use the magnitudes of frame coefficients. We have linked the reason for the existence of these algorithms to the associated rank-one operator valued frames with maximal span. In addition, we have seen that using mutually unbiased bases provides an error-correction mechanism for up to one erasure per basis, as long as at least one basis remains without any erased coefficients.

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