First-order aggregation models with alignment

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Abstract

We include alignment interactions in a well-studied first-order attractive-repulsive macroscopic model for aggregation. The distinctive feature of the extended model is that the equation that specifies the velocity in terms of the population density, becomes *implicit*, and can have non-unique solutions. We investigate the well-posedness of the model and show rigorously how it can be obtained as a macroscopic limit of a second-order kinetic equation. We work within the space of probability measures with compact support and use mass transportation ideas and the characteristic method as essential tools in the analysis. A discretization procedure that parallels the analysis is formulated and implemented numerically in one and two dimensions.

Keywords: aggregation models; nonlocal interactions; kinetic equations; macroscopic limit; mass transport; particle methods

1   Introduction

The literature on self-organizing behaviour or swarming has grown dramatically over the last years. A variety of mathematical models has been proposed, which have origin in biological applications (e.g., self-collective behaviour seen in species such as fish, birds or insects [CDF+03]), as well as in social sciences and engineering (e.g., opinion formation [MT14], social networks [Jac10], robotics and space missions [JE07]). The main aspect is the modelling of the social interactions between the members of a group; due to such inter-individual interactions, self-organization may occur in a physical space (insect swarms, fish schools, robots) or, more abstractly, in an opinion space.

One approach in modelling aggregation is to consider individuals/organisms as point particles and design either an ordinary differential equation (ODE) or a discrete-time equation to model their evolution. Another is to formulate a partial differential equation (PDE) that governs the time evolution of the population density field. These two approaches result from the various descriptions that one can take in modelling aggregation behaviour: particle-based/microscopic or continuum/macroscopic. We refer to [CFTV10] for a recent review of aggregation models, where in particular, it is shown how microscopic models can be related to macroscopic ones via kinetic theory.

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Three types of social interactions have been commonly considered in the literature on mathematical aggregations: attraction, repulsion, and alignment. Consequently, aggregation models can be distinguished in terms of which of these interactions are being accounted for. Some models consider only a subset of these interactions (just attraction and repulsion [MEK99, BCL09, DCBC06, BT11] or just alignment [CS07, CFRT10]), others account for all three of them. Models of the latter type are typically referred to as "three-zone" models, as each particular interaction type acts at different ranges (repulsion acts at short distances, while alignment and attraction are present at intermediate and long ranges, respectively). This class of models has had many successful applications in biological and sociological modelling [CKJ$^+$02, Rey87, MKJNR12].

Aggregation models (discrete or continuous) may also differ in how the velocity field is determined. There are second-order/dynamic models, typically in the form of the Newton's second law, where a differential equation for the evolution of the velocity is being provided [DCBC06, CS07], and first-order/kinematic models where the velocity is prescribed in terms of the spatial configuration [MEK99, TB04, BCL09]. The aim of the present paper is to extend, by including alignment interactions, a first-order continuum model for aggregation that attracted a high amount of interest in recent literature [MEK99, BV06, BL07, KSUB11, LTB09, FHK11]. Below we introduce the extended model and its derivation, then point out the fundamental issues that arise with such an extension, and how we address these issues in the present paper.

Consider the following continuum model for the evolution of the macroscopic density function $\rho(t, x)$ in $\mathbb{R}^d$:

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \qquad \rho\big|_{t=0} = \rho_0(x), \tag{1.1a}$$

$$\Phi(t,x)u(t,x) = \int_{\mathbb{R}^d} \phi(|x - y|)\rho(t,y)u(t,y)\,\mathrm{d}y - \nabla_x K * \rho(t,x), \tag{1.1b}$$

where ϕ is an *influence* function that controls the alignment interactions, $\Phi = \phi * \rho$, and K is an attractive-repulsive interaction potential. The asterisk denotes spatial convolution. Hence, the model consists in an active transport equation for the density ρ, with velocity field u defined by (1.1b). The coefficient $\Phi(t, x)$ in the left-hand-side of (1.1b) has the interpretation of the total influence received at location x and time t from the rest of the group. The right-hand-side of (1.1b) has two terms: the first models alignment, and the second models attraction and repulsion, as detailed below.

Alignment is modelled through an averaging mechanism that allows individuals to adjust their velocities relative to the velocities of the others. Specifically, the velocity at location x and time t is assumed to depend non-locally on the velocities $u(t, y)$ at locations y within the alignment interaction range set by the support of the influence function ϕ. The first-term in the right-hand-side of (1.1b) captures this averaging process, with weights/interaction strengths given by $\phi(|x - y|)$, assumed to depend only on the relative distance between locations x and y.

Attraction and repulsion are modelled by the convolution of the gradient of the interaction potential with the population density. In brief, individuals are assumed to repel each other at short ranges, to create a comfort zone around them, but attract each other once they distance themselves too far apart. Equation (1.1a) with the velocity field given *solely* by this term, i.e., $u(t, x) = -\nabla_x K * \rho(t, x)$, constitutes the aggregation model that has been referred to above and which the present paper generalizes. A variety of issues has been investigated during

the last decade for this explicit attractive-repulsive aggregation model, including the well-posedness of solutions [BV06, BL07, BLR11, CDF$^+$11], the long-time behaviour of solutions [MEK99, BDF08, KSUB11, LTB09, BT11, FHK11], and the derivation of the continuum model as a mean-field limit [CCH14].

We point out an essential feature of equation (1.1b), which is that it is an *implicit* equation in u, and can have *non-unique* solutions (e.g., due to translational invariance, one can add an arbitrary function of t to any solution of (1.1b), and obtain a different solution). This is a key challenge brought up by the inclusion of alignment interactions in the explicit aggregation model from [MEK99]. Addressing this challenge is one of the major goals of the present paper. The non-uniqueness of solutions to models of type (1.1) has been noted in [MKJNR12], but no resolution was offered. To our best knowledge, the present paper is the first systematic study of a *first-order* continuum model for aggregation that includes *both* attractive/repulsive and alignment interactions.

The origin of the macroscopic model (1.1) can be traced back to the following second-order discrete model derived from Newton's second law. Suppose there are N particles in $\mathbb{R}^d$, whose positions and velocities denoted by x_i and v_i, respectively ($i = 1, \ldots, N$), evolve according to the following system of ODE's:

$$\frac{dx_i}{dt} = v_i, \tag{1.2a}$$

$$\epsilon \frac{dv_i}{dt} = \frac{1}{N} \sum_{j \neq i} \phi(|x_j - x_i|)(v_j - v_i) - \frac{1}{N} \sum_{j \neq i} \nabla_{x_i} K(x_i - x_j), \tag{1.2b}$$

for $i = 1, \ldots, N$. Here, it has been assumed that that all particles have the same mass $m_i = \epsilon$. The functions ϕ and K have similar meaning as in (1.1).

Without the attractive-repulsive term modelled by the second term in the right-hand-side of (1.2b), model (1.2) represents the celebrated model of Cucker and Smale [CS07]. It is well-known that for certain influence functions ϕ, the Cucker-Smale model successfully captures the unconditional *flocking* phenomenon, where individuals align their velocities into a certain asymptotic direction [HT08, HL09, CFRT10]. Comprehensive literature also exists on second-order attractive-repulsive models without alignment [DCBC06, CDM$^+$07, CFTV10]. Second-order models with *both* alignment and attraction/repulsion have also been studied [CD10, LLEK08, CCR11, AIR11], though in not as much depth and detail as the models with the two sets of forces considered separately.

The passage from the second-order discrete model (1.2) to the first-order macroscopic model (1.1) is done in two steps. First, for each fixed $\epsilon > 0$, one can take the limit $N \to \infty$ in (1.2) and reach, by BBGKY hierarchies or mean field limits (see e.g., [HT08] and the review in [CFTV10]), the following kinetic equation for the density $f(t, x, v)$ at position $x \in \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$:

$$\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon = \frac{1}{\epsilon} \nabla_v \cdot (F[f_\epsilon] f_\epsilon), \qquad f_\epsilon\big|_{t=0} = f_0(x, v), \tag{1.3a}$$

$$F[f_\epsilon] = \int_{\mathbb{R}^{2d}} \phi(|x - y|)(v - v^*) f_\epsilon(t, y, v^*) \, \mathrm{d}y \, \mathrm{d}v^* + \int_{\mathbb{R}^{2d}} \nabla_x K(x - y) f_\epsilon(t, y, v^*) dy dv^*. \tag{1.3b}$$

Rigorous derivations of mean field limits starting from particle systems is a classical subject, comprising an extensive body of works. Some of the most recent works include the mean field

limit of the Cucker-Smale model [HL09, CFRT10], as well as extensions of these results to include general aggregation models of the form (1.2) [CCR11].

The second step in passing from (1.2) to (1.1) is to send $\epsilon \to 0$ in the kinetic equation (1.3) and derive (1.1) as a hydrodynamic limit. The rigorous treatment of this limit constitutes in fact a major component of the present work. Passage from kinetic to macroscopic equations is also a vast topic, extensively studied for instance in the context of hydrodynamic limits of the nonlinear Botlzmann equations. It is beyond the scope of this introduction to give a detailed account of this well-established research area, we simply refer here to a recent review paper [SR14] and the references therein.

As indicated above, a major issue that arises when one considers the first-order model (1.1) is the *non-uniqueness* of solutions to (1.1b). In the present paper we resolve this non-uniqueness issue for the case when the interaction potential is *symmetric* about the origin, i.e., it satisfies

$$K(x) = K(-x), \qquad \text{for all } x \in \mathbb{R}^d. \tag{1.4}$$

For symmetric potentials, the ODE system (1.2) and the kinetic equation (1.3) conserve the linear momentum: $\sum_{i=1}^{N} v_i$ and $\int_{\mathbb{R}^{2d}} v f_\epsilon(t, x, v) \, \mathrm{d}x \, \mathrm{d}v$, respectively. In terms of macroscopic variables, $\int_{\mathbb{R}^d} \rho_\epsilon u_\epsilon \, \mathrm{d}x$ remains constant through the evolution of (1.3), where ρ_ϵ and u_ϵ are defined by:

$$\rho_\epsilon(t, x) = \int_{\mathbb{R}^d} f_\epsilon(t, x, v) \, \mathrm{d}v, \quad \rho_\epsilon(t, x) u_\epsilon(t, x) = \int_{\mathbb{R}^d} v f_\epsilon(t, x, v) \, \mathrm{d}v.$$

The key idea for dealing with the non-uniqueness of solutions to (1.1b) is to account for the fact that (1.1) is the limit of a second-order model for which linear momentum conservation holds and hence, as a limiting equation, (1.1) should inherit this momentum conservation as well.

Given the considerations above, we append model (1.1) with the constraint/requirement that the linear momentum remains constant through the time evolution. With no loss of generality, we assume that the linear momentum is zero at the initial time and hence, it has to remain zero for all times. For ease of referencing, we list below the macroscopic model (1.1) together with momentum conservation:

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \qquad \rho\big|_{t=0} = \rho_0(x), \tag{1.5a}$$

$$\Phi(t, x) u(t, x) = \int_{\mathbb{R}^d} \phi(|x - y|) \rho(t, y) u(t, y) \, \mathrm{d}y - \nabla_x K * \rho(t, x), \tag{1.5b}$$

$$\int_{\mathbb{R}^d} \rho \, u \, \mathrm{d}x = 0. \tag{1.5c}$$

System (1.5) is the main object of study of this paper.

The layout of the present work is as follows. In Section 2 we establish a well-posedness theory for system (1.5). Solutions are sought in the space $\mathcal{P}_c(\mathbb{R}^d)$ of probability measures with compact support, using the mass transportation framework developed in [CCR11]. The main difficulty is to obtain a Lipschitz bound on u, given that for each ρ, (1.5b) provides multiple solutions. The momentum conservation condition (1.5c) plays an essential role, as it enforces uniqueness. The well-posedness result is stated in Theorem 2.4.

Another major issue of interest investigated in detail in this paper is the zero-inertia limit of solutions of the kinetic model (1.3). We use again the framework from [CCR11]

and consider measure-valued solutions f_ϵ of (1.3) in the mass transportation sense. For this reason, the method of characteristics plays a major role throughout our analysis. In Section 3 we establish uniform bounds in ϵ of solutions to (1.3). We note in particular the key bound on the second-order fluctuation term established by Theorem 3.7. The uniform bounds enable us to derive the main convergence results in Section 4. Specifically, we show that f_ϵ converges weak-$*$ as measures to $f(t,x,v) = \rho(t,x)\,\delta(v - u(t,x))$ as $\epsilon \to 0$, where ρ (and u) represent the unique solution to the macroscopic model (1.5). The precise statements are presented in Theorems 4.1 and 4.5. We also point out the convergence of characteristics paths established by Theorem 4.4, where the subtlety resides in the singular limit, with characteristics of a second-order system collapsing into first-order characteristics.

Finally, in Section 5 we design and implement a numerical procedure for solving (1.1). The numerical method is rooted in the analytical considerations made for the continuum model, as a discrete analogue of the momentum conservation is enforced in order to obtain unique solutions for the discretization of (1.5b). To test our method, we implement and illustrate it in one and two dimensions, for several choices of interaction kernels K for which we know the exact steady states.

We note that for models that include only attraction and repulsion, the analogous zero inertia limits $\epsilon \to 0$ of second-order models[1] such as (1.2) and (1.3), have been investigated recently in [FS14] (an earlier study, restricted to PDE analysis, was done in [Jab00]). As in [FS14], the focus in the present work is to find uniform (in ϵ) bounds on the size of the support of the solutions f_ϵ, as well as to show the vanishing (with ϵ) of a second-order fluctuation term. By adding the alignment term such estimates become significantly more involved, as they require the use of the momentum conservation in (1.3), as well as the coupling with control on high order moments of f_ϵ.

Finally, we point out that we work in this paper with smooth interaction potentials and influence functions. Specifically, we assume $\nabla K \in W^{1,\infty}(\mathbb{R}^d)$ and $\phi \in W^{1,\infty}(\mathbb{R}^+)$, where ϕ is also assumed to be positive, non-increasing, and to decay sufficiently slow at infinity. All these assumptions are used in an essential way in our analysis, in particular to control (uniformly in ϵ) the support of the solutions to (1.3) and the second-order fluctuation term. Removing (some of) these requirements would require a significantly different approach and analysis than the present one. We comment however that, from the point of view of applications, the smoothness requirement is mostly irrelevant, as in numerical simulations one does not normally distinguish between a singular potential and its smooth regularization.

2 Well-posedness of the macroscopic equation

In this section, we establish a well-posedness theory for the macroscopic equation (1.5). Solutions ρ are sought in the space $C([0,T]; \mathcal{P}_c(\mathbb{R}^d))$, where $\mathcal{P}_c(\mathbb{R}^d)$ denotes the space of probability measures with compact support in $\mathbb{R}^d$. We use the measure-transportation framework from [CCR11] and consider the characteristic equations associated to (1.5a), given by

$$\frac{\mathrm{d}}{\mathrm{d}t}x = u[\rho](t, x(t)), \qquad x(0) = x_0. \tag{2.1}$$

In order to specify the solution space for (1.1), we use the concept of flow maps: suppose ρ is a fixed function and the vector field $u[\rho](t,x)$ is locally Lipschitz in x. Then standard ODE

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[1]We refer to the kinetic model (1.3) as being "second-order" since it is based on Newton's second law (1.2); in strict terms, equation (1.3) is a first-order PDE.

theory provides the local existence of the flow map $\mathcal{T}_{u[\rho]}^{t}$ associated to (2.1):

$$x_0 \xrightarrow{\mathcal{T}_{u[\rho]}^{t}} x(t),$$

where $x(t)$ is the unique solution of (2.1) that starts at x_0.

Definition 2.1. *Let $T > 0$ be fixed. A measure-valued function $\rho \in C([0,T]; \mathcal{P}_c(\mathbb{R}^d))$ is said to be a solution of the macroscopic equation* (1.5) *with initial condition $\rho_0 \in \mathcal{P}_c(\mathbb{R}^d)$ if*

$$\rho(t) = \mathcal{T}_{u[\rho]}^{t} \# \rho_0, \tag{2.2}$$

where at each $t \in [0,T]$, the velocity field $u[\rho](t, \cdot) \in W^{1,\infty}(supp\rho)$ satisfies (1.5b) *and* (1.5c).

The push-forward in Definition 2.1 is taken in the mass transportation sense; equation (2.2) is equivalent to:

$$\int_{\mathbb{R}^d} \varphi(x) \rho(t,x) dx = \int_{\mathbb{R}^d} \varphi(\mathcal{T}_{u[\rho]}^{t}(X)) \rho_0(X) dX,$$

for all $\varphi \in C_b(\mathbb{R}^d)$.

The results in [CCR11] establish the existence and uniqueness of measure solutions for various aggregation models via fixed point arguments. These results apply in particular to attractive-repulsive models in the form (1.5a), but there the velocity field u is given *explicitly* in terms of ρ. In [CCR11], the assumptions made on the attractive-repulsive potential K guarantee the local Lipschitz continuity of the velocity field u and hence, the global existence and uniqueness of characteristics paths.

In our model, u is given *implicitly* by (1.5b) and (1.5c). While (1.5b) can have multiple solutions (as discussed above), we show that the momentum conservation property (1.5c) enforces uniqueness. Once a unique solution $u[\rho]$ of (1.5b)-(1.5c) has been identified, this solution is shown to be Lipschitz continuous in the spatial variable so that the characteristic equation is well-posed.

We now establish the necessary a priori bounds for (ρ, u) for deriving the well-posedness of solutions to (1.5). First we show the Lipschitz bound of u for a fixed density function ρ. Similar estimates have also been used in [MT11, TT14] for flocking models.

Proposition 2.1. *Assume K is symmetric (i.e., it satisfies* (1.4)*), $\nabla_x K \in W^{1,\infty}(\mathbb{R}^d)$ and $\phi \in W^{1,\infty}(\mathbb{R}^+)$ is positive and non-increasing. Consider a given probability measure $\rho(t, \cdot) \in \mathcal{P}_c(\mathbb{R}^d)$ at some fixed time t. Then equations* (1.5b)-(1.5c) *uniquely define a bounded, Lipschitz continuous velocity field u on the support of ρ.*

Proof. Since equations (1.5b)-(1.5c) hold at a specific fixed time, we simplify the notation, and drop the time dependence in the proof. Denote

$$S_\rho = \max_{x \in \mathrm{supp}\rho} |x|. \tag{2.3}$$

By our assumption on ρ, S_ρ is finite. The following lower bound on the influence function ϕ can be inferred immediately:

$$\phi(|x-y|) \geq \eta \qquad \text{for any } x, y \in \mathrm{supp}\rho,$$

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where

$$\eta := \phi(2S_\rho) > 0. \tag{2.4}$$

As a consequence, $\Phi(x)$ is bounded from below. Indeed, for any $x \in \text{supp}\rho$,

$$\Phi(x) = \int\limits_{\text{supp}\rho} \phi(|x-y|)\rho(y)\,\mathrm{d}y \geq \eta. \tag{2.5}$$

Moreover, we have

$$\Phi(x) \leq \|\phi\|_{L^\infty(0,\infty)}, \qquad \text{for any } x \in \mathbb{R}^d.$$

Define the following alignment operator $\mathcal{A}$:

$$\mathcal{A}[u](x) = \Phi(x)u(x) - \int\limits_{\mathbb{R}^d} \phi(|x-y|)\rho(y)u(y)\,\mathrm{d}y. \tag{2.6}$$

It is easy to check that $\mathcal{A}$ is a linear bounded operator which maps $L^\infty(\text{supp}\rho)$ to $L^\infty(\text{supp}\rho)$.

Equation (1.5b) can now be expressed as

$$\mathcal{A}[u](x) = -\nabla_x K * \rho(x).$$

As commented above, the operator $\mathcal{A}$ is *not* invertible since $\mathcal{A}$ maps all constant (in x) functions to zero.

To overcome the degeneracy of $\mathcal{A}$, we make use of the momentum conservation condition (1.5c) and define a new operator $\mathcal{M}$:

$$\begin{aligned} \mathcal{M}[u](x) := \mathcal{A}[u](x) + \eta \int\limits_{\mathbb{R}^d} \rho(y)u(y)dy \\ = \Phi(x)u(x) - \int\limits_{\mathbb{R}^d} (\phi(|x-y|) - \eta)\rho(y)u(y)\,\mathrm{d}y. \end{aligned} \tag{2.7}$$

We note that the operators $\mathcal{A}$ and $\mathcal{M}$ depend on ρ; however, we choose not to indicate this dependence explicitly in the notations, as this fact is irrelevant for the main well-posedness result that follows.

A solution u of (1.5b) and (1.5c) also satisfies

$$\mathcal{M}[u](x) = -\nabla_x K * \rho(x). \tag{2.8}$$

We claim that $\mathcal{M}$ is invertible. Indeed, given any $x \in \text{supp}\rho$,

$$\int\limits_{\mathbb{R}^d} (\phi(|x-y|) - \eta)\rho(y)u(y)\,\mathrm{d}y \leq \|u\|_{L^\infty(\text{supp}\rho)} \int\limits_{\text{supp}\rho} (\phi(|x-y|) - \eta)\rho(y)dy$$
$$= \|u\|_{L^\infty(\text{supp}\rho)}(\Phi(x) - \eta).$$

By the definition of $\mathcal{M}$ in (2.7), one has

$$\mathcal{M}[u](x) \geq \Phi(x)u(x) - \|u\|_{L^\infty(\text{supp}\rho)}(\Phi(x) - \eta) \qquad \text{for all } x \in \text{supp}\rho.$$

For any $\delta > 0$, there exists $x_\delta \in \mathrm{supp}\rho$ such that $u(x_\delta) \geq \|u\|_{L^\infty(\mathrm{supp}\rho)} - \delta$ and hence,

$$\mathcal{M}[u](x_\delta) \geq \Phi(x_\delta)\|u\|_{L^\infty(\mathrm{supp}\rho)} - \|u\|_{L^\infty(\mathrm{supp}\rho)}(\Phi(x_\delta) - \eta) - \delta\,\|\phi\|_{L^\infty(0,\infty)}$$
$$= \eta\,\|u\|_{L^\infty(\mathrm{supp}\rho)} - \delta\,\|\phi\|_{L^\infty(0,\infty)}\,.$$

By letting $\delta \to 0$ we obtain

$$\|\mathcal{M}[u]\|_{L^\infty(\mathrm{supp}\rho)} \geq \eta\|u\|_{L^\infty(\mathrm{supp}\rho)}. \tag{2.9}$$

Hence for a given ρ, $\mathcal{M}$ is invertible on $L^\infty(\mathrm{supp}\rho)$ and u is uniquely solvable from (2.8) such that

$$u(x) = (u[\rho])(x) = -\mathcal{M}^{-1}\left(\nabla_x K * \rho(x)\right). \tag{2.10}$$

Next we show that $u \in W^{1,\infty}(\mathrm{supp}\rho)$ and find an explicit bound on its $W^{1,\infty}$-norm. We start with the L^∞ estimate. From (2.9) and (2.10), we have

$$\|u\|_{L^\infty(\mathrm{supp}\rho)} \leq \|\mathcal{M}^{-1}\|\,\|\nabla_x K * \rho\|_{L^\infty(\mathrm{supp}\rho)} \leq \frac{1}{\eta}\|\nabla_x K\|_{L^\infty}. \tag{2.11}$$

Next, we estimate the bound of $\nabla_x u$. The distributional derivative of u satisfies

$$\nabla_x u = \int_{\mathbb{R}^d} \nabla_x\left(\frac{\phi(|x-y|)}{\Phi(x)}\right)\rho(y)u(y)dy + \nabla_x\left(\frac{\nabla_x K}{\Phi}\right).$$

Applying the lower bound estimate (2.5) for Φ and the L^∞ estimate (2.11) for u in each term gives

$$\left\|\int_{\mathbb{R}^d} \nabla_x\left(\frac{\phi(|x-y|)}{\Phi(x)}\right)\rho(y)u(y)dy\right\|_{L^\infty(\mathrm{supp}\rho)} \leq \frac{2}{\eta^2}\|\phi'\|_{L^\infty}\|\nabla_x K\|_{L^\infty},$$

$$\left\|\nabla_x\left(\frac{\nabla_x K}{\Phi}\right)\right\|_{L^\infty(\mathrm{supp}\rho)} \leq \frac{1}{\eta^2}\|\phi'\|_{L^\infty}\|\nabla_x K\|_{L^\infty} + \frac{1}{\eta}\|\nabla_x^2 K\|_{L^\infty}.$$

This shows

$$\|\nabla_x u\|_{L^\infty(\mathrm{supp}\rho)} \leq \frac{3}{\eta^2}\|\phi'\|_{L^\infty}\|\nabla_x K\|_{L^\infty} + \frac{1}{\eta}\|\nabla_x^2 K\|_{L^\infty}. \tag{2.12}$$

Combining (2.11) with (2.12), we have

$$\|u\|_{W^{1,\infty}(\mathrm{supp}\rho)} \leq \left(\frac{1}{\eta} + \frac{3}{\eta^2}\|\phi'\|_{L^\infty(0,\infty)}\right)\|\nabla_x K\|_{L^\infty} + \frac{1}{\eta}\left\|\nabla_x^2 K\right\|_{L^\infty}, \tag{2.13}$$

with η defined in (2.4). $\qquad\square$

Remark 2.2. *The values of the velocity field u outside the support of ρ do not explicitly enter in the dynamics of model (1.5). However, we can extend the definition of u outside of $\mathrm{supp}\rho$ by using (1.5b). This extension is applied in the proof of Theorem 2.4 where a $W^{1,\infty}$-bound on u is used in a domain that is larger than $\mathrm{supp}\rho$. In particular, we note that the right-hand-side of (1.5b) only depends on values of ρ and u inside the support of ρ. For any $R > 0$ such that $\mathrm{supp}\rho \subset B_R$ and $x \in B_R \setminus \mathrm{supp}\rho$, $\Phi(x)$ is bounded from below by $\phi(2R)$ (see (2.5)) and the assumptions made on K and ϕ along with the bound (2.13) immediately imply a bound on $\|u\|_{W^{1,\infty}(B_R)}$ that depends on R, $\|\nabla_x K\|_{W^{1,\infty}}$ and $\|\phi\|_{W^{1,\infty}}$ This bound has a similar expression as in (2.13), but with η replaced by $\phi(2R)$.*

As discussed below, Proposition 2.1 combined with the fixed-point argument developed in [CCR11] is enough to conclude the *local* existence of solutions to (1.5) in the measure-transportation sense of Definition 2.1. In order to show the *global* existence, we need to prove *a priori* that the support of ρ does not become infinite in a finite time. This requires an additional assumption on the influence function ϕ. The precise statement is as follows:

Proposition 2.3. *Assume K and ρ satisfy the hypotheses of Proposition 2.1, together with the following slow decay condition on the influence function ϕ:*

$$\int\limits_0^\infty \phi(r)\,\mathrm{d}r = \infty. \tag{2.14}$$

Let (ρ, u) be a solution to (1.5) with compactly supported initial density ρ_0. Then for any $t > 0$, we have

(a) *$\rho(t, \cdot)$ remains compactly supported and the bound of its support satisfies*

$$S_\rho(t) \leq \frac{1}{2}\Psi^{-1}\left(\Psi(2S_\rho(0)) + 2t\,\|\nabla_x K\|_{L^\infty}\right), \tag{2.15}$$

where $S_\rho(t)$ is defined in (2.3) and $\Psi(r) = \int_0^r \phi(s)\,\mathrm{d}s$.

(b) *$u \in W^{1,\infty}(supp\rho)$ and (2.13) holds with*

$$\eta(t) = \phi(\Psi^{-1}(\Psi(2S_\rho(0)) + 2t\,\|\nabla_x K\|_{L^\infty})).$$

Proof. First, from (2.11) we know that the L^∞-bound of u depends on S_ρ. Specifically,

$$\|u(t,\cdot)\|_{L^\infty(\mathrm{supp}\rho(t))} \leq \frac{1}{\phi(2S_\rho(t))}\,\|\nabla_x K\|_{L^\infty} \tag{2.16}$$

for all t in a local interval of existence $[0, T]$. On the other hand, for any characteristic path $x(t)$ originating at $x_0 \in \mathrm{supp}\rho_0$ (see (2.1)), we have

$$\frac{d}{dt}|x(t)| \leq \|u(t,\cdot)\|_{L^\infty(\mathrm{supp}\rho(t))}.$$

Hence S_ρ is bounded in terms of u as

$$S_\rho(t) \leq S_\rho(0) + \int\limits_0^t \|u(\tau,\cdot)\|_{L^\infty(\mathrm{supp}\rho(\tau))}\,\mathrm{d}\tau.$$

We can now close the loop between S_ρ and u. To this end, denote

$$D(t) = S_\rho(0) + \int\limits_0^t \|u(\tau,\cdot)\|_{L^\infty(\mathrm{supp}\rho(\tau))}\,\mathrm{d}\tau.$$

Then $S_\rho(t) \leq D(t)$. By (2.16) and the monotonicity of ϕ,

$$D'(t) = \|u(t,\cdot)\|_{L^\infty(\mathrm{supp}\rho(t))} \leq \frac{1}{\phi(2D(t))}\,\|\nabla_x K\|_{L^\infty},$$

which implies

$$\frac{d}{dt}\Psi(2D(t)) \le 2\left\|\nabla_x K\right\|_{L^\infty}.$$

By integrating the equation above, we find

$$\Psi(2D(t)) \le \Psi(2S_\rho(0)) + 2t\left\|\nabla_x K\right\|_{L^\infty}. \tag{2.17}$$

Assumption (2.14) implies that $\Psi(r) \to \infty$ as $r \to \infty$. Moreover, $\Psi(0) = 0$ and $\Psi' > 0$. Consequently, Ψ is a bijection from $\mathbb{R}^+$ to $\mathbb{R}^+$. We can thus infer (2.15) from (2.17) and the inequality $S_\rho(t) \le D(t)$. The Lipschitz bound on u in part (b) follows immediately. $\square$

Combining the a priori bounds in Proposition 2.3 with the fixed-point argument in [CCR11], we can now show the global wellposedness of solutions to (1.5). The theorem states

Theorem 2.4 (Well-posedness of the macroscopic model). *Assume K is symmetric, $\nabla_x K \in W^{1,\infty}(\mathbb{R}^d)$ and $\phi \in W^{1,\infty}(\mathbb{R}^+)$ is positive, non-increasing, and satisfies the slow decay condition (2.14). Suppose the initial measure $\rho_0 \in \mathcal{P}_c(\mathbb{R}^d)$. Then for any $T > 0$, there exists a unique solution (ρ, u) to (1.5) such that $\rho \in C([0,T], \mathcal{P}_c(\mathbb{R}^d))$ and $u \in C([0,T]; W^{1,\infty}(supp\rho))$.*

Proof. The proof follows directly from the theory developed in [CCR11]; we only summarize the setup and the main steps. We refer for details to the proof of Theorem 3.10 in [CCR11].

Local well-posedness. Fix $R > 0$ such that the ball B_R contains the initial support of ρ_0, and consider the set $\mathcal{S}$ consisting of all functions in $C([0,T], \mathcal{P}_c(\mathbb{R}^d))$ such that the support of $\rho_t(\cdot) = \rho(t, \cdot)$ is contained in the ball B_R for all $t \in [0,T]$ (T is a positive number to be chosen later). The set $\mathcal{S}$ endowed with the distance $\mathcal{W}_1$ defined by

$$\mathcal{W}_1(\rho_t, \tilde{\rho}_t) = \sup_{t \in [0,T]} W_1(\rho_t, \tilde{\rho}_t),$$

is a complete metric space. Here, W_1 denotes the 1-Wasserstein distance.

For fixed $\rho \in \mathcal{S}$, one can define a map $\mathcal{G}$ on this metric space by

$$\mathcal{G}[\rho](t) := \mathcal{T}^t_{u[\rho]} \# \rho_0.$$

Consequently, solutions of (1.5) in the sense of Definition 2.1 are fixed points of the map $\mathcal{G}$. The essential step is to show that this map is contractive, and hence, conclude that it has a unique fixed point in $\mathcal{S}$.

In our context, the key observation is that for every $\rho \in \mathcal{S}$, one can use the argument in the proof of Proposition 2.1 (with $\eta = \phi(2R)$ independent of $t \in [0,T]$ and $\rho \in \mathcal{S}$) and find a $W^{1,\infty}$-bound on the velocity field $u[\rho]$, bound which depends only on K, ϕ and R (see estimate (2.13) and Remark 2.2).

The Lipschitz continuity of $u[\rho]$ provides well-posedness of characteristic trajectories for each $\rho \in \mathcal{S}$. Also, for T short enough, the support of $\mathcal{G}[\rho](t)$ is contained in B_R. By simple arguments (Lemmas 3.11 and 3.7 in [CCR11]), the following estimate can be shown to hold:

$$W_1(\mathcal{G}[\rho_1](t), \mathcal{G}[\rho_2](t)) \le C(t) \sup_{t \in [0,T]} \|u[\rho_1] - u[\rho_2]\|_{L^\infty(B_R)}.$$

To infer the contractivity of $\mathcal{G}$ we also need an estimate of the form

$$\sup_{t\in[0,T]} \|u[\rho_1] - u[\rho_2]\|_{L^\infty(B_R)} \leq \tilde{C}(T) \sup_{t\in[0,T]} W_1(\rho_1,\rho_2), \qquad (2.18)$$

for some constant $\tilde{C}$ that depends on T, as well as on K, ϕ and R. Requiring that T is small enough such that $\tilde{C} < 1$ concludes the existence of a fixed point of $\mathcal{G}$, and consequently, of a *local* solution to (1.5) in the sense of Definition 2.1.

For proving (2.18), let $\rho_1, \rho_2 \in \mathcal{S}$, and $\eta = \phi(2R)$. Denote $u_1 = u[\rho_1]$ and $u_2 = u[\rho_2]$. Then, by (1.5b) and (1.5c),

$$(\phi * \rho_1)u_1(t,x) = \int_{B_R} (\phi(|x-y|) - \eta)\rho_1(t,y)u_1(t,y)\,dy - \nabla_x K * \rho_1(t,x),$$

$$(\phi * \rho_2)u_2(t,x) = \int_{B_R} (\phi(|x-y|) - \eta)\rho_2(t,y)u_2(t,y)\,dy - \nabla_x K * \rho_2(t,x).$$

By Remark 2.2, we can extend u_1 and u_2 to the whole ball B_R such that $u_1, u_2 \in W^{1,\infty}(B_R)$. Taking the difference of the above two equations, we have

$$(\phi * \rho_1)(u_1 - u_2) = \int_{B_R} (\phi(|x-y|) - \eta)\rho_1(y)(u_1(y) - u_2(y))\,dy + \mathcal{R}_1, \qquad (2.19)$$

where

$$\mathcal{R}_1 = -\nabla_x K * (\rho_1 - \rho_2) - (\phi * (\rho_1 - \rho_2))u_2 + \int_{B_R} (\phi(|x-y|) - \eta)(\rho_1(y) - \rho_2(y))u_2(y)\,dy.$$

By Lemma 3.15 in [CCR11], we have

$$\|\mathcal{R}_1\|_{L^\infty(B_R)} \leq \left(\|\nabla_x^2 K\|_{L^\infty(\mathbb{R}^d)} + \|\phi\|_{W^{1,\infty}(0,\infty)} \|u_2\|_{W^{1,\infty}(B_R)}\right) W_1(\rho_1, \rho_2),$$

where a bound on $\|u_2\|_{W^{1,\infty}(B_R)}$ can be inferred as explained in Remark 2.2.

From (2.19), following an argument similar to that leading to (2.11) from (2.7) and (2.8), we estimate

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^\infty(B_R)} \leq \tilde{C} W_1(\rho_1,\rho_2), \qquad \text{for all } t \in [0,T],$$

where $\tilde{C}$ depends on T, R, $\|\phi\|_{W^{1,\infty}(0,\infty)}$ and $\|\nabla_x K\|_{W^{1,\infty}(\mathbb{R}^d)}$. Hence (2.18) holds and there exists $T > 0$ small enough such that (1.1) has a unique solution (ρ, u) over the time interval $[0,T]$.

Global well-posedness. Note that the argument above does not use the slow decay condition (2.14) on ϕ. Local existence can be extended for as long as the support of ρ remains bounded. Provided that ϕ satisfies the additional assumption (2.14), Proposition 2.3 guarantees that the support of ρ does not blow up in a finite time. In this case the local theory can be extended to $[0,T]$ for any $T > 0$, which gives the global well-posedness. $\square$

Remark 2.5. *It is interesting to note that the slow decay condition* (2.14) *appears in previous literature on the Cucker-Smale model, in connection to asymptotic flocking. Indeed, for influence functions in power-law form, it has been shown in* [HL09] *that the discrete Cucker-Smale model has the* unconditional flocking *property when ϕ satisfies* (2.14). *In addition, the same property* (2.14) *is needed for unconditional flocking in the corresponding kinetic and macroscopic Cucker-Smale models* [CFRT10, Tan14, TT14]. *In our context, condition* (2.14) *is first needed to guarantee global-in-time well-posedness of solutions to* (1.5), *as discussed above. Then, condition* (2.14) *is needed again, in an essential way, to show that solutions f_ϵ of the kinetic model* (1.3) *remain compactly supported for all finite times, with a bound on their support that is independent of ϵ (see Proposition 3.4 and its proof). This estimate is essential in the $\epsilon \to 0$ limit of the kinetic model.*

Note that by [AGS06] (Lemma 8.1.6 in Chapter 8), the unique measure-valued solution in Theorem 2.4 is exactly the unique weak solution for (1.5), i.e., it satisfies

$$\int\limits_0^T \int\limits_{\mathbb{R}^d} \partial_t \varphi(t,x) \rho(t,x) \, \mathrm{d}x \, \mathrm{d}t + \int\limits_0^T \int\limits_{\mathbb{R}^d} \nabla_x \varphi \cdot \rho u \, \mathrm{d}x \, \mathrm{d}t + \int\limits_{\mathbb{R}^d} \varphi(0,x) \rho_0(x,v) \, \mathrm{d}x = 0, \qquad (2.20)$$

for any $\varphi \in C_c^1([0,T]; C_b^1(\mathbb{R}^d))$.

3 Well-posedness and uniform bounds of the kinetic model

In this section we make preparations for proving the convergence of the kinetic equation (1.3) to the macroscopic model (1.5) by showing several uniform bounds in ϵ. These include an uniform bound on the support of the density function f_ϵ, along with uniform bounds on its moments. A key step is to show that a certain "second-order fluctuation" term vanishes with ϵ, which is the main ingredient that allows the passage from the second-order kinetic equation (1.3) to the first order macroscopic model (1.5).

3.1 Well-posedness setup for the kinetic model (1.3)

Model (1.3) with $\epsilon > 0$ fixed is a particular case of the general class of models studied in [CCR11]. For completeness, we present briefly the well-posedness setup from [CCR11], which is similar in fact to the measure transportation framework used for the macroscopic equation in Section 2.

The solution space for equation (1.3) is $C([0,T], \mathcal{P}_c(\mathbb{R}^d \times \mathbb{R}^d))$, where $\mathcal{P}_c(\mathbb{R}^d \times \mathbb{R}^d)$ represents the space of probability measures with compact support in $\mathbb{R}^d \times \mathbb{R}^d$. The approach is to consider the characteristic equations associated with (1.3) and define measure-valued solutions f_ϵ to (1.3) as the mass-transport (or push-forward) of the initial density f_0 along the flow defined by characteristics.

Specifically, the characteristic equations for (1.3) are given by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v,$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{\epsilon}\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\phi(|x-y|)(v-v^*)f_\epsilon(t,y,v^*)\,\mathrm{d}y\,\mathrm{d}v^* + \nabla_x K * \rho_\epsilon(t,x)\right), \qquad (3.1)$$

$$(x,v)\big|_{t=0} = (x_0, v_0) \in \mathrm{supp}f_0.$$

In short-hand notation, one can write the characteristic system (3.1) as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v,$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \mathcal{H}[f_\epsilon](t,x,v), \qquad (3.2)$$

$$(x,v)\big|_{t=0} = (x_0, v_0) \in \mathrm{supp}f_0,$$

where $\mathcal{H}[f_\epsilon]$ denotes the entire right-hand-side of the v-equation in (3.1).

Suppose that f_ϵ is a fixed function and the vector field $\mathcal{H}[f_\epsilon](t,x,v)$ is locally Lipschitz in x and v. Then, standard ODE theory provides the local existence of the flow map $\mathcal{T}^{t,\epsilon}_{\mathcal{H}[f_\epsilon]}$ associated to (3.2):

$$(x_0, v_0) \xrightarrow{\mathcal{T}^{t,\epsilon}_{\mathcal{H}[f_\epsilon]}} (x,v), \qquad (x,v) = (x_\epsilon(t), v_\epsilon(t)),$$

where $(x_\epsilon(t), v_\epsilon(t))$ is the unique solution of (3.2) that starts at (x_0, v_0).

Similar to Definition 2.1, we consider measure-valued solutions of (1.3) in the mass-transportation sense [CCR11]:

Definition 3.1. *Let $T > 0$ be fixed. A measure-valued function $f_\epsilon \in C([0,T]; \mathcal{P}_c(\mathbb{R}^d \times \mathbb{R}^d))$ is said to be a solution of the kinetic equation (1.3) with initial condition $f_0 \in \mathcal{P}_c(\mathbb{R}^d \times \mathbb{R}^d)$ if*

$$f_\epsilon(t) = \mathcal{T}^{t,\epsilon}_{\mathcal{H}[f_\epsilon]}\#f_0, \qquad (3.3)$$

for each $t \in [0,T]$.

The push-forward in (3.3) is in the mass transportation sense:

$$\int_{\mathbb{R}^{2d}} \zeta(x,v)f_\epsilon(t,x,v)dxdv = \int_{\mathbb{R}^{2d}} \zeta(\mathcal{T}^{t,\epsilon}_{\mathcal{H}[f_\epsilon]}(X,V))f_0(X,V)dXdV, \qquad (3.4)$$

for all $\zeta \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$.

The results in [CCR11, Section 4] apply to the kinetic equation (1.3) with $\epsilon > 0$ fixed. The well-posedness result that can be inferred directly from there is the following.

Theorem 3.1 (Well-posedness of the kinetic model (1.3) [CCR11])**.** *Assume the following properties on the influence function ϕ and the interaction potential K:*

ϕ *and* ∇K *are locally Lipschitz, and* $|\phi(x)|, |\nabla K(x)| \leq C(1 + |x|)$ *for all* $x \in \mathbb{R}^d$,

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for some $C > 0$. Take an initial measure $f_0 \in \mathcal{P}_c(\mathbb{R}^d \times \mathbb{R}^d)$. Then there exists a unique solution $f_\epsilon \in C([0, \infty), \mathcal{P}_c(\mathbb{R}^d \times \mathbb{R}^d))$ of (1.3), in the sense of Definition 3.1, whose support grows at a controlled rate. Specifically, there exists an increasing function $R_\epsilon(T)$ such that for all $T > 0$,

$$\text{supp}\, f_\epsilon(t) \subset B_{R_\epsilon(T)} \quad \text{for all } t \in [0, T].$$

We also note that by results in [AGS06, CDF$^+$11], f_ϵ given by Theorem 3.1 is also a weak solution of (1.3), i.e., it satisfies

$$
\int\limits_0^T \int\limits_{\mathbb{R}^{2d}} \partial_t \psi(t, x, v) f_\epsilon(t, x, v) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + \int\limits_0^T \int\limits_{\mathbb{R}^{2d}} \nabla_x \psi \cdot v f_\epsilon \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t
$$
$$
- \frac{1}{\epsilon} \int\limits_0^T \int\limits_{\mathbb{R}^{2d}} \nabla_v \psi \cdot F[f_\epsilon] f_\epsilon \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + \int\limits_{\mathbb{R}^{2d}} \psi(0, x, v) f_0(x, v) \, \mathrm{d}x \, \mathrm{d}v = 0,
$$

(3.5)

for any $\psi \in C_c^1([0, T); C_b^1(\mathbb{R}^d \times \mathbb{R}^d))$.

Remark 3.2. *An essential assumption made throughout our work is that the interaction potential K is symmetric. For such potentials, the kinetic model (1.3) conserves linear momentum through time evolution. To see this, one can take in the weak formulation (3.5) the test function $\psi(t, x, v) = \psi_1(t) v \psi_2(v)$, where $\psi_1 \in C_c^1(0, T)$ and $v \psi_2(v) \in C_b^1(\mathbb{R}^d)$, with $\psi_2 = 1$ on the set $\{v \in \mathbb{R}^d : (x, v) \in \text{supp} f_\epsilon\}$. Due to the symmetry of K, $\int_{\mathbb{R}^{2d}} F[f_\epsilon] f_\epsilon \, \mathrm{d}x \, \mathrm{d}v = 0$, and we obtain that*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\mathbb{R}^{2d}} v f_\epsilon(t, x, v) \, \mathrm{d}x \, \mathrm{d}v = 0, \qquad \text{in the distributional sense.}$$

Throughout the paper we assume, with no loss of generality, that the initial density f_0 has zero momentum, i.e.,

$$\int\limits_{\mathbb{R}^{2d}} v f_0(x, v) \, \mathrm{d}x \, \mathrm{d}v = 0.$$

(3.6)

Hence, the solutions f_ϵ provided by Theorem 3.1 satisfy:

$$\int\limits_{\mathbb{R}^{2d}} v f_\epsilon(t, x, v) \, \mathrm{d}x \, \mathrm{d}v = 0,$$

(3.7)

for all finite times.

3.2 Uniform bound of the support

We prove here that starting from a compactly supported probability density function f_0, the solution $f_\epsilon(t, \cdot, \cdot)$ to (1.3) remains compactly supported in (x, v) for any finite time. Moreover, the size of $\text{supp} f_\epsilon$ is independent of ϵ.

We first prove a rough bound on the first moment[2] of f_ϵ.

---

[2]We adopt here a common terminology used for kinetic equations and refer to *moments* of f_ϵ the moments with respect with the velocity variable v.

Proposition 3.3. *Let f_ϵ be the solution to (1.3), as provided by Theorem 3.1. Suppose $\phi \geq 0$ and that the initial data f_0 has a finite first moment. Then for each finite time $T > 0$ and each fixed $\epsilon > 0$, $f_\epsilon(t)$ has a finite first moment for any $t \in [0, T]$. The bound of the first moment only depends on T, ϵ, and K. In particular, it is independent of ϕ.*

Proof. By symmetry of the integral we have

$$\int\limits_{\mathbb{R}^{2d}} \frac{v}{|v|} \cdot F[f_\epsilon] f_\epsilon \, \mathrm{d}x \, \mathrm{d}v = \int\limits_{\mathbb{R}^{4d}} \phi(|x - y|)(v - v^*) \cdot \frac{v}{|v|} f_\epsilon(t, y, v^*) f_\epsilon(t, x, v) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}v \, \mathrm{d}v^*$$

$$+ \int\limits_{\mathbb{R}^{2d}} \frac{v}{|v|} \cdot \left(\nabla_x K * f_\epsilon \right) f_\epsilon \, \mathrm{d}x \, \mathrm{d}v$$

$$= \frac{1}{2} \int\limits_{\mathbb{R}^{4d}} \phi(|x - y|)(v - v^*) \cdot \left(\frac{v}{|v|} - \frac{v^*}{|v^*|} \right) f_\epsilon(t, y, v^*) f_\epsilon(t, x, v) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}v \, \mathrm{d}v^*$$

$$+ \int\limits_{\mathbb{R}^{2d}} \frac{v}{|v|} \cdot \left(\nabla_x K * f_\epsilon \right) f_\epsilon \, \mathrm{d}x \, \mathrm{d}v.$$

Notice that

$$(v - v^*) \cdot \left(\frac{v}{|v|} - \frac{v^*}{|v^*|} \right) = |v| + |v^*| - \frac{v \cdot v^*}{|v|} - \frac{v^* \cdot v}{|v^*|} \geq 0.$$

Hence,

$$\int\limits_{\mathbb{R}^{2d}} \frac{v}{|v|} \cdot F[f_\epsilon] f_\epsilon \, \mathrm{d}x \, \mathrm{d}v \geq \int\limits_{\mathbb{R}^{2d}} \frac{v}{|v|} \cdot \left(\nabla_x K * f_\epsilon \right) f_\epsilon \, \mathrm{d}x \, \mathrm{d}v.$$

This gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\mathbb{R}^{2d}} |v| f_\epsilon(t, x, v) \, \mathrm{d}x \, \mathrm{d}v \leq -\frac{1}{\epsilon} \int\limits_{\mathbb{R}^{2d}} \frac{v}{|v|} \cdot \left(\nabla_x K * f_\epsilon \right) f_\epsilon \, \mathrm{d}x \, \mathrm{d}v \leq \frac{1}{\epsilon} \left\| \nabla_x K \right\|_{L^\infty}.$$

Therefore, for $t \in [0, T]$, we have

$$\int\limits_{\mathbb{R}^{2d}} |v| f_\epsilon(t, x, v) \, \mathrm{d}x \, \mathrm{d}v \leq \int\limits_{\mathbb{R}^{2d}} |v| f_0(x, v) \, \mathrm{d}x \, \mathrm{d}v + \frac{T}{\epsilon} \left\| \nabla_x K \right\|_{L^\infty}.$$

Hence the first moment of f_ϵ is bounded for each ϵ and T, and the bound is independent of ϕ. $\square$

The following proposition shows that $f_\epsilon(t, \cdot, \cdot)$ is compactly supported and the size of its support is independent of ϵ.

Proposition 3.4 (Uniform bound on the support of f_ϵ). *Suppose K is symmetric, $\nabla_x K \in W^{1,\infty}(\mathbb{R}^d)$ and $\phi \in W^{1,\infty}(\mathbb{R}^+)$ is positive, non-increasing, and satisfies the slow decay condition (2.14). Let f_ϵ be the solution to (1.3) with an initial data $f_0 \in \mathcal{P}_c(\mathbb{R}^d \times \mathbb{R}^d)$ that satisfies (3.6). Then for each finite time T, $f_\epsilon(t, \cdot, \cdot)$ has a compact support for any $t \in [0, T]$. The size of the support only depends on T, ϕ, and K; in particular, it is independent of ϵ.*

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Proof. Consider the characteristic equations (3.1) associated with (1.3). Using the boundedness of the first moment of f_ϵ on the time interval $[0, T]$, as shown in Proposition 3.3, we immediately derive from (3.1) that for each fixed ϵ and T, f_ϵ has a compact support in (x, v). Our goal is to show that the size of this compact support is independent of ϵ.

To indicate the dependence on the initial location, we use the notation $(x_\epsilon(t; x_0, v_0), v_\epsilon(t; x_0, v_0))$ for the characteristic path that originates at (x_0, v_0). Consider now the entire family of characteristic paths initialized at points (x_0, v_0) inside the support of f_0. Since $x_\epsilon(t; x_0, v_0)$ and $v_\epsilon(t; x_0, v_0)$ are continuous in (x_0, v_0) for each $t \in [0, T]$, the maxima of the supports of x_ϵ and v_ϵ are attained, and one can define the sizes $S_\epsilon(t)$ and $V_\epsilon(t)$ of these supports:

$$S_\epsilon(t) = \max_{(x_0,v_0)\in \mathrm{supp} f_0} |x_\epsilon(t; x_0, v_0)|, \qquad V_\epsilon(t) = \max_{(x_0,v_0)\in \mathrm{supp} f_0} |v_\epsilon(t; x_0, v_0)|. \tag{3.8}$$

Denote

$$D_\epsilon(t) = S_\epsilon(0) + \int_0^t |v_\epsilon(\tau)| \, \mathrm{d}\tau. \tag{3.9}$$

Then

$$S_\epsilon(t) \le D_\epsilon(t). \tag{3.10}$$

Denote

$$\Phi_\epsilon(t, x) = \int_{\mathbb{R}^d} \phi(|x - y|) \rho_\epsilon(t, y) \, \mathrm{d}y. \tag{3.11}$$

Let $v_M(t)$ be the velocity with the largest magnitude at time t, that is,

$$|v_M(t)| = V_\epsilon(t), \tag{3.12}$$

and let $\eta(t) = \phi(2D_\epsilon(t))$. Then the monotonicity of ϕ gives

$$0 < \eta(t) \le \phi(|x - y|) \qquad \text{for any } (x, \cdot), (y, \cdot) \in \mathrm{supp} f_\epsilon(t, \cdot, \cdot).$$

The time evolution of V_ϵ is given by

$$\frac{\mathrm{d}V_\epsilon}{\mathrm{d}t} = -\frac{1}{\epsilon} \frac{v_M}{|v_M|} \cdot \left(\int_{\mathbb{R}^{2d}} \phi(|x_\epsilon - y|)(v_M - v^*) f_\epsilon(t, y, v^*) \, \mathrm{d}y \, \mathrm{d}v^* + \nabla_x K * \rho_\epsilon \right)$$

$$\le -\frac{1}{\epsilon} \left(|v_M| \int_{\mathbb{R}^d} \phi(|x_\epsilon - y|) \rho_\epsilon(t, y) \, \mathrm{d}y - \frac{v_M}{|v_M|} \cdot \int_{\mathbb{R}^{2d}} \phi(|x_\epsilon - y|) v^* f_\epsilon(t, y, v^*) \, \mathrm{d}y \, \mathrm{d}v^* \right)$$

$$\tag{3.13}$$

$$+ \frac{1}{\epsilon} \|\nabla_x K\|_{L^\infty} \overset{\triangle}{=} RHS.$$

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By the momentum conservation (3.7), we have

$$RHS = -\frac{1}{\epsilon}\left(|v_M|\Phi_\epsilon(x_\epsilon) - \frac{v_M}{|v_M|}\cdot\int\limits_{\mathbb{R}^{2d}}(\phi(|x_\epsilon - y|) - \eta(t))v^*f_\epsilon(t, y, v^*)\,\mathrm{d}y\,\mathrm{d}v^*\right) + \frac{1}{\epsilon}\|\nabla_x K\|_{L^\infty}$$

$$\leq -\frac{1}{\epsilon}\left(|v_M|\Phi_\epsilon(x_\epsilon) - |v_M|\int\limits_{\mathbb{R}^{2d}}(\phi(|x_\epsilon - y|) - \eta(t))f_\epsilon(t, y, v^*)\,\mathrm{d}y\,\mathrm{d}v^*\right) + \frac{1}{\epsilon}\|\nabla_x K\|_{L^\infty}$$

$$= -\frac{\eta(t)}{\epsilon}\left(|v_M| - \frac{\|\nabla_x K\|_{L^\infty}}{\eta(t)}\right).$$

Using the estimate above in (3.13), along with (3.12) and the fact that $\eta(t)$ is non-increasing, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(V_\epsilon - \frac{\|\nabla_x K\|_{L^\infty}}{\eta(t)}\right) \leq -\frac{\eta(t)}{\epsilon}\left(|v_M| - \frac{\|\nabla_x K\|_{L^\infty}}{\eta(t)}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\|\nabla_x K\|_{L^\infty}}{\eta(t)}\right).$$

$$\leq -\frac{\eta(t)}{\epsilon}\left(V_\epsilon - \frac{\|\nabla_x K\|_{L^\infty}}{\eta(t)}\right).$$

Therefore for $t \in [0, T]$ we have

$$V_\epsilon(t) \leq V_\epsilon(0) - \frac{\|\nabla_x K\|_{L^\infty}}{\eta(0)} + \frac{\|\nabla_x K\|_{L^\infty}}{\eta(t)} \leq V_\epsilon(0) + \frac{\|\nabla_x K\|_{L^\infty}}{\eta(t)}. \tag{3.14}$$

Combining (3.14) with the definition of $D_\epsilon(t)$ in (3.9), we obtain the differential inequality

$$D_\epsilon'(t) \leq V_\epsilon(0) + \frac{\|\nabla_x K\|_{L^\infty}}{\eta(t)} = V_\epsilon(0) + \frac{\|\nabla_x K\|_{L^\infty}}{\phi(2D_\epsilon(t))}. \tag{3.15}$$

Note that D_ϵ is an increasing function in t and $\phi(r) \downarrow 0$ as $r \to \infty$. To show that D_ϵ is bounded independent of ϵ, we let

$$\Lambda(r) = \int\limits_0^r \frac{\phi(s)}{c_2 + c_1\phi(s)}\,\mathrm{d}s,$$

where $c_1 = V_\epsilon(0)$ and $c_2 = \|\nabla_x K\|_{L^\infty}$. Then Λ is an increasing function and it has the bounds

$$\frac{1}{c_2 + c_1\|\phi\|_{L^\infty}}\int\limits_0^r \phi(s)\,\mathrm{d}s \leq \Lambda(r) \leq \frac{1}{c_2}\int\limits_0^r \phi(s)\,\mathrm{d}s.$$

By (2.14) we have $\mathrm{Range}(\Lambda) = [0, \infty)$. Meanwhile, for each fixed ϵ and T such that D_ϵ is well-defined on $[0, T]$, solving (3.15) gives

$$\Lambda(2D_\epsilon(t)) \leq \Lambda(2D_\epsilon(0)) + 2t.$$

Note that $D_\epsilon(0) = S_\epsilon(0)$ is independent of ϵ, as it represents the size of the spatial projection of the support of f_0 (see (3.8)). Therefore we obtain the following uniform in ϵ bound:

$$D_\epsilon(t) \leq \frac{1}{2}\Lambda^{-1}\left(\Lambda(2|\mathrm{supp}f_0|) + 2T\right) \qquad \text{for any } t \in [0, T].$$

We conclude that by (3.10), we have that $S_\epsilon(t)$ is uniformly bounded in ϵ on $[0, T]$. Moreover, by (3.14) and the definition of η, we also have that the support of v is uniformly bounded in ϵ on $[0, T]$. $\qquad\square$

Throughout the rest of the paper we denote by $\Omega(T) \subset \mathbb{R}^d \times \mathbb{R}^d$ the common support of $f_\epsilon(t, \cdot, \cdot)$ for all $\epsilon > 0$ and $t \in [0, T]$. Also, we use the following notation for the spatial and velocity projections of $\Omega(T)$:

$$\Omega_1(T) = \{x \in \mathbb{R}^d : (x, v) \in \Omega(T)\}, \qquad \Omega_2(T) = \{v \in \mathbb{R}^d : (x, v) \in \Omega(T)\}. \qquad (3.16)$$

Using the uniform bound of v_ϵ, we can immediately infer the uniform (in ϵ) bounds for all the moments of f_ϵ:

Corollary 3.5. *Suppose the assumptions in Proposition 3.4 hold. Then for any $k \in \mathbb{N}$ there exists a constant $C(T, k, \phi, K)$ such that*

$$\int_{R^{2d}} |v|^k f_\epsilon(t, x, v) \,\mathrm{d}x \,\mathrm{d}v \leq C(T, k, \phi, K) < \infty, \qquad k \geq 1. \qquad (3.17)$$

Remark 3.6. *By doing a more careful estimate, one can show that (3.17) holds as long as the initial measure f_0 has bounded moments (with no requirement on f_0 having compact support). Nevertheless, we content ourselves with Corollary 3.5 since the main setting here requires that the initial data is compact supported.*

3.3 Uniform bound of the fluctuation

In this part we show a uniform in ϵ bound on the second order fluctuation term defined below by (3.20). This bound (and in fact the rate of vanishing of the second order fluctuation with ϵ) is essential to show the convergence of f_ϵ [Jab00, FS14]. The estimates here are a priori. However, one can apply a density argument (see for example, Section 4 in [FS14]) to show that they hold for measure-valued solutions $f_\epsilon(t, \cdot, \cdot) \in \mathcal{P}_c(\mathbb{R}^d \times \mathbb{R}^d)$ at each $t \in [0, T]$.

By previous estimates, we have a priori uniform-in-ϵ bounds for all the moments and the size of the support of f_ϵ over an arbitrary finite time interval $[0, T]$. Hence, for $T > 0$ fixed, there exists a constant $C_0(T)$ that depends on ϕ and K (but independent of ϵ) such that, for $k = 1, 2$,

$$\int_{\mathbb{R}^{2d}} |v|^k f_\epsilon(t, x, v) \,\mathrm{d}x \,\mathrm{d}v + |\mathrm{supp} f_\epsilon(t, \cdot, \cdot)| \leq C_0(T), \qquad \text{for all } t \in [0, T]. \qquad (3.18)$$

Also, there exists a constant $\tilde{C}_0(T)$ independent of ϵ such that

$$\phi(|x - y|) \geq \tilde{C}_0(T) > 0, \qquad \text{for any } x, y \in \mathrm{supp} f_\epsilon, \text{ and } t \in [0, T]. \qquad (3.19)$$

Theorem 3.7. *Suppose the interaction potential K and the influence function ϕ satisfy the assumptions in Proposition 3.4. Let f_ϵ be the solution to (1.3) starting from an initial data $f_0 \in \mathcal{P}_c(\mathbb{R}^d \times \mathbb{R}^d)$ that satisfies (3.6). Let I_ϵ be the second order fluctuation defined by*

$$I_\epsilon(t) = \int_{\mathbb{R}^{2d}} |F[f_\epsilon]|^2 \, f_\epsilon \,\mathrm{d}x \,\mathrm{d}v. \qquad (3.20)$$

Then for any $t \in [0, T]$, there exists $C(T)$ independent of ϵ such that

$$I_\epsilon(t) \leq C(T) \,\epsilon^2 + I_\epsilon(0) \, e^{-\frac{2}{\epsilon} t}.$$

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Proof. We first make the simple observation that by Cauchy-Schwartz and (3.18), for $k = 0, 1$, we have

$$\int\limits_{\mathbb{R}^{2d}} |v|^k \, |F[f_\epsilon]| \, f_\epsilon \, \mathrm{d}x \, \mathrm{d}v \leq \left(\int\limits_{\mathbb{R}^{2d}} |v|^{2k} f_\epsilon \, \mathrm{d}x \, \mathrm{d}v \right)^{1/2} I_\epsilon(t)^{1/2} < C_0(T) \, I_\epsilon(t)^{1/2}. \tag{3.21}$$

Now we estimate the rate of change of I_ϵ. To this end, we multiply $\frac{1}{2} |F[f_\epsilon]|^2$ to equation (1.3a) and integrate in x, v. This gives

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} I_\epsilon = \underbrace{\frac{1}{2} \int\limits_{\mathbb{R}^{2d}} v f_\epsilon \cdot \nabla_x \left(|F[f_\epsilon]|^2 \right) \mathrm{d}x \, \mathrm{d}v}_{R_1} + \underbrace{\frac{1}{2} \int\limits_{\mathbb{R}^{2d}} \left(\partial_t |F[f_\epsilon]|^2 \right) f_\epsilon \, \mathrm{d}x \, \mathrm{d}v}_{R_2} \tag{3.22}$$

$$\underbrace{- \frac{1}{2\epsilon} \int\limits_{\mathbb{R}^{2d}} \nabla_v \left(|F[f_\epsilon]|^2 \right) \cdot (F[f_\epsilon] f_\epsilon) \, \mathrm{d}x \, \mathrm{d}v}_{D_0} .$$

Below we estimate separately the terms R_1, R_2 and D_0 indicated above.

Estimate of D_0: We have

$$D_0 = -\frac{1}{\epsilon} \int\limits_{\mathbb{R}^{2d}} \nabla_v F : (F \otimes F) f_\epsilon \, \mathrm{d}x \, \mathrm{d}v$$

$$= -\frac{1}{\epsilon} \int\limits_{\mathbb{R}^{2d}} \Phi_\epsilon(x - y) \, |F[f_\epsilon]|^2 \, f_\epsilon \, \mathrm{d}x \, \mathrm{d}v, \tag{3.23}$$

where for the second equality we used (1.3b) and the definition (3.11) of $\Phi_\epsilon(x)$. By (3.19) we conclude that D_0 provides dissipation in (3.22); the exact expression of D_0 is used below to control a certain contribution from R_2.

Estimate of R_1: From the expression of R_1 and (1.3b), we infer immediately:

$$R_1 \leq \int\limits_{\mathbb{R}^{2d}} |\nabla_x F[f_\epsilon]| \, |F[f_\epsilon]| \, |v| \, f_\epsilon \, \mathrm{d}x \, \mathrm{d}v,$$

where

$$|\nabla_x F[f_\epsilon]| \leq \|\nabla_x \phi\|_{L^\infty(\mathbb{R}^d)} \left(|v| + \|v f_\epsilon\|_{L^1(\mathbb{R}^{2d})} \right) + \left\| \nabla_x^2 K \right\|_{L^\infty(\mathbb{R}^d)}.$$

Hence, by (3.18) and Cauchy-Schwartz (see also (3.21)), there exists $C_1(T)$ independent of ϵ such that

$$R_1 \leq C_1(T) \, I_\epsilon^{1/2}(t). \tag{3.24}$$

Estimate of R_2: The estimate of R_2 is more delicate. The expression of R_2 and (1.3b) yields

$$R_2 = \int\limits_{\mathbb{R}^{2d}} (F[f_\epsilon] \cdot \partial_t F[f_\epsilon]) f_\epsilon \, \mathrm{d}x \, \mathrm{d}v$$

$$\overset{\triangle}{=} R_{21} + R_{22},$$

where

$$R_{21} = \int\limits_{\mathbb{R}^{4d}} \phi(|x-y|)(v-v^*)\,\partial_t f_\epsilon(t,y,v^*) \cdot F[f_\epsilon](t,x,v) f_\epsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}v^*$$

and

$$R_{22} = \int\limits_{\mathbb{R}^{2d}} \nabla_x K * \partial_t \rho_\epsilon(t,x) \cdot F[f_\epsilon](t,x,v) f_\epsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

We first estimate R_{22}. Integrating (1.3a) in v gives

$$\partial_t \rho_\epsilon = -\nabla_x \cdot \langle v f_\epsilon \rangle.$$

Angle brackets denote integration with respect to v.

Hence by (3.18) and (3.21), there exists $C_2(T)$ independent of ϵ such that

$$
\begin{aligned}
|R_{22}| &\leq \int\limits_{\mathbb{R}^{2d}} \left|\nabla_x^2 K * \langle v f_\epsilon \rangle\right| \, |F[f_\epsilon]|\,(t,x,v) f_\epsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \\
&\leq \left\|\nabla_x^2 K * \langle v f_\epsilon \rangle\right\|_{L^\infty(\mathbb{R}^d)} \|F[f_\epsilon] f_\epsilon\|_{L^1(\mathbb{R}^{2d})} \\
&\leq C_2(T)\, I_\epsilon^{1/2}(t).
\end{aligned}
\tag{3.25}
$$

Next we estimate R_{21}. Using the kinetic equation (1.3a), we have

$$
\begin{aligned}
R_{21} = &-\int\limits_{\mathbb{R}^{4d}} \phi(|x-y|)(v-v^*)\,(v^* \cdot \nabla_y f_\epsilon(t,y,v^*)) \cdot F[f_\epsilon](t,x,v) f_\epsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}v^* \\
&+ \frac{1}{\epsilon} \int\limits_{\mathbb{R}^{4d}} \phi(|x-y|)(v-v^*)\nabla_{v^*} \cdot (F[f_\epsilon] f_\epsilon)\,(t,y,v^*) \cdot F[f_\epsilon](t,x,v) f_\epsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}v^* \\
\triangleq &\ R_{21,1} + R_{21,2}\,.
\end{aligned}
$$

Integration by parts in $R_{21,1}$ gives

$$R_{21,1} = \int\limits_{\mathbb{R}^{4d}} \nabla_y \phi(|x-y|) \cdot (v^* f_\epsilon(t,y,v^*))\,(v-v^*) \cdot F[f_\epsilon](t,x,v) f_\epsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}v^*.$$

Therefore, by (3.18) and (3.21), there exists $C_3(T)$ independent of ϵ such that

$$
\begin{aligned}
|R_{21,1}| &\leq \|\nabla\phi\|_{L^\infty} \left(\|v f_\epsilon\|_{L^1(\mathbb{R}^{2d})} \|v F[f_\epsilon] f_\epsilon\|_{L^1(\mathbb{R}^{2d})} + \||v|^2 f_\epsilon\|_{L^1(\mathbb{R}^{2d})} \|F[f_\epsilon] f_\epsilon\|_{L^1(\mathbb{R}^{2d})} \right) \\
&\leq C_3(T)\, I_\epsilon^{1/2}(t).
\end{aligned}
\tag{3.26}
$$

We are left to estimate $R_{21,2}$. Integration by parts in v^* gives

$$R_{21,2} = \frac{1}{\epsilon} \int\limits_{\mathbb{R}^{4d}} \phi(|x-y|)\,(F[f_\epsilon] f_\epsilon)\,(t,y,v^*) \cdot (F[f_\epsilon] f_\epsilon)\,(t,x,v)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}v^*\,.$$

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Combining $R_{21,2}$ with D_0 we get

$$D_0 + R_{21,2} = \frac{1}{\epsilon} \int_{\mathbb{R}^{4d}} \phi(|x-y|)\,(F^* - F) \cdot F f_\epsilon^* f_\epsilon \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}v^*,$$

where $F^* = F[f_\epsilon](t,y,v^*)$ and $f_\epsilon^* = f_\epsilon(t,y,v^*)$. By symmetry, (3.19) and the conservation property $\int_{\mathbb{R}^{2d}} F[f_\epsilon] f_\epsilon \,\mathrm{d}x\,\mathrm{d}v = 0$, we can infer further:

$$D_0 + R_{21,2} = -\frac{1}{2\epsilon} \int_{\mathbb{R}^{4d}} \phi(|x-y|)\,|F^* - F|^2\, f_\epsilon^* f_\epsilon \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}v^* \tag{3.27}$$

$$\leq -\frac{1}{2\epsilon} \tilde{C}_0(T) \int_{\mathbb{R}^{4d}} |F^* - F|^2\, f_\epsilon^* f_\epsilon \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}v^*$$

$$= -\frac{1}{\epsilon} \tilde{C}_0(T) \int_{\mathbb{R}^{2d}} |F[f_\epsilon]|^2\, f_\epsilon \,\mathrm{d}x\,\mathrm{d}v.$$

Putting together (3.24), (3.25), (3.26), and (3.27) we get

$$R_1 + R_2 + D_0 \leq C_5(T) I_\epsilon^{1/2}(t) - \frac{1}{\epsilon} \tilde{C}_0(T) I_\epsilon(t). \tag{3.28}$$

The following differential inequality for I_ϵ can now be obtained from (3.22) and (3.28):

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} I_\epsilon \leq C_5(T) I_\epsilon^{1/2}(t) - \frac{1}{\epsilon} \tilde{C}_0(T) I_\epsilon(t), \qquad \text{for } t \in [0,T].$$

By Gronwall's inequality, we thus have

$$I_\epsilon(t) \leq C(T)\,\epsilon^2 + I_\epsilon(0)\,e^{-\frac{2}{\epsilon}t},$$

for some constant $C(T)$ independent of ϵ. $\qquad\qquad\square$

4 Passage to the limit in the kinetic equation

In this section we show the convergence of solutions f_ϵ of the kinetic equation (1.3) to solutions ρ_ϵ of the macroscopic equation (1.5), as well as the convergence of the characteristics of (1.3) to those of (1.5).

4.1 Convergence to the macroscopic equation

The convergence to the macroscopic solution is given in the following theorem.

Theorem 4.1 (Convergence to the macroscopic model)**.** *Suppose K is symmetric, $\nabla_x K \in W^{1,\infty}(\mathbb{R}^d)$ and $\phi \in W^{1,\infty}(\mathbb{R}^+)$ is positive, non-increasing, and satisfies the slow decay condition (2.14). For any fixed $T > 0$ and $\epsilon > 0$, let $f_\epsilon \in C([0,T]; \mathcal{P}_c(\mathbb{R}^d \times \mathbb{R}^d))$ be the unique solution to (1.3) with initial data $f_0 \in \mathcal{P}_c(\mathbb{R}^d \times \mathbb{R}^d)$ that satisfies (3.6). Then there exists a unique $\rho \in C([0,T]; \mathcal{P}_c(\mathbb{R}^d))$ such that*

$$\rho_\epsilon(t, \cdot) \xrightarrow{w^*} \rho(t, \cdot) \qquad \text{in } \mathcal{P}(\mathbb{R}^d) \text{ as } \epsilon \to 0, \text{ for each } t \in [0,T].$$

Moreover, the limiting measure ρ is the unique solution to (1.5) with initial density $\rho_0 = \int_{\mathbb{R}^d} f_0 \,\mathrm{d}v$.

Proof. We apply a similar argument as in [Jab00] (see [FS14] as well). The key ingredient is to study the limit involving the term $F[f_\epsilon]$ defined in (1.3b). For ease of notation, denote

$$J_\epsilon(t,x) = \int\limits_{\mathbb{R}^d} v f_\epsilon(t,x,v)\, \mathrm{d}v\,. \tag{4.1}$$

Then $F[f_\epsilon]$ can be written as

$$F[f_\epsilon](t,x,v) = v\, \phi * \rho_\epsilon(t,x) - \phi * J_\epsilon(t,x) + \nabla_x K * \rho_\epsilon(t,x). \tag{4.2}$$

We divide the proof into four steps.

Step 1. First we study the passage to the limit of $\phi * J_\epsilon$. To this end, take the test function in (3.5) as $\psi(t,x,v) = \psi_1(t)\psi_2(x)\, v\psi_3(v)$ where $\psi_1 \in C_c^1(0,T)$, $\psi_2(x), v\psi_3(v) \in C_b^1(\mathbb{R}^d)$, and $\psi_3 = 1$ on $\Omega_2(T)$ (recall that $\Omega_2(T)$ is defined in (3.16)). Using this ψ in (3.5), we have

$$\int\limits_0^T \psi_1'(t) \int\limits_{\mathbb{R}^d} \psi_2(x) J_\epsilon(t,x)\, \mathrm{d}x\, \mathrm{d}t + \int\limits_0^T \psi_1(t) \int\limits_{\mathbb{R}^{2d}} \nabla_x \psi_2 \cdot v \otimes v f_\epsilon\, \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}t$$

$$- \frac{1}{\epsilon} \int\limits_0^T \psi_1(t) \int\limits_{\mathbb{R}^{2d}} \psi_2(x) F[f_\epsilon] f_\epsilon\, \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}t = 0.$$

Therefore, if we denote

$$\xi_\epsilon(t) = \int\limits_{\mathbb{R}^d} \psi_2(x) J_\epsilon(t,x)\, \mathrm{d}x,$$

then its weak derivative is given by

$$\xi_\epsilon'(t) = \int\limits_{\mathbb{R}^{2d}} \nabla_x \psi_2 \cdot v \otimes v f_\epsilon\, \mathrm{d}x\, \mathrm{d}v - \frac{1}{\epsilon} \int\limits_{\mathbb{R}^{2d}} \psi_2(x) F[f_\epsilon] f_\epsilon\, \mathrm{d}x\, \mathrm{d}v.$$

By Theorem 3.7, there exists a constant $C(t_1,T)$ independent of ϵ such that

$$\frac{1}{\epsilon} \int\limits_{\mathbb{R}^{2d}} |F[f_\epsilon]|\, f_\epsilon\, \mathrm{d}x\, \mathrm{d}v \le \frac{1}{\epsilon} \left(I_\epsilon(t) \right)^{1/2} \le C(t_1,T),$$

for any $t \in [t_1,T]$ where $0 < t_1 < T$. Using this fact, together with Corollary 3.5, we conclude that $\xi_\epsilon \in W^{1,\infty}(t_1,T)$ for any $\psi_2 \in C_b^1(\mathbb{R}^d)$, and

$$\|\xi_\epsilon\|_{W^{1,\infty}(t_1,T)} \le C(t_1,T)\, \|\psi_2\|_{W^{1,\infty}} \qquad \text{for any } t_1 > 0.$$

Hence, $\{\xi_\epsilon(t)\}_{\epsilon>0}$ is uniformly bounded and equicontinuous in $C([t_1,T])$, and by Ascoli-Arzelà Theorem it converges uniformly on a subsequence. We conclude that for any $\psi_2 \in C_b^1(\mathbb{R}^d)$ and $t_1 \in (0,T]$, there exists a subsequence ϵ_k and $\xi(t)$ such that

$$\int\limits_{\mathbb{R}^d} \psi_2(x) J_{\epsilon_k}(t,x)\, \mathrm{d}x \to \xi(t) \qquad \text{in } C([t_1,T]) \quad \text{as } \epsilon_k \to 0. \tag{4.3}$$

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On the other hand, Proposition 3.4 provides a uniform (in ϵ) bound for the support of f_ϵ, which in turn gives a uniform bound of J_ϵ. Hence, for each $t \in [0, T]$, there exists $J(t, \cdot) \in \mathcal{M}(\mathbb{R}^d)$ and a subsequence of J_{ϵ_k}, denoted as $J_{\epsilon_{k_l}}(t, \cdot)$, such that

$$J_{\epsilon_{k_l}}(t, \cdot) \xrightarrow{w^*} J(t, \cdot) \qquad \text{in } \mathcal{M}(\mathbb{R}^d) \quad \text{as } \epsilon_{k_l} \to 0. \tag{4.4}$$

Note that in general, ϵ_{k_l} could depend on t. However, comparing (4.3) with (4.4), we have that along the subsequence ϵ_k which is independent of t,

$$\int_{\mathbb{R}^d} \psi_2(x) J_{\epsilon_k}(t, x)\, \mathrm{d}x \to \int_{\mathbb{R}^d} \psi_2(x) J(t, x)\, \mathrm{d}x \qquad \text{in } C([t_1, T]) \quad \text{as } \epsilon_k \to 0. \tag{4.5}$$

By a density argument applied to ψ_2, we then have

$$J_{\epsilon_k}(t, \cdot) \xrightarrow{w^*} J(t, \cdot) \qquad \text{in } \mathcal{M}(\mathbb{R}^d) \quad \text{as } \epsilon_k \to 0, \tag{4.6}$$

where $t \in [t_1, T]$, $t_1 > 0$ is arbitrary, and ϵ_k is independent of t.

Furthermore, the assumption $\phi \in W^{1,\infty}(\mathbb{R}^d)$ guarantees that the sequence $\{\phi * J_\epsilon\}_{\epsilon > 0}$ is uniformly bounded and equicontinuous in $C([t_1, T] \times \overline{\Omega_1(T)})$, where $\Omega_1(T)$ is defined in (3.16). We conclude that along a subsequence which is still denoted as ϵ_k,

$$\phi * J_{\epsilon_k} \to \phi * J \qquad \text{in } C\left([t_1, T] \times \overline{\Omega_1(T)}\right) \quad \text{as } \epsilon_k \to 0, \tag{4.7}$$

where again, $t_1 > 0$ is arbitrary, and ϵ_k is independent of t.

Step 2. One can use similar arguments to infer convergence results for the other two terms entering the expression of $F[f_\epsilon]$ in (4.2). These arguments would in fact be identical to those used in [FS14] for the attractive-repulsive model with no alignment. We summarize the key ideas briefly and refer to the proof of Theorem 5.1 in [FS14] for details.

By taking a test function of the form $\psi(t, x, v) = \psi_1(t)\psi_2(x)$ in (3.5), one can show, by similar calculations as in Step 1, that $\int_{\mathbb{R}^d} \psi_2(x)\rho_\epsilon(t, x)$ is uniformly bounded in $W^{1,\infty}(0, T)$. Together with the tightness of the sequence $\rho_\epsilon(t, \cdot)$ and the application of Prokhorov's theorem (cf. [Bil71, Theorem 4.1]), it can be inferred that there exists a probability measure $\rho(t, \cdot) \in \mathcal{P}(\mathbb{R}^d)$ such that, along a subsequence of ϵ_k (still denoted as ϵ_k),

$$\rho_{\epsilon_k}(t, \cdot) \xrightarrow{w^*} \rho(t, \cdot) \qquad \text{in } \mathcal{P}(\mathbb{R}^d) \quad \text{as } \epsilon_k \to 0. \tag{4.8}$$

The convergence above holds for each $t \in (0, T]$ and the sequence ϵ_k is independent of t.

Also as in Step 1, one can get

$$\phi * \rho_{\epsilon_k} \to \phi * \rho \qquad \text{in } C\left([t_1, T] \times \overline{\Omega_1(T)}\right) \quad \text{as } \epsilon_k \to 0, \tag{4.9}$$

where the uniformity with respect to x follows from the equicontinuity in x of $\{\phi * \rho_\epsilon(t, \cdot)\}$.

Finally, regarding the convergence of $\{\nabla_x K * \rho_{\epsilon_k}\}$, the equicontinuity in x is immediate (by the assumption made on the potential). The equicontinuity with respect to time is slightly more delicate, as $\nabla_x K$ does not have enough regularity to be used as a test function. Nevertheless, one can overcome this difficulty by a regularization of the kernel (as shown in [FS14]) and it can be inferred that

$$\nabla_x K * \rho_{\epsilon_k} \to \nabla_x K * \rho \qquad \text{in } C\left([t_1, T] \times \overline{\Omega_1(T)}\right) \quad \text{as } \epsilon_k \to 0, \tag{4.10}$$

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with $t_1 > 0$ arbitrary.

Step 3. We now proceed to deriving the limiting equation (1.5b) at each $t \in [0, T]$. For all $(x, v) \in \Omega(T)$, we use (4.2), (4.7), (4.9) and (4.10) to derive that

$$F[f_{\epsilon_k}](t, \cdot, \cdot) \to F_0(t, \cdot, \cdot) \qquad \text{strongly in } C_b(\Omega(T)) \quad \text{for each } t \in (0, T],$$

where

$$F_0(t, x, v) = v \, \phi * \rho(t, x) - \phi * J(t, x) + \nabla_x K * \rho(t, x).$$

Therefore, for each time $t \in (0, T]$,

$$\int_{\mathbb{R}^{2d}} (F[f_{\epsilon_k}] - F_0)^2 \, f_{\epsilon_k} \, \mathrm{d}x \, \mathrm{d}v \leq \|F[f_{\epsilon_k}] - F_0\|_{C_b(\Omega_T)}^2 \to 0 \qquad \text{as } \epsilon_k \to 0.$$

Combining the above convergence with the vanishing in ϵ of $I_\epsilon(t)$, as established in Theorem 3.7, we have

$$\int_{\mathbb{R}^{2d}} F_0^2 \, f_{\epsilon_k} \, \mathrm{d}x \, \mathrm{d}v \to 0 \qquad \text{as } \epsilon_k \to 0.$$

Let $\varphi_1 \in C_b^1(\mathbb{R}^d)$ be arbitrary. Then, by Cauchy-Schwarz inequality,

$$\int_{\mathbb{R}^{2d}} \varphi_1(x) F_0 f_{\epsilon_k} \, \mathrm{d}v \, \mathrm{d}x \to 0 \qquad \text{as } \epsilon_k \to 0,$$

and hence,

$$\int_{\mathbb{R}^d} \varphi_1(x) \left(\Phi J_{\epsilon_k} - \rho_{\epsilon_k} \left(\phi * J + \nabla_x K * \rho \right) \right) \mathrm{d}x \to 0 \qquad \text{as } \epsilon_k \to 0.$$

where $\Phi = \phi * \rho \in C_b^1(\mathbb{R}^d)$. By (4.6) and (4.8) we can then pass to the limit and get

$$\int_{\mathbb{R}^d} \varphi_1(x) \left(\Phi J - \rho \left(\phi * J + \nabla_x K * \rho \right) \right) \mathrm{d}x = 0 \quad \text{for any } \varphi_1 \in C_b^1(\mathbb{R}^d).$$

This implies that for each $t \in (0, T]$,

$$\Phi(t, x) J(t, x) = \rho(t, x) \left(\int_{\mathbb{R}^d} \phi(|x - y|) J(t, y) \, \mathrm{d}y - \nabla_x K * \rho(t, x) \right). \tag{4.11}$$

By denoting

$$u(t, x) = J(t, x) / \rho(t, x), \qquad \text{on the support of } \rho(t, \cdot), \tag{4.12}$$

then equation (1.5b) holds on suppρ.

Step 4. In this last step we show that the limit ρ satisfies the macroscopic equation (1.5a), as well as the momentum conservation (1.5c). First, following arguments used in [FS14], one

can generalize (4.5) and show that ψ_2 can be allowed to depend on t as well. Specifically, the following convergence holds:

$$\int_{\mathbb{R}^d} \psi_2(t,x) J_{\epsilon_k}(t,x)\,\mathrm{d}x \to \int_{\mathbb{R}^d} \psi_2(t,x) J(t,x)\,\mathrm{d}x \qquad \text{in } C([t_1,T]) \quad \text{as } \epsilon_k \to 0, \qquad (4.13)$$

for all $\psi_2 \in C_c([t_1,T), C_b^1(\mathbb{R}^d))$.

Similarly, an analogous result can be shown for the convergence of ρ_{ϵ_k}:

$$\int_{\mathbb{R}^d} \psi_2(t,x) \rho_{\epsilon_k}(t,x)\,\mathrm{d}x \to \int_{\mathbb{R}^d} \psi_2(t,x) \rho(t,x)\,\mathrm{d}x \qquad \text{in } C([t_1,T]) \quad \text{as } \epsilon_k \to 0, \qquad (4.14)$$

for all $\psi_2 \in C_c([t_1,T), C_b^1(\mathbb{R}^d))$.

Now, in the weak formulation (3.5) for f_{ϵ_k}, take a function ψ of the form $\psi(t,x,v) = \varphi(t,x) \in C_c^1([0,T), C_b^1(\mathbb{R}^d))$ to get

$$\int_0^T \int_{\mathbb{R}^d} \partial_t \varphi(t,x) \rho_{\epsilon_k}(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} \nabla_x \varphi \cdot J_{\epsilon_k}\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} \varphi(0,x) \rho_0(x,v)\,\mathrm{d}x = 0, \qquad (4.15)$$

where $\rho_0 = \int_{\mathbb{R}^d} f_0(x,v)\,\mathrm{d}v$.

Fix $\epsilon_1 > 0$ small. Then, one can choose $t_1 > 0$ small enough such that

$$\int_0^{t_1} \int_{\mathbb{R}^d} |\partial_t \varphi(t,x) \rho_{\epsilon_k}(t,x)|\,\mathrm{d}x\,\mathrm{d}t + \int_0^{t_1} \int_{\mathbb{R}^d} |\nabla_x \varphi \cdot J_{\epsilon_k}|\,\mathrm{d}x\,\mathrm{d}t \leq C\,\|\varphi\|_{W_{t,x}^{1,\infty}} t_1 < \epsilon_1. \qquad (4.16)$$

With t_1 fixed, such that (4.16) is satisfied, break the time integrals $\int_0^T$ in (4.15) into two pieces: $\int_0^{t_1}$ and $\int_{t_1}^T$. Due to (4.13) and (4.14) one can pass to the limit $\epsilon_k \to 0$ in the time integrals $\int_{t_1}^T$. Hence, we infer from (4.15) and (4.16) that for every $\epsilon_1 > 0$, there exists $0 < t_1 < \epsilon_1/C\,\|\varphi\|_{W_{t,x}^{1,\infty}}$ such that

$$\left| \int_{t_1}^T \int_{\mathbb{R}^d} \partial_t \varphi(t,x) \rho(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^T \int_{\mathbb{R}^d} \nabla_x \varphi \cdot J\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} \varphi(0,x) \rho_0(x,v)\,\mathrm{d}x \right| \leq \epsilon_1.$$

We conclude

$$\int_0^T \int_{\mathbb{R}^d} \partial_t \varphi(t,x) \rho(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} \nabla_x \varphi \cdot J\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} \varphi(0,x) \rho_0(x,v)\,\mathrm{d}x = 0, \qquad (4.17)$$

for all $\varphi(t,x) \in C_c^1([0,T), C_b^1(\mathbb{R}^d))$. Note that (4.17) is equivalent to (2.20) since $J = \rho u$ on the support of ρ and $J = 0$ outside the support of ρ.

The conservation of momentum (1.5c) holds since it holds for every J_{ϵ_k}. Indeed, by (3.7), we then have

$$\int_{\mathbb{R}^d} J_{\epsilon_k}(t,x)\,\mathrm{d}x = 0 \qquad \text{for all } t \in [0,T] \text{ and } \epsilon_k > 0.$$

Consequently, by (4.5) and (4.12), we infer that (1.5c) is satisfied.

We conclude by noting that the convergence above was derived on a subsequence ϵ_k. However, by the uniqueness of the weak solution to (1.5), as established in Theorem 2.4, the *full* sequence $\rho_\epsilon(t, \cdot)$ converges to $\rho(t, \cdot)$ for each $t \in [0, T]$. $\qquad\square$

4.2 Convergence of characteristic paths

In this part we investigate in more detail the convergence of f_ϵ. In particular, we show that the characteristic paths of (1.3) converge to those of (1.5). This provides a geometric point of view of the singular limiting process, which also provides an explicit formula for the limit of f_ϵ.

The characteristic system (3.1) along which the solution f_ϵ is transported has the form

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= v, \\
\epsilon \frac{\mathrm{d}v}{\mathrm{d}t} &= -\phi * \rho_\epsilon(t, x) \, v + \phi * J_\epsilon(t, x) - \nabla_x K * \rho_\epsilon(t, x), \\
(x, v)\big|_{t=0} &= (x_0, v_0) \in \mathrm{supp} f_0.
\end{aligned}
\tag{4.18}
$$

The goal here is to pass the limit $\epsilon \to 0$ in the characteristic system (4.18) and relate the limit to the characteristic paths of the limiting macroscopic equation (1.5).

The main tool in showing this limit is a classical result in singular perturbation theory due to Tikhonov [Tik52]. A brief account of this result is the following. Using notations from [Vas63], consider the general system

$$
\begin{cases}
\dfrac{dx}{dt} = v, \\[2mm]
\epsilon \dfrac{dv}{dt} = \mathcal{F}(x, v, t),
\end{cases}
\tag{4.19}
$$

where $x, v \in \mathbb{R}^d$ and $\epsilon > 0$.

System (4.19) is a two-scale equation, where t and $\tau = t/\epsilon$ represent the slow and fast time scales, respectively. The slow dynamics of (4.19) is given by

$$
\begin{cases}
\dfrac{dx}{dt} = v, \\[2mm]
v = \Gamma(x, t),
\end{cases}
\tag{4.20}
$$

where $v = \Gamma(x, t)$ is a root of the equation

$$
\mathcal{F}(x, v, t) = 0.
\tag{4.21}
$$

Roots $v = \Gamma(x, t)$ of (4.21) are in general non-unique. For a fixed configuration x^* and time t^*, the fast dynamics is defined by

$$
\frac{dv}{d\tau} = \mathcal{F}(x^*, v, t^*).
\tag{4.22}
$$

Note that in the fast system (4.22), x^* and t^* are being regarded as parameters.

Tikhonov's theorem establishes conditions on a root $v = \Gamma(x,t)$ which guarantee that solutions of the two-scale system (4.19) converge, via a fast initial layer governed by (4.22), to solutions of the degenerate/slow system (4.20).

A root $v = \Gamma(x,t)$, with Γ defined on a closed and bounded set $D \subset \mathbb{R}^{d+1}$, is called *isolated* if there is a $\delta > 0$ such that for all $(x,t) \in D$, the only element in $B(\Gamma(x,t), \delta)$ that satisfies $\mathcal{F}(x,v,t) = 0$ is $v = \Gamma(x,t)$. An isolated root Γ is called *positively stable* in D, if $v^* = \Gamma(x^*, t^*)$ is an asymptotically stable stationary point of (4.22) as $\tau \to \infty$, for each $(x^*, t^*) \in D$. The *domain of influence* of an isolated positively stable root Γ is the set of points $(x^*, \tilde{v}, t^*)$ such that the solution of (4.22) satisfying $v|_{\tau=0} = \tilde{v}$ tends to $v^* = \Gamma(x^*, t^*)$ as $\tau \to \infty$.

Tikhonov's theorem [Tik52] states the following:

Theorem 4.2 (Tikhonov [Tik52, Vas63])**.** *Assume that a root $v = \Gamma(x,t)$ of (4.21) is isolated positively stable in some bounded closed domain D. Consider a point (x_0, v_0, t_0) in the domain of influence of this root, and assume that the slow equation (4.20) has a solution $x(t)$ initialized at $x(t_0) = x_0$, such that $(x(t), t)$ lies in D for all $t \in [t_0, T]$. Then, as $\epsilon \to 0$, the solution $(x_\epsilon(t), v_\epsilon(t))$ of (4.19) initialized at (x_0, v_0), converges to $(x(t), v(t)) := (x(t), \Gamma(x(t), t))$ in the following sense:*

i) $\lim_{\epsilon \to 0} v_\epsilon(t) = v(t)$ *for all* $t \in (t_0, T^*]$, *and*

ii) $\lim_{\epsilon \to 0} x_\epsilon(t) = x(t)$ *for all* $t \in [t_0, T^*]$,

for some $T^* < T$.

Remark 4.3. *The convergence of $v_\epsilon(t)$ to $v(t)$ occurs via a fast initial layer and normally does not occur at the initial time t_0, unless the initial data satisfies $v_0 = \Gamma(x_0, t_0)$.*

Our main result concerning convergence of characteristic paths is the following theorem.

Theorem 4.4 (Convergence of characteristic paths)**.** *Assume K and ϕ satisfy the same assumptions as in Theorem 4.1. Consider the measure-valued solution f_ϵ to (1.3) and the characteristic path $(x_\epsilon(t), v_\epsilon(t))$ that originates from some $(x_0, v_0) \in \mathrm{supp}\, f_0$ at $t = 0$. Then,*

$$\lim_{\epsilon \to 0} (x_\epsilon(t), v_\epsilon(t)) = (x(t), u(t, x(t))) \quad \text{for all} \quad 0 \le t \le T, \tag{4.23}$$

where $x(t)$ is the characteristic trajectory of the limiting macroscopic equation (1.5) that starts at x_0, and $u(t,x)$ is the velocity field defined in (4.12). In particular, $x(t)$ satisfies (2.1), and u solves (1.5b).

Proof. The characteristic system (4.18) has a right-hand-side that depends on ϵ and Tikhonov's theorem does not apply directly. To circumvent this, we replace $(\rho_\epsilon, J_\epsilon)$ by (ρ, J) in (4.18) to arrive at the following system:

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= v, \\
\epsilon \frac{\mathrm{d}v}{\mathrm{d}t} &= -\phi * \rho(t,x)\, v + \phi * J(t,x) - \nabla_x K * \rho(t,x), \\
(x,v)\big|_{t=0} &= (x_0, v_0) \in \mathrm{supp} f_0.
\end{aligned}
\tag{4.24}
$$

Theorem 4.2 applies to system (4.24). Indeed, (4.24) can be written in the form (4.19), with

$$\mathcal{F}(x,v,t) = -\phi * \rho(t,x)\, v + \phi * J(t,x) - \nabla_x K * \rho(t,x).$$

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For a fixed spatial configuration x and time t, the root $v = \Gamma(x,t)$ given by

$$\Gamma(x,t) = \frac{1}{\phi * \rho(t,x)}(\phi * J(t,x) - \nabla_x K * \rho(t,x)) \qquad (4.25)$$

is unique, hence isolated. It is also immediate that for a fixed spatial configuration x^* and time t^*, the corresponding fast equation (4.22) has a globally attracting equilibrium $v^* = \Gamma(x^*, t^*)$. Consequently, v^* is positively stable and its domain of influence is $\{x^*\} \times \mathbb{R}^d \times \{t^*\}$.

Denote by $(\tilde{x}_\epsilon(t), \tilde{v}_\epsilon(t))$ the solution of (4.24) that originates from (x_0, v_0). Then, by Theorem 4.2, the convergence in (4.23), which needs to be shown for $x_\epsilon(t)$ and $v_\epsilon(t)$, holds for $\tilde{x}_\epsilon(t)$ and $\tilde{v}_\epsilon(t)$ (note that by (4.11), the root Γ in (4.25) is in fact the velocity u defined by (4.12)). Hence, it would be enough to show that for a fixed $t > 0$,

$$\lim_{\epsilon \to 0} |x_\epsilon(t) - \tilde{x}_\epsilon(t)| = 0 \quad \text{and} \quad \lim_{\epsilon \to 0} |v_\epsilon(t) - \tilde{v}_\epsilon(t)| = 0. \qquad (4.26)$$

Indeed, from (4.18) and (4.24) we get

$$\epsilon \frac{\mathrm{d}}{\mathrm{d}t}(v_\epsilon(t) - \tilde{v}_\epsilon(t)) = -\phi * \rho_\epsilon(t, x_\epsilon(t))v_\epsilon(t) + \phi * \rho(t, \tilde{x}_\epsilon(t))\tilde{v}_\epsilon(t)$$
$$+ \phi * J_\epsilon(t, x_\epsilon(t)) - \phi * J(t, \tilde{x}_\epsilon(t)) - \nabla_x K * \rho_\epsilon(t, x_\epsilon(t)) + \nabla_x K * \rho(t, \tilde{x}_\epsilon(t)).$$

To the first line on the right-hand-side we add and subtract $\phi * \rho_\epsilon(t, x_\epsilon(t))\tilde{v}_\epsilon(t)$. Then, we can write

$$\epsilon \frac{\mathrm{d}}{\mathrm{d}t}(v_\epsilon(t) - \tilde{v}_\epsilon(t)) = -\phi * \rho_\epsilon(t, x_\epsilon(t))\left(v_\epsilon(t) - \tilde{v}_\epsilon(t)\right) + G_\epsilon(t), \qquad (4.27)$$

where

$$G_\epsilon(t) = \left(\phi * \rho_\epsilon(t, x_\epsilon(t)) - \phi * \rho(t, \tilde{x}_\epsilon(t))\right)\tilde{v}_\epsilon(t) \qquad (4.28)$$
$$+ \phi * J_\epsilon(t, x_\epsilon(t)) - \phi * J(t, \tilde{x}_\epsilon(t)) - \nabla_x K * \rho_\epsilon(t, x_\epsilon(t)) + \nabla_x K * \rho(t, \tilde{x}_\epsilon(t)).$$

By integrating (4.27) one finds

$$v_\epsilon(t) - \tilde{v}_\epsilon(t) = \frac{1}{\epsilon} \int_0^t e^{-\frac{1}{\epsilon}\int_s^t \phi * \rho_\epsilon(\tau, x_\epsilon(\tau))d\tau} G_\epsilon(s)ds, \qquad \text{for all } t \in [0, T]. \qquad (4.29)$$

Using Proposition 3.4, we have

$$\phi * \rho_\epsilon(\tau, x_\epsilon(\tau)) \geq C_1, \qquad \text{for all } \tau \in [0, T],$$

where the constant C_1 depends on T, ϕ and K, but not on ϵ. Consequently, from (4.29), we get

$$|v_\epsilon(t) - \tilde{v}_\epsilon(t)| \leq \frac{1}{\epsilon} \int_0^t e^{-\frac{C_1}{\epsilon}(t-s)} |G_\epsilon(s)|ds, \qquad \text{for all } t \in [0, T]. \qquad (4.30)$$

We now focus on estimating the right-hand-side of (4.30). Inspect G_ϵ given by (4.28), in particular the term on the first line of the right-hand-side. The term $\tilde{v}_\epsilon(t)$ is uniformly bounded in ϵ (to show this, one can follow for instance the arguments used to prove Proposition

3.4). To estimate the term in round brackets, add and subtract $\phi * \rho(t, x_\epsilon(t))$. Then, by triangle inequality and the Mean Value Theorem, we get

$$
\begin{aligned}
|\phi * \rho_\epsilon(t, x_\epsilon(t)) - \phi * \rho(t, \tilde{x}_\epsilon(t))| \leq{}& \sup_{t \in [0,T]} \|\phi * (\rho_\epsilon - \rho)\|_{L^\infty(\Omega_1(T))} \\
& + \sup_{t \in [0,T]} \|\nabla\phi * \rho\|_{L^\infty(\Omega_1(T))} |x_\epsilon(t) - \tilde{x}_\epsilon(t)|
\end{aligned}
$$

Estimates entirely similar to the one above can be made for the terms on the second line of the right-hand-side of (4.28), which lead to:

$$
\begin{aligned}
|\phi * J_\epsilon(t, x_\epsilon(t)) - \phi * J(t, \tilde{x}_\epsilon(t))| \leq{}& \sup_{t \in [0,T]} \|\phi * (J_\epsilon - J)\|_{L^\infty(\Omega_1(T))} \\
& + \sup_{t \in [0,T]} \|\nabla\phi * J\|_{L^\infty(\Omega_1(T))} |x_\epsilon(t) - \tilde{x}_\epsilon(t)|,
\end{aligned}
$$

and

$$
\begin{aligned}
|\nabla_x K * \rho_\epsilon(t, x_\epsilon(t)) - \nabla_x K * \rho(t, \tilde{x}_\epsilon(t))| \leq{}& \sup_{t \in [0,T]} \|\nabla_x K * (\rho_\epsilon - \rho)\|_{L^\infty(\Omega_1(T))} \\
& + \sup_{t \in [0,T]} \|\nabla_x^2 K * \rho\|_{L^\infty(\Omega_1(T))} |x_\epsilon(t) - \tilde{x}_\epsilon(t)|.
\end{aligned}
$$

where $\Omega_1(T)$ is defined in (3.16).

By the uniform convergences of $\phi * \rho_\epsilon$, $\phi * J_\epsilon$, and $\nabla_x K * \rho_\epsilon$, as established in the proof of Theorem 4.1, and by assumptions we made on ϕ and K, we can group the three estimates from above and get from (4.28),

$$
|G_\epsilon(t)| \leq C_\epsilon + C_2 |x_\epsilon(t) - \tilde{x}_\epsilon(t)|, \tag{4.31}
$$

where C_ϵ and C_2 are constants that depends on T, ϕ and K. Moreover, $C_\epsilon \to 0$ as $\epsilon \to 0$.

Applying (4.31) in (4.30), we get

$$
|v_\epsilon(t) - \tilde{v}_\epsilon(t)| \leq \frac{C_\epsilon}{C_1} + \frac{C_2}{\epsilon} \int_0^t e^{-\frac{C_1}{\epsilon}(t-s)} |x_\epsilon(s) - \tilde{x}_\epsilon(s)| ds, \qquad \text{for all } t \in [0,T].
$$

Using (4.18) and (4.24), we further find

$$
|v_\epsilon(t) - \tilde{v}_\epsilon(t)| \leq \frac{C_\epsilon}{C_1} + \frac{C_2}{\epsilon} \int_0^t e^{-\frac{C_1}{\epsilon}(t-s)} \int_0^s |v_\epsilon(\tau) - \tilde{v}_\epsilon(\tau)| \, d\tau \, ds, \qquad \text{for all } t \in [0,T]. \tag{4.32}
$$

Change order of integration in the double integral on the right-hand-side of (4.32) to get

$$
|v_\epsilon(t) - \tilde{v}_\epsilon(t)| \leq \frac{C_\epsilon}{C_1} + \frac{C_2}{\epsilon} \int_0^t |v_\epsilon(\tau) - \tilde{v}_\epsilon(\tau)| \int_\tau^t e^{-\frac{C_1}{\epsilon}(t-s)} \, ds \, d\tau, \qquad \text{for all } t \in [0,T].
$$

After evaluating the integral in s we finally arrive at

$$
|v_\epsilon(t) - \tilde{v}_\epsilon(t)| \leq \frac{C_\epsilon}{C_1} + \frac{C_2}{C_1} \int_0^t |v_\epsilon(\tau) - \tilde{v}_\epsilon(\tau)| d\tau, \qquad \text{for all } t \in [0,T].
$$

The second limit in (4.26) now follows from the integral form of Gronwall's inequality, given that $C_\epsilon \to 0$ as $\epsilon \to 0$. The convergence of trajectories follows from here as well. $\qquad \square$

The convergence of characteristic paths yields the limiting flow map $\mathcal{T}^t : x_0 \to x(t)$. It is convenient in the calculations below to use the notation $x(t; x_0)$ to denote the limiting characteristic path $x(t)$ that starts at x_0.

The next result characterizes the limiting densities.

Theorem 4.5 (Characterization of the limiting densities). *The limiting macroscopic density ρ identified in Theorem 4.1 is the push-forward of the initial density ρ_0 by the limiting flow map $\mathcal{T}^t$,*

$$\rho = \mathcal{T}^t \# \rho_0. \tag{4.33}$$

In addition, for each $t \in [0, T)$, f_ϵ converges weak-* to a probability density $f(t, \cdot, \cdot) \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$:

$$f_\epsilon \xrightarrow{w^*} f \qquad in \ \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \quad as \ \epsilon \to 0. \tag{4.34}$$

The limiting density f, with first marginal ρ, is given explicitly by:

$$f(t, x, v) = \rho(t, x)\delta(v - u(t, x)), \tag{4.35}$$

where (ρ, u) is the unique solution of (1.5).

Proof. The first part, expressed by equation (4.33), follows from considerations made in Theorem 4.1. However, it also follows directly, as a consequence of the argument below.

The limiting behaviour of f_ϵ was not explicitly stated or needed in Theorem 4.1, but follows by arguments similar to those used for J_ϵ and ρ_ϵ in the proof of Theorem 4.1. Let us sketch this argument briefly.

Fix $\psi_1 \in C^1_c(0, T)$ and $\psi_2 \in C^1_b(\mathbb{R}^d \times \mathbb{R}^d)$, and let $\varphi(t, x, v) = \psi_1(t)\psi_2(x, v)$ in (3.5). Find

$$\int_0^T \psi_1'(t) \int_{\mathbb{R}^{2d}} \psi_2(x, v) f_\epsilon(t, x, v) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t =$$

$$- \int_0^T \psi_1(t) \int_{\mathbb{R}^{2d}} \left[\nabla_x \psi_2 \cdot v - \frac{1}{\epsilon} \nabla_v \psi_2 \cdot F[f_\epsilon] \right] f_\epsilon \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t. \tag{4.36}$$

Denoting by

$$\tilde{\xi}_\epsilon(t) = \int_{\mathbb{R}^{2d}} \psi_2(x, v) f_\epsilon(t, x, v) \, \mathrm{d}x \, \mathrm{d}v,$$

then, by (4.36), the weak derivative of $\tilde{\xi}_\epsilon$ is given by

$$\tilde{\xi}_\epsilon'(t) = \int_{\mathbb{R}^{2d}} \left[\nabla_x \psi_2 \cdot v - \frac{1}{\epsilon} \nabla_v \psi_2 \cdot F[f_\epsilon] \right] f_\epsilon \, \mathrm{d}x \, \mathrm{d}v \in L^\infty(0, T).$$

The right-hand-side of the equation above is bounded (the boundedness of the term that contains ϵ follows from Theorem 3.7).

Since $\tilde{\xi}_\epsilon$ is uniformly bounded in $W^{1,\infty}(0, T)$, it converges uniformly on a subsequence. On the other hand, similar to the arguments used for J_ϵ and ρ_ϵ in the proof of Theorem 4.1, we note that the sequence $f_\epsilon(t, \cdot, \cdot) \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ is tight, and hence, for each $t \in [0, T)$, $f_\epsilon(t, \cdot, \cdot)$ converges weak-* as measures, on a subsequence ϵ_k, to a probability measure

$f(t, \cdot, \cdot) \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$. Note also that the subsequence ϵ_k does not depend on t, due to the equicontinuity of $\tilde{\xi}_\epsilon(t)$ derived above.

Hence,

$$f_{\epsilon_k}(t, \cdot, \cdot) \xrightarrow{w^*} f(t, \cdot, \cdot) \qquad \text{in } \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \quad \text{as } \epsilon_k \to 0, \tag{4.37}$$

which proves the convergence (4.34) on a subsequence. To show its convergence on the full sequence $\epsilon \to 0$ we use the uniqueness of f, as derived from the arguments below.

Since $f_\epsilon(t) = \mathcal{T}_{\mathcal{H}[f_\epsilon]}^{t,\epsilon} \# f_0$ by Definition 3.1, (3.4) holds for f_{ϵ_k}:

$$\int\limits_{\mathbb{R}^{2d}} \zeta(x, v) f_{\epsilon_k}(t, x, v) dx dv = \int\limits_{\mathbb{R}^{2d}} \zeta(\mathcal{T}_{\mathcal{H}[f_{\epsilon_k}]}^{t,\epsilon_k}(X, V)) f_0(X, V) dX dV, \tag{4.38}$$

for all $\zeta \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$.

By the weak-* convergence of f_{ϵ_k}, the left-hand-side of (4.38) converges as $\epsilon_k \to 0$:

$$\int\limits_{\mathbb{R}^{2d}} \zeta(x, v) f_{\epsilon_k}(t, x, v) dx dv \to \int\limits_{\mathbb{R}^{2d}} \zeta(x, v) f(t, x, v) dx dv.$$

Due to convergence of trajectories (4.23), the right-hand-side of (4.38) converges by Lebesgue's dominated convergence theorem,

$$\int\limits_{\mathbb{R}^{2d}} \zeta(\mathcal{T}_{\mathcal{H}[f_{\epsilon_k}]}^{t,\epsilon_k}(X, V)) f_0(X, V) dX dV \to \int\limits_{\mathbb{R}^{2d}} \zeta(x(t; X), u(t, x(t; X))) f_0(X, V) dX dV,$$

as $\epsilon_k \to 0$. Combining these two, we find

$$\int\limits_{\mathbb{R}^{2d}} \zeta(x, v) f(t, x, v) dx dv = \int\limits_{\mathbb{R}^{2d}} \zeta(x(t; X), u(t, x(t; X))) f_0(X, V) dX dV, \tag{4.39}$$

for all $\zeta \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$.

First note that (4.33) can be derived from (4.39). Indeed, choose $\zeta(x, v) = \varphi(x)$ in (4.39) to find

$$\int\limits_{\mathbb{R}^d} \varphi(x) \rho(t, x) dx = \int\limits_{\mathbb{R}^d} \varphi(x(t; X)) \rho_0(X) dX,$$

for all $\varphi \in C_b(\mathbb{R}^d)$; this represents exactly the mass transport given by (4.33).

Now, observe that (4.35) is equivalent to

$$\int\limits_{\mathbb{R}^{2d}} f(t, x, v) \zeta(x, v) dx dv = \int\limits_{\mathbb{R}^d} \zeta(x, u(t, x)) \rho(t, x) dx,$$

for all test functions $\zeta \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$, which can be inferred immediately from (4.33) and (4.39).

The *unique* explicit representation of the limiting density f implies that the convergence in (4.37) holds on the full sequence f_ϵ, as stated in (4.34). $\qquad\square$

5 Numerical implementation and large time behaviour

In this section, we present and implement a numerical scheme for the macroscopic system (1.5). The numerical approach taken here follows closely the analytical considerations made in Section 2. In particular, it provides a discrete analogue for replacing the singular operator $\mathcal{A}$ by an invertible operator $\mathcal{M}$.

5.1 Discrete setting

The numerical discretization of the evolution equation (1.5a) for the density ρ can be done by standard methods. For instance, spatial discretization can be performed by a finite volume method [CCH15] or by a semi-Lagrangian particle method [DHPRF04]. For this reason, we focus in this section on the numerical solution of the velocity equation (1.5b). The subtlety is that for a given ρ, this equation does not have a unique solution for u, and hence, a direct discretization would lead to a singular system.

We illustrate this degeneracy in one dimension. To this end, let h be a fixed mesh size and $x_i = ih$ be equally distributed nodes. The range of i is $-N, \ldots, N$, with $N \in \mathbb{N}$ large enough such that the support of the density is within the computational domain $[-x_N, x_N]$. The time dependence is irrelevant for the discretization of (1.5b) and we drop it in the calculations below.

Denote by ρ_i and u_i, $i = -N, \ldots, N$, the numerical approximations of $\rho(x_i)$ and $u(x_i)$, respectively. Also, denote by $\vec{\rho}$ and $\vec{u}$ the column vectors containing these values:

$$\vec{\rho} = (\rho_{-N}, \ldots, \rho_N)^T, \qquad \vec{u} = (u_{-N}, \ldots, u_N)^T.$$

We choose the midpoint rule to approximate the integrals in (1.5b); note however that the argument below can be adapted to apply to higher order quadrature rules. This gives

$$\Phi(x_i) = \int_{\mathbb{R}} \phi(|x_i - y|)\rho(y)dy \approx h \sum_k \phi(|i - k|h)\rho_k,$$

$$\int_{\mathbb{R}} \phi(|x_i - y|)\rho(y)u(y)dy \approx h \sum_k \phi(|i - k|h)\rho_k u_k,$$

$$\int_{\mathbb{R}} K'(x_i - y)\rho(y)dy \approx h \sum_k K'((i - k)h)\rho_k.$$

Introduce notations

$$\phi_i = \phi(|i|h), \quad K_i' = K'(ih), \qquad i = -N, \ldots, N.$$

The following symmetries, respectively antisymmetries, hold:

$$\phi_i = \phi_{-i}, \qquad K_i' = -K_{-i}', \qquad \text{for all } i = -N, \ldots, N,$$

where the antisymmetry of K_i' follows from (1.4). Also, since the influence function ϕ is assumed to be positive and non-increasing (in accord with hypotheses made throughout the paper), ϕ has a lower bound on the computational domain and we have $\phi_i \geq \eta$, for all $i = -N, \ldots, N$, where $\eta = \phi(2x_N)$. Note that we made an abuse of notation here, as a

variable η, with a very similar meaning and role, was defined in (2.4); however we try to parallel the analytical considerations made in the proof of Proposition 2.1, and refrain from introducing unnecessary extra notations.

The discretization of (1.5b) reduces then to solving the linear system

$$A\vec{u} = \vec{b}, \tag{5.1}$$

where

$$A = \begin{bmatrix} \sum_k \phi_{-N-k}\rho_k & & \\ & \ddots & \\ & & \sum_k \phi_{N-k}\rho_k \end{bmatrix} - \begin{bmatrix} \phi_{-N-(-N)}\rho_{-N} & \cdots & \phi_{-N-N}\rho_N \\ \vdots & \ddots & \vdots \\ \phi_{N-(-N)}\rho_{-N} & \cdots & \phi_{N-N}\rho_N \end{bmatrix}, \tag{5.2}$$

and

$$\vec{b} = (b_{-N}, \ldots, b_N)^T, \qquad b_i = -\sum_k K'_{i-k}\rho_k, \quad i = -N, \ldots, N. \tag{5.3}$$

The matrix A is the discrete analogue of the operator $\mathcal{A}$ defined in (2.6). Similar to the continuous case, matrix A is singular, and solutions of the linear system are not unique. This fact is detailed in the following proposition.

Proposition 5.1 (Existence and non-uniqueness)**.** *Matrix A given by* (5.2) *is singular with* $dim(Null(A)) = 1$. *Moreover, the discrete linear system* (5.1)-(5.3) *has infinitely many solutions.*

Proof. It is easy to check that the entries of each row of A add up to zero; therefore, A is singular. To show that $dim(Null(A)) = 1$ we simply note that if we remove the i-th row and the i-th column of A (for an index i such that $\rho_i \neq 0$), the remaining matrix is strictly diagonally dominant, and hence it has full rank.

To prove existence of infinitely many solutions, as opposed to no solution, we need to check $\vec{b} \in Range(A)$. Indeed, by symmetry of ϕ_i, $A^T\vec{\rho} = 0$, and hence

$$Null(A^T) = span\{\vec{\rho}\}.$$

Finally, by the antisymmetry of K'_i, one has $\vec{b} \perp \vec{\rho}$, which, by Fredholm alternative, yields the conclusion. $\square$

Using Propostion 5.1, we immediately have

Proposition 5.2 (Uniqueness with momentum conservation)**.** *There exists a unique solution $\vec{u}$ of* (5.1)-(5.3)*, subject to the (discrete) momentum conservation condition*

$$\vec{\rho}^T\vec{u} = 0. \tag{5.4}$$

Proof. Although this result can be derived directly from Proposition 5.1, using the structure of $Null(A)$ deduced in its proof, we show here a more constructive way, that is also more relevant to the numerical implementation. Let $\vec{e} = (1, \cdots, 1)^T$. Then, a solution $\vec{u}$ of (5.1)-(5.3) which satisfies the discrete momentum conservation condition (5.4), also solves

$$M\vec{u} = \vec{b}, \tag{5.5}$$

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with

$$M = A + \eta\, \vec{e}\vec{\rho}^{\,T}. \tag{5.6}$$

Here, η represents the positive lower bound of ϕ_i, $i = -N, \ldots, N$, mentioned above.

The matrix M has the form

$$M = \begin{bmatrix} \sum_k \phi_{-N-k}\rho_k & & \\ & \ddots & \\ & & \sum_k \phi_{N-k}\rho_k \end{bmatrix} - \begin{bmatrix} (\phi_{-N-(-N)} - \eta)\rho_{-N} & \cdots & (\phi_{-N-N} - \eta)\rho_N \\ \vdots & \ddots & \vdots \\ (\phi_{N-(-N)} - \eta)\rho_{-N} & \cdots & (\phi_{N-N} - \eta)\rho_N \end{bmatrix}.$$

Since all the off-diagonal entries of M are negative, M can be shown to be strictly diagonally dominant. Hence, M is an invertible matrix, and (5.5) has a unique solution. $\qquad\square$

Remark 5.3. *The discrete theory aligns perfectly well with the results in Section 2. In particular, matrices A and M are the discrete analogues of the singular, respectively invertible, operators $\mathcal{A}$ and $\mathcal{M}$ defined in (2.6) and (2.7), while the conservation (5.4) of the discrete linear momentum is used as an auxiliary condition that enforces uniqueness of the numerical solution.*

To conclude, our numerical discretization of (1.5b) consists in the following procedure. First, form the matrix A using appropriate quadrature rules on integrations. Next, change the singular matrix A into the invertible matrix M given by (5.6), and solve the linear system (5.5). For an efficient numerical implementation, it is important to note that the velocities u_i at grid points with nontrivial density values $\rho_i \neq 0$ do *not* depend on values u_j for which $\rho_j = 0$; this comment is in fact the discrete counterpart of Remark 2.2. Hence, one can consider only the nodes (or indices) that correspond to nontrivial densities and reduce the size of the system.

5.2 Numerical examples

Below we illustrate our method with three numerical simulations: two examples in 1D with smooth and non-smooth potentials, respectively, and a 2D example with a smooth potential. We make choices of potentials for which explicit steady states of (1.5) can be calculated. The influence function in all the cases is chosen as

$$\phi(r) = (1 + r^2)^{-1/2}. \tag{5.7}$$

Example 1 (1D with smooth potential). We consider a power-law interaction potential with quartic attraction and quadratic repulsion:

$$K(x) = \frac{x^4}{4} - \frac{x^2}{2}. \tag{5.8}$$

The derivative K' is smooth, but not bounded, which means that K' lies in $W_{loc}^{1,\infty}(\mathbb{R})$, but not in $W^{1,\infty}(\mathbb{R})$, as assumed by theoretical results in previous sections. In simulations however, we deal with a bounded computational domain, on which K' and K'' are of course, bounded.

The initial condition is chosen to be a smooth bump function

$$\rho_0(x) = C_s e^{-\frac{1}{s^2-x^2}} \mathbf{1}_{(-s,s)}(x), \tag{5.9}$$

where s is a positive integer which determines the support of ρ_0, **1** denotes the indicator function, and C_s is a normalization constant such that $\|\rho_0\|_{L^1} = 1$.

Figure 5.1 shows the time evolution of ρ starting from initial data (5.9) with $s = 0.5$ (top row) and $s = 1.5$ (bottom row). The computational domain is set to be $[-2, 2]$ and the mesh size is $h = .02$. In both cases we observe that the solution approaches (by expanding, respectively compressing) the same steady state ρ_∞ given by two Dirac delta distributions located at $-1/2$ and $1/2$:

$$\rho_\infty(x) = \frac{1}{2}\left[\delta(x - 1/2) + \delta(x + 1/2)\right]. \tag{5.10}$$

We note that (5.10) is in fact a steady state of the attraction-repulsion model (with no alignment), i.e,

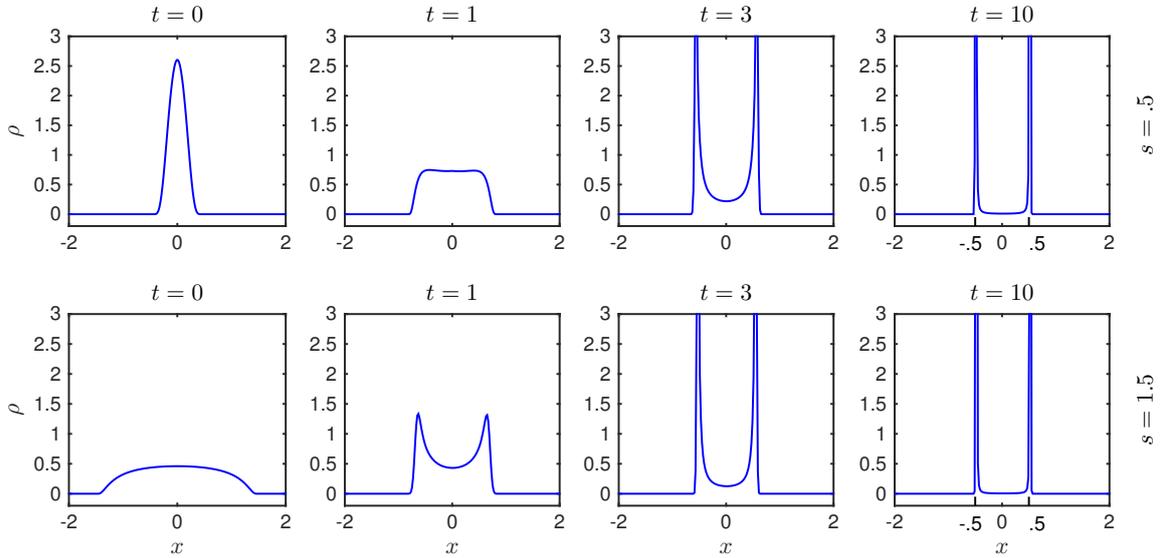$$\partial_t \rho - \nabla_x \cdot (\rho(\nabla_x K * \rho)) = 0. \tag{5.11}$$

Figure 5.1: Time evolution of the solution $\rho(t, x)$ of (1.5) in one dimension, starting from the initial density (5.9) with $s = 0.5$ (top row) and $s = 1.5$ (bottom row). The influence function is given by (5.7) and the interaction potential by (5.8). The two density profiles expand, respectively compress, toward the same equilibrium state given by (5.10).

Example 2 (1D with Morse potential). We now take a Morse-type interaction potential:

$$K(x) = -e^{-|x|/2} + e^{-|x|}. \tag{5.12}$$

Note again that due to the singularity at the origin, $K' \notin W^{1,\infty}(\mathbb{R})$. In terms of numerical simulations however, singularities in the potential are not an issue of concern (at discrete level, a pointy potential and a very localized regularization of it, are essentially the same).

In this simulation we choose a more general non-smooth and non-symmetric initial density:

$$\rho_0(x) = \frac{1}{2}(1 - x)\mathbf{1}_{(-1,1)}(x). \tag{5.13}$$

Figure 5.2 shows the time evolution of the solution (solid line) at $t = 0, 2, 5, 20$. In the same plot we show (dashed-line) the steady state of the attractive-repulsive equation (5.11) with the Morse-type potential (5.12). An exact, explicit form of this equilibrium solution was derived in [BT11]:

$$\rho_\infty(x) = C\left(\cos(\mu(x-c)) - \lambda\right) \cdot \mathbf{1}_{|x-c|<H}(x), \tag{5.14}$$

where the constants are given as follows: $\mu = 1/\sqrt{2}$, $\lambda = -1/(3 + \pi/\sqrt{2})$, $H = \pi/\sqrt{2}$. Also, $C = 3/(2(\pi + 3\sqrt{2}))$ is a scaling that sets the mass to one, and $c = 1/3$ is a shift of the centre of mass to the origin. Similar to Example 1, we observe that the solution approaches asymptotically the steady state ρ_∞.
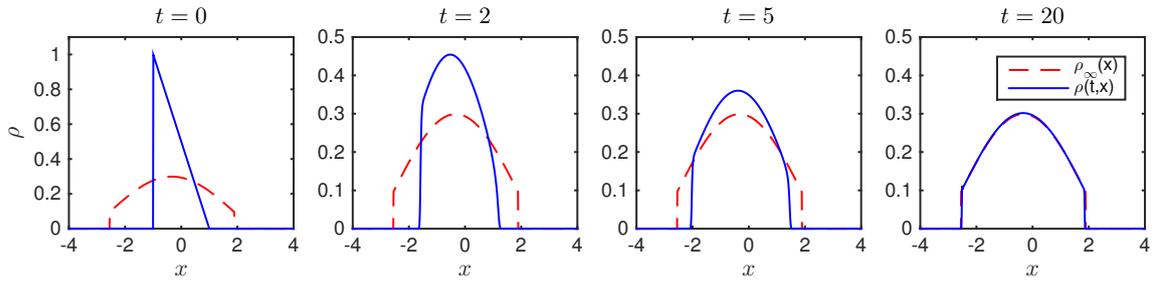

Figure 5.2: Time evolution of the solution $\rho(t,x)$ of (1.5) in one dimension, starting from the initial density (5.13). The influence function is given by (5.7) and the interaction potential by (5.12). The solution (solid curve) approaches asymptotically the equilibrium state ρ_∞ (dashed line) of the attractive-repulsive model (5.11), given explicitly by (5.14).

Example 3 (2D with Newtonian-quadratic potential). In the final example, we check the asymptotic behaviour of solutions to (1.5) in 2D. The algorithm presented above can be immediately extended to two dimensions. We take an interaction potential K with Newtonian repulsion and quadratic attraction:

$$K(x) = -\log|x| + \frac{1}{2}x^2. \tag{5.15}$$

With this choice of K, it has been shown that the steady state of the attractive-repulsive model (5.11) is the indicator function of a disk:

$$\rho_\infty(x) = C\mathbf{1}_{|x|<1}(x-c), \tag{5.16}$$

where the scale C is set by conservation of mass, and c is a shift of the centre of mass [FHK11].

We take the following radially symmetric initial density:

$$\rho_0(r) = (r-1)^2(r-2)^2\mathbf{1}_{1<r<2}(r). \tag{5.17}$$

The numerical time evolution illustrated in Figure 5.3 indicates again, that the solution approaches asymptotically ρ_∞ (here $C = 0.1$ and $c = 0$).

In this experiment we set the computational domain to be $[-2.2, 2.2] \times [-2.2, 2.2]$ and fix an Eulerian computational grid with $dx = dy = 0.04$. We use a semi-Lagrangian scheme to evolve the density equation in time [DHPRF04]. Specifically, we evolve numerically, using a forward Euler integrator with $dt = 0.005$, the characteristic trajectories (2.1) and the density equation
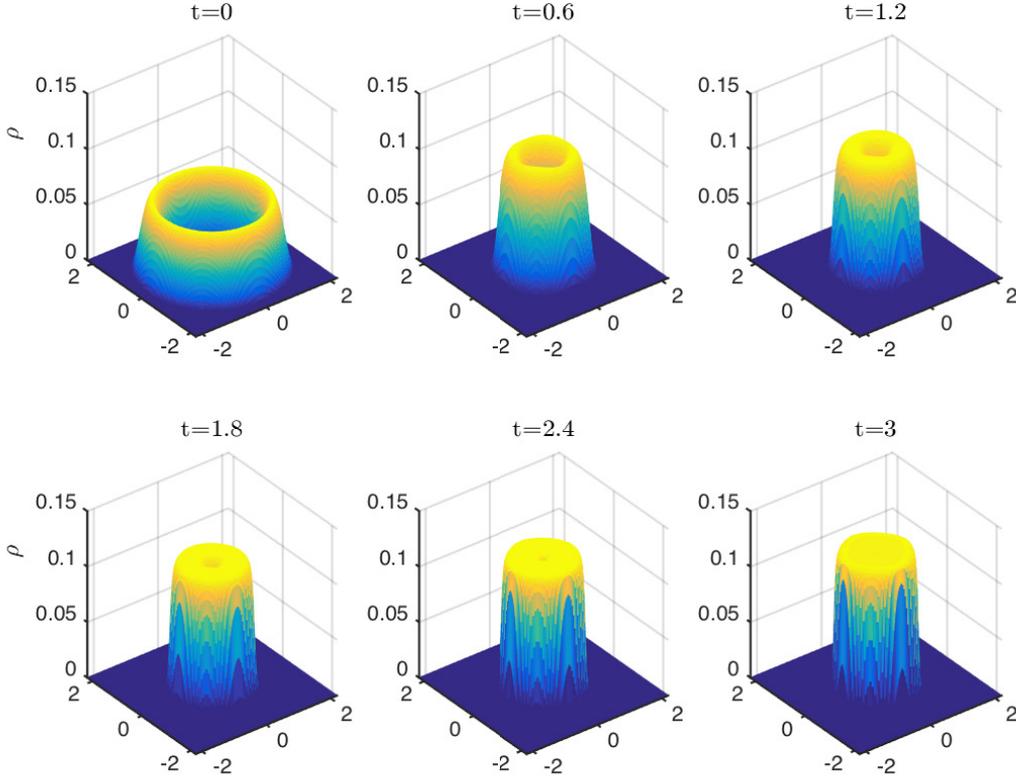
Figure 5.3: Time evolution of the density $\rho(t, x)$ of the aggregation model (1.5) in two dimensions, starting from the initial density (5.17). The influence function is given by (5.7) and the interaction potential by (5.15). The solution approaches asymptotically the equilibrium state (5.16) consisting of a constant distribution in the unit disk.

(1.5a) written in characteristic form. After each time step, we interpolate the density values from the Lagrangian (distorted) grid back to the Eulerian grid, calculate the new velocity field according to the method described in Section 5.1, and repeat the procedure.

We conclude this section by noting that our preliminary numerical investigations of model (1.5) (as presented here) suggest that the large time behaviour of solutions to (1.5) is:

$$\rho(t, x) \to \rho_\infty(x), \qquad \text{for } x \in \text{supp}\rho(t), \quad \text{as } t \to \infty,$$

where ρ_∞ is a steady state of (5.11). Consequently,

$$u(t, x) \to 0, \qquad \text{for all } x \in \text{supp}\rho(t), \quad \text{as } t \to \infty.$$

This means that the macroscopic velocity u converges to its (zero) average as time goes to infinity, which corresponds to *flocking*. Whether the model always approaches asymptotically the flocking limit is left for future investigation.

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