Acoustic Limit for the Boltzmann equation in Optimal Scaling

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Abstract

Based on a recent L^2-L^∞ framework, we establish the acoustic limit of the Boltzmann equation for general collision kernels. The scaling of the fluctuations with respect to Knudsen number is optimal. Our approach is based on a new analysis of the compressible Euler limit of the Boltzmann equation, as well as refined estimates of Euler and acoustic solutions.

1 Introduction and Main Results

We study the Boltzmann equation

$$\partial_t F^\varepsilon + v\cdot\nabla_x F^\varepsilon = \frac{1}{\varepsilon}\mathcal{Q}(F^\varepsilon, F^\varepsilon) \tag{1.1}$$

where $F^\varepsilon(t,x,v)\geq 0$ is the density of particles of velocity $v\in\mathbf{R}^3$, and position $x\in\Omega=\mathbf{R}^3$ or $\mathbf{T}^3$, a periodic box. The positive parameter ε is the Knudsen number. Throughout this paper, the collision operator takes the form

$$\begin{aligned}\mathcal{Q}(F_1\,,F_2)(v) = &\int_{\mathbf{R}^3}\int_{\mathbf{S}^2}|v-u|^\gamma F_1(u')F_2(v')B(\theta)\,d\omega\,du\\ &-\int_{\mathbf{R}^3}\int_{\mathbf{S}^2}|v-u|^\gamma F_1(u)F_2(v)B(\theta)\,d\omega\,du\,,\end{aligned} \tag{1.2}$$

where $-3<\gamma\leq 1$, $u' = u+[(v-u)\cdot\omega]\omega$, $v' = v-[(v-u)\cdot\omega]\omega$, $\cos\theta = (u-v)\cdot\omega/|v-u|$, and $0<B(\theta)\leq C|\cos(\theta)|$. Such collision operators cover both the hard-sphere interaction and inverse power law with an angular cutoff. The hard potential means $0\leq\gamma\leq 1$, while soft potential means $-3<\gamma<0$.

1.1 Hilbert Expansion

We define a family of special distribution functions $\mu(t, x, v)$ the *local Maxwellians* by

$$\mu(t\,, x\,, v) \equiv \frac{\rho(t, x)}{[2\pi T(t, x)]^{3/2}} \exp\left\{-\frac{[v - \mathfrak{u}(t, x)]^2}{2T(t, x)}\right\} \tag{1.3}$$

which are equilibrium of the collision process:

$$\mathcal{Q}(\mu\,, \mu) = 0\,.$$

$(\rho\,, \mathfrak{u}\,, T)$ represent the macroscopic density, bulk velocity, and temperature respectively. If $(\rho\,, \mathfrak{u}\,, T)$ are constant in t and x, μ is called a *global Maxwellian*. It was shown in [6, 15] that for hard-sphere interaction, namely $\gamma = 1$, as $\varepsilon \to 0$, $\{F^\varepsilon\}$ solutions to the Boltzmann equation (1.1) converge to a local Maxwellian μ induced by a solution to the compressible Euler system:

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot (\rho \mathfrak{u}) &= 0 \\ \partial_t(\rho \mathfrak{u}) + \nabla_x \cdot (\rho \mathfrak{u} \otimes \mathfrak{u}) + \nabla_x p &= 0 \\ \partial_t \left[\rho(e + \tfrac{1}{2}|\mathfrak{u}|^2)\right] + \nabla_x \cdot \left[\rho \mathfrak{u}(e + \tfrac{1}{2}|\mathfrak{u}|^2)\right] + \nabla_x \cdot (p\mathfrak{u}) &= 0 \end{aligned} \tag{1.4}$$

with the equation of state

$$p = \rho R T = \tfrac{2}{3}\rho e \tag{1.5}$$

as long as the solution stays smooth. Let $(\rho(t, x), \mathfrak{u}(t, x), T(t, x))$ be a smooth solution of the Euler equations (1.4) for $t \in [0, \tau]$, $x \in \Omega$. Consider the local Maxwellian μ from $(\rho, \mathfrak{u}, T)$ as in (1.3). As in [6], we take the Hilbert expansion of solutions around $F_0 \equiv \mu$ with the form

$$F^\varepsilon = \sum_{n=0}^{5} \varepsilon^n F_n + \varepsilon^3 F_R^\varepsilon\,,$$

where $F_0, ..., F_5$ are the first 6 terms of the Hilbert expansion, independent of ε, which solve the equations

$$\begin{aligned} 0 &= \mathcal{Q}(F_0, F_0), \\ \{\partial_t + v \cdot \nabla_x\} F_0 &= \mathcal{Q}(F_0, F_1) + \mathcal{Q}(F_1, F_0), \\ \{\partial_t + v \cdot \nabla_x\} F_1 &= \mathcal{Q}(F_0, F_2) + \mathcal{Q}(F_2, F_0) + \mathcal{Q}(F_1, F_1), \\ &\ldots \\ \{\partial_t + v \cdot \nabla_x\} F_5 &= \mathcal{Q}(F_0, F_6) + \mathcal{Q}(F_6, F_0) + \sum_{\substack{i+j=6 \\ 1 \le i \le 5, 1 \le j \le 5}} \mathcal{Q}(F_i, F_j). \end{aligned}$$

We can construct smooth $F_1(t, x, v), F_2(t, x, v), ..., F_6(t, x, v)$ for $0 \le t \le \tau$. For more detailed discussion, see [6]. Now we put $F^\varepsilon = \sum_{n=0}^{5} \varepsilon^n F_n + \varepsilon^3 F_R^\varepsilon$ into the Boltzmann equation (1.1) to derive the remainder equation for F_R^ε

$$\begin{aligned}&\partial_t F_R^\varepsilon + v\cdot\nabla_x F_R^\varepsilon - \frac{1}{\varepsilon}\{\mathcal{Q}(\mu, F_R^\varepsilon) + \mathcal{Q}(F_R^\varepsilon,\mu)\}\\ =&\,\varepsilon^2\mathcal{Q}(F_R^\varepsilon, F_R^\varepsilon) + \sum_{i=1}^{5}\varepsilon^{i-1}\{\mathcal{Q}(F_i, F_R^\varepsilon) + \mathcal{Q}(F_R^\varepsilon, F_i)\} + \varepsilon^2 A\end{aligned} \tag{1.6}$$

where

$$A = -\{\partial_t + v\cdot\nabla_x\}F_5 + \sum_{i+j\geq 6, 1\leq i,j\leq 5}\varepsilon^{i+j-6}\mathcal{Q}(F_i, F_j). \tag{1.7}$$

The acoustic system is the linearization about the homogeneous state of the compressible Euler system. After a suitable choice of units, the fluid fluctuations (σ, u, θ) satisfy

$$\begin{aligned}\partial_t\sigma + \nabla_x\!\cdot u = 0\,, &\qquad \sigma(x,0) = \sigma^0(x)\,,\\ \partial_t u + \nabla_x(\sigma+\theta) = 0\,, &\qquad u(x,0) = u^0(x)\,,\\ \tfrac{3}{2}\partial_t\theta + \nabla_x\!\cdot u = 0\,, &\qquad \theta(x,0) = \theta^0(x)\,.\end{aligned} \tag{1.8}$$

Such acoustic system (1.8) can be formally derived from the Boltzmann equation (1.1) by letting

$$F^\varepsilon = \mu^0 + \delta G^\varepsilon \tag{1.9}$$

where μ^0 is the global Maxwellian which corresponds to $\rho = T = 1$ and $\mathfrak{u} = 0$:

$$\mu^0 \equiv \tfrac{1}{(2\pi)^{3/2}}\exp(-\tfrac{|v|^2}{2})$$

and the fluctuation amplitude δ is a function of ε satisfying

$$\delta \to 0 \;\text{ as }\; \varepsilon \to 0\,. \tag{1.10}$$

For instance, one can take

$$\delta = \varepsilon^m \text{ for any } m > 0\,.$$

With the above scalings, G^ε formally converges to

$$G = \left\{\sigma + v\cdot u + \left(\tfrac{|v|^2-3}{2}\right)\theta\right\}\mu^0 \tag{1.11}$$

as $\varepsilon \to 0$, where σ, u, θ satisfy the acoustic system (1.8). For detailed formal derivation, see [1, 10].

1.2 Main Theorems

The endeavor to understand how fluid dynamical equations for both compressible and incompressible flows can be derived from kinetic theory goes back to the founding work of Maxwell [26] and Boltzmann [5]. Most of these derivations are well understood at several formal levels by now, and yet their full mathematical justifications are still incomplete. In fact, the purpose of the Hilbert's sixth problem [19] is to seek a unified theory of the gas dynamics including various levels of descriptions from a mathematical standpoint. So far, there are basically three different approaches mathematically. The first is based on spectral analysis of the semi-group generated by the linearized Boltzmann equation, see [4, 21, 27].

The second is based on Hilbert or Chapman-Enskog expansions [6, 7], see more recent work in [13, 16], [12, 14]. The third approach was the program initiated from [1, 2], working in the framework of global renormalized solutions after the celebrated work of DiPerna-Lions [8], to justify global weak solutions of incompressible flows (Navier-Stokes, Stokes, and Euler), and (compressible) acoustic system, see [1, 2, 3, 10, 11, 18, 22, 23, 24, 28].

The authors in [10] proved the convergence of the acoustic limit from DiPerna-Lions solutions of the Boltzmann equation (1.1) with the restriction on the size of fluctuations: $m > \frac{1}{2}$. Recently, in [18], this restriction has been relaxed to the borderline case $m = \frac{1}{2}$ by employing some new nonlinear estimates developed in [22] and a new L^1 averaging lemma in [11]. However, due to some technical difficulties mainly caused by the lack of local conservation laws and regularity of renormalized solutions, the case for $m < \frac{1}{2}$ remains an open question. On the other hand, in the framework of classical solutions, in [17], the authors have established the global-in-time uniform energy estimates and proven the strong convergence for $m = 1$, by adapting the nonlinear energy method of [13, 16]. Although this method displays in a clear way how the dissipation disappears in the acoustic limit in terms of instant energies and dissipation rates, it does not cover other interesting cases $0 < m < 1$ due to weak dissipations.

The purpose of this article is to establish the acoustic limit for $0 < m < 1$ via a recent L^2-L^∞ framework. We will use δ instead of ε^m to denote the fluctuation amplitude. Since our interest is the case of $0 < m < 1$ towards the optimal scaling, throughout the paper, we assume that in addition to (1.10),

$$\frac{\varepsilon}{\delta} \to 0 \ \text{ as } \ \varepsilon \to 0\,. \tag{1.12}$$

Theorem 1.1. *Let $\tau > 0$ be any given finite time and let*

$$\sigma(0,x) = \sigma^0(x),\ u(0,x) = u^0(x),\ \theta(0,x) = \theta^0(x)\ \in H^s,\ s \geq 4 \tag{1.13}$$

be any given initial data to the acoustic system (1.8)*. Then there exist an $\varepsilon_0 > 0$ and a $\delta_0 > 0$ such that for each $0 < \varepsilon \leq \varepsilon_0$ and $0 < \delta \leq \delta_0$, there exists a constant $C > 0$ so that*

$$\sup_{0\leq t\leq\tau} \|G^\varepsilon(t) - G(t)\|_\infty + \sup_{0\leq t\leq\tau} \|G^\varepsilon(t) - G(t)\|_2 \leq C\{\frac{\varepsilon}{\delta} + \delta\} \tag{1.14}$$

where $\frac{\varepsilon}{\delta} \to 0$ as $\varepsilon \to 0$, and G^ε and G are defined in (1.9) *and* (1.11)*, and C depends only on τ and the initial data σ^0, u^0, θ^0.*

Our proof is different from the previous approach. Instead of estimating $G^\varepsilon - G$ directly, we make a detour to control of $G^\varepsilon - G$ in two steps. The first step (Section 2) is to show as $\varepsilon \to 0$, F^ε is close to the local Maxwellian μ^δ, constructed from the smooth solution of the compressible Euler equation. In fact, we are able to establish (Theorem 1.2)

$$F^\varepsilon - \mu^\delta = O(\varepsilon),$$

before the time of possible shock formation, which is of the order of $\frac{1}{\delta}$ in the acoustic scaling (longer than any fixed time τ!). The second step (Section 3) is to show that (Lemma 3.3), within the time scale of $\frac{1}{\delta}$,

$$\mu^\delta - \mu^0 = \delta G + O(\delta^2).$$

Such an estimate confirms that the solution of the acoustic equation G is the first order (linear) approximation of that to the Euler equations. Combining these two estimates and comparing with (1.9), we deduce our theorem by dividing δ. Our proof relies on the existence of global in-time smooth solutions to the linear acoustic system (1.8).

Our main technical contribution is a new analysis of the classical compressible Euler limit to complete step one above. To precisely state our result, we use the standard notation H^s to denote the Sobolev space $W^{s,2}(\Omega)$ with corresponding norm $\|\cdot\|_{H^s}$. We also use the standard notation $\|\cdot\|_2$ and $\|\cdot\|_\infty$ to denote L^2 norm and L^∞ norm in both $(x,v)\in\Omega\times\mathbf{R}^3$ variables. We use $\langle\cdot\,,\cdot\rangle$ to denote the standard L^2 inner product. We also define a weighted L^2 norm

$$\|g\|_\nu^2=\int_{\Omega\times\mathbf{R}^3} g^2(x\,,v)\nu(v)\,dx\,dv\,,$$

where the collision frequency $\nu(v)\equiv\nu(\mu)(v)$ is defined as

$$\nu(\mu)=\int_{\mathbf{R}^3} B(\theta)|v-v'|^\gamma\mu(v')\,dv'd\omega\,.$$

Note that for given $-3<\gamma\le 1$,

$$\nu(\mu)\sim\rho\,(1+|v|)^\gamma.$$

Define the linearized collision operator $\mathcal{L}$ by

$$\mathcal{L}g=-\frac{1}{\sqrt{\mu}}\{\mathcal{Q}(\mu,\sqrt{\mu}g)+\mathcal{Q}(\sqrt{\mu}g,\mu)\}.$$

Let $\mathbf{P}g$ be the L_v^2 projection with respect to $[\sqrt{\mu},v\sqrt{\mu},|v|^2\sqrt{\mu}]$. Then it is well-known that there exists a positive number $c_0>0$ such that

$$\langle\mathcal{L}g,g\rangle\ge c_0\|\{\mathbf{I}-\mathbf{P}\}g\|_\nu^2\,. \tag{1.15}$$

The solutions to the Boltzmann equation (1.1) are constructed near the local Maxwellian of the compressible Euler system. So it is natural to rewrite the remainder

$$F_R^\varepsilon=\sqrt{\mu}f^\varepsilon. \tag{1.16}$$

Since μ is a local Maxwellian, the equation of the remainder includes the new term $\sqrt{\mu}^{-1}(\partial_t+v\cdot\nabla_x)\sqrt{\mu}f^\varepsilon$. At large velocities, the distribution functions may be growing rapidly due to streaming. To remedy this difficulty, following Caflisch [6], we introduce a global Maxwellian

$$\mu_M=\frac{1}{(2\pi T_M)^{3/2}}\exp\left\{-\frac{|v|^2}{2T_M}\right\}.$$

where T_M satisfies the following condition

$$T_M<\max_{t\in[0,\tau],x\in\Omega}T(t,x)<2T_M\,. \tag{1.17}$$

Note that under the assumption (1.17), there exist constants $c_1\,,c_2$ such that for some $1/2<\alpha<1$ and for each $(t\,,x\,,v)\in[0,\tau]\times\Omega\times\mathbf{R}^3$, the following holds

$$c_1\mu_M\le\mu\le c_2\mu_M^\alpha. \tag{1.18}$$

We further define

$$F_R^\varepsilon = \{1+|v|^2\}^{-\beta}\sqrt{\mu_M}h^\varepsilon \equiv \frac{1}{w(v)}\sqrt{\mu_M}h^\varepsilon \tag{1.19}$$

for any fixed

$$\beta \geq \frac{9-2\gamma}{2}.$$

We now state the result on the compressible Euler limit:

Theorem 1.2. *Assume that the solution to the Euler equations* $[\rho(t,x), u(t,x), T(t,x)]$ *is smooth and* $\rho(t,x)$ *has a positive lower bound for* $0 \leq t \leq \tau$. *Furthermore, assume that the temperature* $T(t,x)$ *satisfies the condition* (1.17). *Let*

$$F^\varepsilon(0,x,v) = \mu(0,x,v) + \sum_{n=1}^{5}\varepsilon^n F_n(0,x,v) + \varepsilon^3 F_R^\varepsilon(0,x,v) \geq 0. \tag{1.20}$$

Then there is an $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon \leq \varepsilon_0$, *and for any* $\beta \geq \frac{9-2\gamma}{2}$, *there exists a constant* $C_\tau(\mu, F_0, F_1, ..F_6)$ *such that*

$$\begin{aligned} &\sup_{0\leq t\leq\tau} \varepsilon^{\frac{3}{2}} \left\| \sqrt{\mu}^{-1}(1+|v|^2)^\beta F_R^\varepsilon(t)\right\|_\infty + \sup_{0\leq t\leq\tau}\left\|\sqrt{\mu}^{-1}F_R^\varepsilon(t)\right\|_2 \\ &\leq C_\tau\left\{\varepsilon^{\frac{3}{2}}\left\|\sqrt{\mu}^{-1}(1+|v|^2)^\beta F_R^\varepsilon(0)\right\|_\infty + \left\|\sqrt{\mu}^{-1}F_R^\varepsilon(0)\right\|_2 + 1\right\}, \end{aligned} \tag{1.21}$$

where F_R^ε *is the solution to the remainder equation* (1.6).

Remark: Applying the bound (1.18), Lemma A.1 and Lemma A.2 of [13], we can carefully choose $F_R^\varepsilon(0,x,v)$ in (1.20) so that the initial data (1.20) are non-negative. Because the argument is quite similar, we omit the details here.

Based on the a priori estimates given in Theorem 1.2, following the arguments in the pioneering work of Caflisch [6], we can immediately derive the compressible Euler limit as well as the existence of the solutions to the Boltzmann equation. As in [6], the Hilbert expansion provides a natural way to establish an uniform in ε control for the Euler limit. However, it was well-known [6] that an $|v|^3 f^\varepsilon$ term due to streaming in the L^2 estimate creates an unpleasant analytical difficulty. We employ both L^2 and L^∞ estimate with polynomial velocity weight [14, 15] to control such a term with a high power of velocity v. On the one hand, our analysis requires an additional assumption of moderate temperature variation (1.17). On the other hand, we do not need the truncation of the Hilbert expansion as in [6], so that the positivity of the solution is guaranteed. In particular, our theorem is designed to apply to the acoustic limit because the temperature variation is only of the order δ. Moreover, a cutoff trick used in [30] enables us to treat all soft potentials $-3 < \gamma \leq 1$ with an angular cutoff.

2 Compressible Euler Limit

In this section, we prove Theorem 1.2. Note that it suffices to estimate $\|f^\varepsilon(t)\|_2$ and $\|h^\varepsilon(t)\|_\infty$ to conclude the theorem. The proof relies on an interplay between L^2 and L^∞ estimates for the Boltzmann equation [14, 15]: L^2 norm of f^ε is controlled by the L^∞ norm of the high velocity part and vice versa. These uniform L^2-L^∞ estimates are stated in the following two lemmas:

Lemma 2.1. (L^2-Estimate): *Let $(\rho, \mathfrak{u}, T)$ be a smooth solution to the Euler equations such that ρ has a positive lower bound and T satisfies the condition* (1.17). *Let $f^\varepsilon, h^\varepsilon$ be defined in* (1.16) *and* (1.19), *and $c_0 > 0$ be as in the coercivity estimate* (1.15). *Then there exists $\varepsilon_0 > 0$ and a positive constant $C = C(\mu, F_0, F_1, \cdots, F_6) > 0$, such that for all $\varepsilon < \varepsilon_0$*

$$\frac{d}{dt}\|f^\varepsilon\|_2^2 + \frac{c_0}{2\varepsilon}\|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|_\nu^2 \le C\{\sqrt{\varepsilon}\|\varepsilon^{3/2}h^\varepsilon\|_\infty + 1\}(\|f^\varepsilon\|_2^2 + \|f^\varepsilon\|_2)\,. \tag{2.1}$$

Lemma 2.2. (L^∞-Estimate): *Let $(\rho, \mathfrak{u}, T)$ be a smooth solution to the Euler equations such that ρ has a positive lower bound and T satisfies the condition* (1.17). *Let $f^\varepsilon, h^\varepsilon$ and $c_0 > 0$ be the same as in Lemma 2.1. Then there exist $\varepsilon_0 > 0$ and a positive constant $C = C(\mu, c_0, F_1, \cdots, F_6) > 0$, such that for all $\varepsilon < \varepsilon_0$*

$$\sup_{0\le s\le\tau}\{\varepsilon^{3/2}\|h^\varepsilon(s)\|_\infty\} \le C\{\|\varepsilon^{3/2}h_0\|_\infty + \sup_{0\le s\le\tau}\|f^\varepsilon(s)\|_2 + \varepsilon^{7/2}\}. \tag{2.2}$$

The proof of Theorem 1.2 is a direct consequence of Lemmas 2.1 and 2.2.

Proof. of Theorem 1.2:

$$\begin{aligned}&\frac{d}{dt}\|f^\varepsilon\|_2^2 + \frac{c_0}{2\varepsilon}\|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|_\nu^2\\ \le& C\left\{\sqrt{\varepsilon}\left[\|\varepsilon^{3/2}h_0\|_\infty + \sup_{0\le s\le\tau}\|f^\varepsilon(s)\|_2 + \varepsilon^{7/2}\right] + 1\right\}\left(\|f^\varepsilon\|_2^2 + \|f^\varepsilon\|_2\right).\end{aligned}$$

A simple Gronwall inequality yields

$$\|f^\varepsilon(t)\|_2 + 1 \le (\|f^\varepsilon(0)\|_2 + 1)e^{Ct\{2+\sqrt{\varepsilon}\|\varepsilon^{3/2}h_0\|_\infty + \sqrt{\varepsilon}\sup_{0\le s\le\tau}\|f^\varepsilon(s)\|_2\}}\,.$$

For ε small, using the Taylor expansion of the exponential function in the above inequality, we have

$$\|f^\varepsilon\|_2 \le C_1(\|f^\varepsilon(0)\|_2 + 1)\left\{1 + \sqrt{\varepsilon}\|\varepsilon^{3/2}h_0\|_\infty + \sqrt{\varepsilon}\sup_{0\le s\le\tau}\|f^\varepsilon(s)\|_2\right\}. \tag{2.3}$$

For $t \le \tau$, letting ε small, we conclude the proof of our main theorem as:

$$\sup_{0\le t\le\tau}\|f^\varepsilon(t)\|_2 \le C_\tau\{1 + \|f^\varepsilon(0)\|_2 + \|\varepsilon^{3/2}h_0\|_\infty\}.$$

□

2.1 L^2 Estimate For f^ε

Proof. of Lemma 2.1: In terms of f^ε, we obtain

$$\begin{aligned}&\partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon + \frac{1}{\varepsilon}\mathcal{L}f^\varepsilon\\ =& \frac{\{\partial_t + v\cdot\nabla_x\}\sqrt{\mu}}{\sqrt{\mu}}f^\varepsilon + \varepsilon^2\Gamma(f^\varepsilon, f^\varepsilon) + \sum_{i=1}^{5}\varepsilon^{i-1}\{\Gamma(\frac{F_i}{\sqrt{\mu}}, f^\varepsilon) + \Gamma(f^\varepsilon, \frac{F_i}{\sqrt{\mu}})\} + \varepsilon^2\bar{A}\end{aligned}$$

where $\bar{A} = -\frac{\{\partial_t + v\cdot\nabla_x\}F_5}{\sqrt{\mu}} + \sum_{i+j\geq 6, i\leq 5, j\leq 5}\varepsilon^{i+j-6}\Gamma(\frac{F_i}{\sqrt{\mu}}, \frac{F_j}{\sqrt{\mu}})$.

Taking L^2 inner product with f^ε on both sides, since $\frac{\{\partial_t+v\cdot\nabla_x\}\sqrt{\mu}}{\sqrt{\mu}}$ is a cubic polynomial in v, we have for any $\kappa > 0$ and $a = 1/(3-\gamma)$,

$$
\begin{aligned}
&\left\langle \frac{\{\partial_t + v\cdot\nabla_x\}\sqrt{\mu}}{\sqrt{\mu}} f^\varepsilon, f^\varepsilon \right\rangle \\
&= \int_{|v|\geq \frac{\kappa}{\varepsilon^a}} + \int_{|v|\leq \frac{\kappa}{\varepsilon^a}} \\
&\leq \{\|\nabla_x\rho\|_2 + \|\nabla_x \mathfrak{u}\|_2 + \|\nabla_x T\|_2\} \times \|\{1+|v|^2\}^{3/2} f^\varepsilon \mathbf{1}_{|v|\geq \frac{\kappa}{\varepsilon^a}}\|_\infty \times \|f^\varepsilon\|_2 \\
&\quad + \{\|\nabla_x\rho\|_\infty + \|\nabla_x \mathfrak{u}\|_\infty + \|\nabla_x T\|_\infty\} \times \|\{1+|v|^2\}^{3/4} f^\varepsilon \mathbf{1}_{|v|\leq \frac{\kappa}{\varepsilon^a}}\|_2^2 \\
&\leq C_\kappa \varepsilon^2 \|h^\varepsilon\|_\infty \|f^\varepsilon\|_2 \\
&\quad + C\|\{1+|v|^2\}^{3/4}\mathbf{P}f^\varepsilon \mathbf{1}_{|v|\leq \frac{\kappa}{\varepsilon^a}}\|_2^2 + C\|\{1+|v|^2\}^{3/4}\{\mathbf{I}-\mathbf{P}\}f^\varepsilon \mathbf{1}_{|v|\leq \frac{\kappa}{\varepsilon^a}}\|_2^2 \\
&\leq C_\kappa \varepsilon^2 \|h^\varepsilon\|_\infty \|f^\varepsilon\|_2 + C\|f^\varepsilon\|_2^2 + \frac{C\kappa^{3-\gamma}}{\varepsilon}\|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|_\nu^2.
\end{aligned}
$$

Here we have used the fact $\{1+|v|^2\}^{3/2} f^\varepsilon \leq \{1+|v|^2\}^{\gamma-3} h^\varepsilon$, for $\beta \geq 3/2 + (3-\gamma)$ in (1.19), and the fact $\mu_M < C\mu$ in (1.18) under the assumption (1.17).

By the same proof as in Lemma 2.3 of [12] and (1.19),

$$
\varepsilon^2 \langle \Gamma(f^\varepsilon, f^\varepsilon), f^\varepsilon\rangle \leq C\varepsilon^2\{\|\nu(\mu) f^\varepsilon\|_\infty\}\|f^\varepsilon\|_2^2 \leq C\sqrt{\varepsilon}\|\varepsilon^{3/2}h^\varepsilon\|_\infty \|f^\varepsilon\|_2^2.
$$

Similarly, by the same proof as in Lemma 2.3 of [12] and (1.19),

$$
\begin{aligned}
&\sum_{i=1}^5 \varepsilon^{i-1}\{\langle \Gamma(\frac{F_i}{\sqrt{\mu}}, f^\varepsilon), f^\varepsilon\rangle + \langle \Gamma(f^\varepsilon, \frac{F_i}{\sqrt{\mu}}), f^\varepsilon\rangle\} \\
&\leq C\sum_{i=1}^5 \varepsilon^{i-1}\|f^\varepsilon\|_\nu^2 \|\int_{\mathbf{R}^3}\frac{F_i}{\sqrt{\mu}}dv\|_\infty \\
&\leq C\{\|\mathbf{P}f^\varepsilon\|_\nu^2 + \|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|_\nu^2\} \\
&\leq C\{\|f^\varepsilon\|_2^2 + \|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|_\nu^2\}.
\end{aligned}
$$

Clearly, $\langle \varepsilon^2 \bar{A}, f^\varepsilon\rangle \leq C\|f^\varepsilon\|_2$. We therefore conclude our lemma by choosing κ small. □

2.2 L^∞ Estimate For h^ε

As in [6], we define

$$
\mathcal{L}_M g = -\frac{1}{\sqrt{\mu_M}}\{\mathcal{Q}(\mu, \sqrt{\mu_M}g) + \mathcal{Q}(\sqrt{\mu_M}g, \mu)\} = \{\nu(\mu) + K\}g,
$$

where $Kg = K_1g - K_2g$ with

$$K_1g = \int_{\mathbf{B}^3\times\mathbf{S}^2} B(\theta)|u-v|^\gamma \sqrt{\mu_M(u)}\frac{\mu(v)}{\sqrt{\mu_M(v)}}g(u)dud\omega$$

$$K_2g = \int_{\mathbf{B}^3\times\mathbf{S}^2} B(\theta)|u-v|^\gamma \mu(u')\frac{\sqrt{\mu_M(v')}}{\sqrt{\mu_M(v)}}g(v')dud\omega + \int_{\mathbf{B}^3\times\mathbf{S}^2} B(\theta)|u-v|^\gamma \mu(v')\frac{\sqrt{\mu_M(u')}}{\sqrt{\mu_M(v)}}g(u')dud\omega\,.$$

Consider a smooth cutoff function $0 \le \chi_m \le 1$ such that for any $m > 0$,

$$\chi_m(s) \equiv 1,\ \text{for } s \le m\,;\ \chi_m(s) \equiv 0,\ \text{for } s \ge 2m.$$

Then define

$$\begin{aligned}K^m g = &\int_{\mathbf{B}^3\times\mathbf{S}^2} B(\theta)|u-v|^\gamma \chi_m(|u-v|)\sqrt{\mu_M(u)}\frac{\mu(v)}{\sqrt{\mu_M(v)}}g(u)dud\omega \\ &- \int_{\mathbf{B}^3\times\mathbf{S}^2} B(\theta)|u-v|^\gamma \chi_m(|u-v|)\mu(u')\frac{\sqrt{\mu_M(v')}}{\sqrt{\mu_M(v)}}g(v')dud\omega \\ &- \int_{\mathbf{B}^3\times\mathbf{S}^2} B(\theta)|u-v|^\gamma \chi_m(|u-v|)\mu(v')\frac{\sqrt{\mu_M(u')}}{\sqrt{\mu_M(v)}}g(u')dud\omega\,,\end{aligned}$$

and also define

$$K^c g = K - K^m.$$

Lemma 2.3.

$$|K^m g(v)| \le C m^{3+\gamma}\nu(\mu)||g||_\infty. \tag{2.4}$$

And $K^c g(v) = \int_{\mathbf{R}^3} l(v,v')g(v')dv'$ where the kernel l satisfies for some $c > 0$,

$$l(v,v') \le C_m \frac{\exp\{-c|v-v'|^2\}}{|v-v'|(1+|v|+|v'|)^{1-\gamma}}. \tag{2.5}$$

Proof. Since $\mu \le C\mu_M^\alpha$ for $\alpha > \frac{1}{2}$ and $|u|^2+|v|^2 = |u'|^2+|v'|^2$, we first have

$$\sqrt{\mu_M(u)}\frac{\mu(v)}{\sqrt{\mu_M(v)}} \le C\sqrt{\mu_M(u)}\mu_M^{\alpha-\frac{1}{2}}(v),$$

$$\mu(u')\frac{\sqrt{\mu_M(v')}}{\sqrt{\mu_M(v)}} + \mu(v')\frac{\sqrt{\mu_M(u')}}{\sqrt{\mu_M(v)}} \le C\{\mu^{\alpha-\frac{1}{2}}(u')\mu_M^{\frac{1}{2}}(u) + \mu_M^{\alpha-\frac{1}{2}}(v')\mu_M^{\frac{1}{2}}(u)\}.$$

Since $|v-u| \le 2m$, $\mu_M(u) \sim \mu_M(v)$ and thus $\mu_M^{\frac{1}{2}}(u) \le C\nu(\mu)$. And since $\gamma > -3$, (2.4) follows.

To show (2.5), clearly the kernel for K_1^c satisfies (2.5), since $\alpha > \frac{1}{2}$. For K_2^c, we can use the Carleman change of variable and apply the proof of Lemma 1 in [30] (one can extend the result to cover all $-3 < \gamma \le 1$). $\square$

We are now ready to prove Lemma 2.2.

Proof. *of Lemma* 2.2: Letting $K_w g \equiv wK(\frac{g}{w})$, from (1.6) and (1.19), we obtain

$$\begin{aligned}
&\partial_t h^\varepsilon + v\cdot\nabla_x h^\varepsilon + \frac{\nu(\mu)}{\varepsilon}h^\varepsilon + \frac{1}{\varepsilon}K_w h^\varepsilon \\
=&\frac{\varepsilon^2 w}{\sqrt{\mu_M}}\mathcal{Q}(\frac{h^\varepsilon\sqrt{\mu_M}}{w},\frac{h^\varepsilon\sqrt{\mu_M}}{w}) + \sum_{i=1}^{5}\varepsilon^{i-1}\frac{w}{\sqrt{\mu_M}}\{\mathcal{Q}(F_i,\frac{h^\varepsilon\sqrt{\mu_M}}{w}) + \mathcal{Q}(\frac{h^\varepsilon\sqrt{\mu_M}}{w},F_i)\} \\
&+\varepsilon^2\tilde{A},
\end{aligned}$$

where $\tilde{A} = -\frac{w\{\partial_t + v\cdot\nabla_x\}F_5}{\sqrt{\mu_M}} + \sum_{i+j\geq 6, i\leq 5, j\leq 5}\varepsilon^{i+j-6}\frac{w}{\sqrt{\mu_M}}\mathcal{Q}(F_i,F_j)$.

By Duhamel's principle, we have $h^\varepsilon(t,x,v) =$

$$\begin{aligned}
&\exp\{-\frac{\nu t}{\varepsilon}\}h^\varepsilon(0,x-vt,v) - \int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\left(\frac{1}{\varepsilon}K_w^m h^\varepsilon\right)(s,x-v(t-s),v)ds \\
&-\int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\left(\frac{1}{\varepsilon}K_w^c h^\varepsilon\right)(s,x-v(t-s),v)ds \\
&+\int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\left(\frac{\varepsilon^2 w}{\sqrt{\mu_M}}\mathcal{Q}(\frac{h^\varepsilon\sqrt{\mu_M}}{w},\frac{h^\varepsilon\sqrt{\mu_M}}{w})\right)(s,x-v(t-s),v)ds \\
&+\int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\left(\sum_{i=1}^{5}\varepsilon^{i-1}\frac{w}{\sqrt{\mu_M}}\mathcal{Q}(F_i,\frac{h^\varepsilon\sqrt{\mu_M}}{w})\right)(s,x-v(t-s),v)ds \\
&+\int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\left(\sum_{i=1}^{5}\varepsilon^{i-1}\frac{w}{\sqrt{\mu_M}}\mathcal{Q}(\frac{h^\varepsilon\sqrt{\mu_M}}{w},F_i)\right)(s,x-v(t-s),v)ds \\
&+\int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\varepsilon^2\tilde{A}(s,x-v(t-s),v)ds.
\end{aligned} \tag{2.6}$$

First note that

$$\nu(\mu) \sim \int |v-u|^\gamma \mu du \sim (1+|v|)^\gamma \rho(t,x) \sim \nu_M(v),$$

$$\int_0^t \exp\{-\frac{\nu(\mu)(t-s)}{\varepsilon}\}\nu(\mu)ds \leq c\int_0^t \exp\{-\frac{c\nu_M(t-s)}{\varepsilon}\}\nu_M ds = O(\varepsilon).$$

Then from (2.4), the second term in (2.6) is bounded by

$$Cm^{3+\gamma}\int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\nu ds \sup_{0\leq t\leq\tau}||h^\varepsilon(t)||_\infty \leq Cm^{3+\gamma}\varepsilon\sup_{0\leq t\leq\tau}||h^\varepsilon(t)||_\infty.$$

By $\mu_M \leq C\mu$, and since $|\frac{w}{\sqrt{\mu_M}}Q(\frac{h^\varepsilon\sqrt{\mu_M}}{w},\frac{h^\varepsilon\sqrt{\mu_M}}{w})| \leq C\nu(\mu)\|h^\varepsilon\|_\infty^2$ from the same proof as in Lemma 10 of [14], the third line in (2.6) is bounded by

$$\begin{aligned}
&C\varepsilon^2\int_0^t \exp\{-\frac{\nu(\mu)(t-s)}{\varepsilon}\}\nu(\mu)\|h^\varepsilon(s)\|_\infty^2 ds \\
&\leq C\varepsilon^3\sup_{0\leq s\leq t}\|h^\varepsilon(s)\|_\infty^2.
\end{aligned} \tag{2.7}$$

From the same proof as in Lemma 10 of [14] again,

$$
\begin{aligned}
&\sum_{i=1}^{5}\varepsilon^{i-1}\frac{w}{\sqrt{\mu_M}}\{\mathcal{Q}(F_i,\frac{h^\varepsilon\sqrt{\mu_M}}{w})+\mathcal{Q}(\frac{h^\varepsilon\sqrt{\mu_M}}{w},F_i)\}\\
&\leq \nu_M(v)\|h^\varepsilon\|_\infty\|\frac{w}{\sqrt{\mu_M}}\sum_{i=1}^{5}\varepsilon^{i-1}F_i\|_\infty
\end{aligned}
$$

so that the fourth and fifth lines in (2.6) are bounded by

$$
C\int_0^t \exp\{-\frac{\nu(\mu)(t-s)}{\varepsilon}\}\nu_M(v)\|h^\varepsilon(s)\|_\infty ds\leq C\varepsilon\sup_{0\leq s\leq t}\|h^\varepsilon(s)\|_\infty. \tag{2.8}
$$

The last line in (2.6) is clearly bounded by $C\varepsilon^3$.

We shall mainly concentrate on the third term in the right hand side of (2.6). Let $l_w(v,v')$ be the corresponding kernel associated with K_w^c. Recalling (2.5), we have

$$
|l_w(v,v')|\leq \frac{Cw(v')\exp\{-c|v-v'|^2\}}{|v-v'|w(v)(1+|v|+|v'|)^{1-\gamma}}\leq \frac{C\exp\{-\widetilde{c}|v-v'|^2\}}{|v-v'|(1+|v|+|v'|)^{1-\gamma}} \tag{2.9}
$$

with a smaller $\widetilde{c}>0$. Since $\nu(\mu)\sim\nu_M$, we bound the second line in (2.6) by

$$
\frac{1}{\varepsilon}\int_0^t\exp\{-\frac{\nu(t-s)}{\varepsilon}\}\int_{\mathbf{R}^3}|l_w(v,v')h^\varepsilon(s,x-v(t-s),v')|dv'ds,
$$

We now use (2.6) again to evaluate h^ε. By (2.7) and (2.8), we can bound the above by

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t\exp\{-\frac{\nu(t-s)}{\varepsilon}\}\sup_v\int_{\mathbf{R}^3}|l_w(v,v')|dv'\exp\{-\frac{\nu s}{\varepsilon}\}h^\varepsilon(0,x-v(t-s)-v's,v')ds\\
&+\frac{1}{\varepsilon^2}\int_0^t\exp\{-\frac{\nu(t-s)}{\varepsilon}\}\int_{\mathbf{R}^3}|l_w(v,v')|\\
&\times\int_0^s\exp\{-\frac{\nu(v')(s-s_1)}{\varepsilon}\}|\{K^m h^\varepsilon\}(s_1,x-v(t-s)-v'(s-s_1),v')|dv'ds_1ds\\
&+\frac{1}{\varepsilon^2}\int_0^t\exp\{-\frac{\nu(t-s)}{\varepsilon}\}\int_{\mathbf{R}^3\times\mathbf{R}^3}|l_w(v,v')l_w(v',v'')\\
&\times\int_0^s\exp\{-\frac{\nu(v')(s-s_1)}{\varepsilon}\}h^\varepsilon(s_1,x-v(t-s)-v'(s-s_1),v'')|dv'dv''ds_1ds\\
&+\frac{C}{\varepsilon}\int_0^t\exp\{-\frac{\nu(t-s)}{\varepsilon}\}ds\times\int_{\mathbf{R}^3}|l_w(v,v')|dv'\times\{\varepsilon^3\sup_{0\leq s\leq t}\|h^\varepsilon(s)\|_\infty^2\}\\
&+\frac{C}{\varepsilon}\int_0^t\exp\{-\frac{\nu(t-s)}{\varepsilon}\}ds\times\int_{\mathbf{R}^3}|l_w(v,v')|dv'\times\{\varepsilon\sup_{0\leq s\leq t}\|h^\varepsilon(s)\|_\infty\}\\
&+\frac{C}{\varepsilon}\int_0^t\exp\{-\frac{\nu(t-s)}{\varepsilon}\}ds\times\int_{\mathbf{R}^3}|l_w(v,v')|dv'\times\{\varepsilon^2\sup_{0\leq s\leq t}\|\tilde{A}\|_\infty\}.
\end{aligned} \tag{2.10}
$$

By (2.4), the second term is bounded as follows:

$$
\begin{aligned}
&\frac{Cm^{\gamma+3}}{\varepsilon^2}\sup_{0\le\tau\le t}\|h^\varepsilon(\tau)\|_\infty\int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\int_{\mathbf{R}^3}|l_w(v,v')|\\
&\quad\times\int_0^s\exp\{-\frac{\nu(v')(s-s_1)}{\varepsilon}\}\nu(v')dv'ds_1ds\\
&\le\frac{Cm^{\gamma+3}}{\varepsilon}\sup_{0\le\tau\le t}\|h^\varepsilon(\tau)\|_\infty\int_0^t\exp\{-\frac{\nu(t-s)}{\varepsilon}\}\int_{\mathbf{R}^3}|l_w(v,v')|dv'ds\\
&=Cm^{\gamma+3}\sup_{0\le\tau\le t}\|h^\varepsilon(\tau)\|_\infty,
\end{aligned}
$$

where we have used the fact $\int_{\mathbf{R}^3}|l_w(v,v')|dv'<\nu(v)$ from (2.9). Similar arguments for other terms except the third term yield the following bound:

$$
C\{\|h^\varepsilon(0)\|_\infty+\varepsilon^3\sup_{0\le s\le t}\|h^\varepsilon(s)\|_\infty^2+\varepsilon\sup_{0\le s\le t}\|h^\varepsilon(s)\|_\infty+C\varepsilon^3\}.
$$

We now concentrate on the third term in (2.10), which will be estimated as in the proof of Theorem 20 in [14].

CASE 1: For $|v|\ge N$. By (2.9),

$$
\int_{\mathbf{R}^3}|l_w(v,v')|dv'\le C\frac{\nu(v)}{N}\quad\text{and}\quad\int_{\mathbf{R}^3}|l_w(v',v'')|dv''\le C\nu(v')
$$

and thus we have the following bound

$$
\begin{aligned}
&\frac{C}{\varepsilon}\sup_{0\le s\le t}\|h^\varepsilon(s)\|_\infty\int_0^t\int_{\mathbf{R}^3}\exp\{-\frac{\nu(v)(t-s)}{\varepsilon}\}|l_w(v,v')|\\
&\qquad\times\int_0^s\exp\{-\frac{\nu(v')(s-s_1)}{\varepsilon}\}\frac{\nu(v')}{\varepsilon}ds_1dv'ds\\
&\le\frac{C}{N}\sup_{0\le s\le t}\|h^\varepsilon(s)\|_\infty.
\end{aligned}
$$

CASE 2: For $|v|\le N$, $|v'|\ge 2N$, or $|v'|\le 2N$, $|v''|\ge 3N$. Notice that we have either $|v'-v|\ge N$ or $|v'-v''|\ge N$, and either one of the following is valid correspondingly for some small $\eta>0$:

$$
\begin{aligned}
|l_w(v,v')|&\le e^{-\frac{\eta}{8}N^2}|l_w(v,v')e^{\frac{\eta}{8}|v-v'|^2}|,\\
|l_w(v',v'')|&\le e^{-\frac{\eta}{8}N^2}|l_w(v',v'')e^{\frac{\eta}{8}|v'-v''|^2}|.
\end{aligned}\tag{2.11}
$$

From (2.9), we obtain

$$
\int|l_w(v,v')e^{\frac{\eta}{8}|v-v'|^2}|dv'\le C\nu(v),\ \text{and}\ \int|l_w(v',v'')e^{\frac{\eta}{8}|v'-v''|^2}|dv''\le C\nu(v').
$$

We use (2.11) to combine the cases of $|v'-v|\geq N$ or $|v'-v''|\geq N$ as:

$$
\begin{aligned}
&\int_0^t\int_0^s\Big\{\int_{|v|\leq N,|v'|\geq 2N}+\int_{|v'|\leq 2N,|v''|\geq 3N}\Big\}\\
&\leq \frac{C_\eta}{\varepsilon^2}e^{-\frac{\eta}{8}N^2}\sup_{0\leq s\leq t}\|h^\varepsilon(s)\|_\infty\int_0^t\int_0^s\int|l_w(v,v')|\exp\{-\frac{\nu(v)(t-s)}{\varepsilon}\}\\
&\qquad\qquad\exp\{-\frac{\nu(v')(s-s_1)}{\varepsilon}\}\nu(v')dv'ds_1ds\\
&\leq C_\eta e^{-\frac{\eta}{8}N^2}\sup_{0\leq s\leq t}\{\|h^\varepsilon(s)\|_\infty\}.
\end{aligned}
\tag{2.12}
$$

CASE 3a: $|v|\leq N$, $|v'|\leq 2N, |v''|\leq 3N$. This is the last remaining case because if $|v'|>2N$, it is included in Case 2; while if $|v''|>3N$, either $|v'|\leq 2N$ or $|v'|\geq 2N$ are also included in Case 2. We further assume that $s-s_1\leq\varepsilon\kappa$, for $\kappa>0$ small. We bound the third term in (2.10) by

$$
\begin{aligned}
&\frac{1}{\varepsilon^2}\int_0^t\int_{s-\varepsilon\kappa}^s C\exp\{-\frac{\nu(v)(t-s)}{\varepsilon}\}\exp\{-\frac{\nu(v')(s-s_1)}{\varepsilon}\}\|h^\varepsilon(s_1)\|_\infty ds_1ds\\
&\leq C_N\sup_{0\leq s\leq t}\{\|h^\varepsilon(s)\|_\infty\}\times\frac{1}{\varepsilon}\int_0^t\exp\{-\frac{\nu(v)(t-s)}{\varepsilon}\}ds\times\int_{s-\varepsilon\kappa}^s\frac{1}{\varepsilon}ds_1\\
&\leq \kappa C_N\sup_{0\leq s\leq t}\{\|h^\varepsilon(s)\|_\infty\}.
\end{aligned}
\tag{2.13}
$$

CASE 3b: $|v|\leq N$, $|v'|\leq 2N, |v''|\leq 3N$, and $s-s_1\geq\varepsilon$. We now can bound the third term in (2.10) by

$$
\begin{aligned}
C\int_0^t\int_B\int_0^{s-\varepsilon\kappa}&e^{-\frac{\nu(v)(t-s)}{\varepsilon}}e^{-\frac{\nu(v')(s-s_1)}{\varepsilon}}|l_{M,w}(v,v')l_{M,w}(v',v'')\\
&h^\varepsilon(s_1,x_1-(s-s_1)v',v'')|ds_1dv'dv''ds
\end{aligned}
$$

where $B=\{|v'|\leq 2N,\ |v''|\leq 3N\}$. By (2.9), $l_w(v,v')$ has possible integrable singularity of $\frac{1}{|v-v'|}$, we can choose $l_N(v,v')$ smooth with compact support such that

$$
\sup_{|p|\leq 3N}\int_{|v'|\leq 3N}|l_N(p,v')-l_w(p,v')|dv'\leq\frac{1}{N}.
\tag{2.14}
$$

Splitting

$$
\begin{aligned}
l_w(v,v')l_w(v',v'')&=\{l_w(v,v')-l_N(v,v')\}l_w(v',v'')\\
&\quad+\{l_w(v',v'')-l_N(v',v'')\}l_N(v,v')+l_N(v,v')l_N(v',v''),
\end{aligned}
$$

we can use such an approximation (2.14) to bound the above s_1,s integration by

$$
\begin{aligned}
&\frac{C}{N}\sup_{0\leq s\leq t}\{\|h^\varepsilon(s)\|_\infty\}\times\{\sup_{|v'|\leq 2N}\int|l_w(v',v'')|dv''+\sup_{|v|\leq 2N}\int|l_N(v,v')|dv'\}\\
&\quad+C\int_0^t\int_B\int_0^{s-\varepsilon\kappa}e^{-\frac{\nu(v)(t-s)}{\varepsilon}}e^{-\frac{\nu(v')(s-s_1)}{\varepsilon}}|l_N(v,v')l_N(v',v'')\\
&\qquad\qquad h^\varepsilon(s_1,x_1-(s-s_1)v',v'')|ds_1dv'dv''ds.
\end{aligned}
\tag{2.15}
$$

Since $l_N(v,v')l_N(v',v'')$ is bounded, we first integrate over v' to get

$$\begin{aligned}
& C_N \int_{|v'|\leq 2N} |h^\varepsilon(s_1, x_1-(s-s_1)v', v'')|dv' \\
& \leq C_N \left\{ \int_{|v'|\leq 2N} \mathbf{1}_\Omega(x_1-(s-s_1)v')|h^\varepsilon(s_1, x_1-(s-s_1)v', v'')|^2 dv' \right\}^{1/2} \\
& \leq \frac{C_N}{\kappa^{3/2}\varepsilon^{3/2}} \left\{ \int_{|y-x_1|\leq (s-s_1)3N} |h^\varepsilon(s_1, y, v'')|^2 dy \right\}^{1/2} \\
& \leq \frac{C_N\{(s-s_1)^{3/2}+1\}}{\kappa^{3/2}\varepsilon^{3/2}} \left\{ \int_{\mathbf{\Omega}} |h^\varepsilon(s_1, y, v'')|^2 dy \right\}^{1/2}.
\end{aligned}$$

Here we have made a change of variable $y = x_1-(s-s_1)v'$, and for $s-s_1 \geq \kappa\varepsilon$, $|\frac{dy}{dv'}| \geq \frac{1}{\kappa^3\varepsilon^3}$. In the case of $\Omega = \mathbf{R}^3$, the factor $\{(s-s_1)^{3/2}+1\}$ is not needed. By (1.19) and (1.16), we then further control the last term in (2.15) by:

$$\begin{aligned}
& \frac{C_{N,\kappa}}{\varepsilon^{7/2}} \int_0^t \int_0^{s-\kappa\varepsilon} e^{-\frac{\nu(v)(t-s)}{\varepsilon}} e^{-\frac{\nu(v')(s-s_1)}{\varepsilon}} \{(s-s_1)^{3/2}+1\} \\
& \qquad \int_{|v''|\leq 3N} \left\{ \int_{\mathbf{\Omega}} |h^\varepsilon(s_1, y, v'')|^2 dy \right\}^{1/2} dv'' ds_1 ds \\
& \leq \frac{C_{N,\kappa}}{\varepsilon^{7/2}} \int_0^t \int_0^{s-\kappa\varepsilon} e^{-\frac{\nu(v)(t-s)}{\varepsilon}} e^{-\frac{\nu(v')(s-s_1)}{\varepsilon}} \{(s-s_1)^{3/2}+1\} \\
& \qquad \left\{ \int_{|v''|\leq 3N} \int_\Omega |f^\varepsilon(s_1, y, v'')|^2 dy dv'' \right\}^{1/2} ds_1 ds \\
& \leq \frac{C_{N,\kappa}}{\varepsilon^{3/2}} \sup_{0\leq s\leq t} \|f^\varepsilon(s)\|_2.
\end{aligned}$$

In summary, we have established, for any $\kappa > 0$ and large $N > 0$,

$$\begin{aligned}
\sup_{0\leq s\leq t} \{\varepsilon^{3/2}\|h^\varepsilon(s)\|_\infty\} &\leq \{Cm^{\gamma+3} + C_{N,m}\kappa + \frac{C_m}{N}\} \sup_{0\leq s\leq t} \{\varepsilon^{3/2}\|h^\varepsilon(s)\|_\infty\} + \varepsilon^{7/2}C \\
&\quad + C_{\varepsilon,N}\|\varepsilon^{3/2}h_0\|_\infty + \sqrt{\varepsilon}C \sup_{0\leq s\leq t} \{\varepsilon^{3/2}\|h^\varepsilon(s)\|_\infty\}^2 + C_{m,N,\kappa} \sup_{0\leq s\leq t} \|f^\varepsilon(s)\|_2.
\end{aligned}$$

For sufficiently small $\varepsilon > 0$, first choosing m small, then N sufficiently large, and finally letting κ small so that $\{Cm^{\gamma+3} + C_{N,m}\kappa + \frac{C_m}{N}\} < \frac{1}{2}$, we get

$$\sup_{0\leq s\leq \tau} \{\varepsilon^{3/2}\|h^\varepsilon(s)\|_\infty\} \leq C\{\|\varepsilon^{3/2}h_0\|_\infty + \sup_{0\leq s\leq \tau} \|f^\varepsilon(s)\|_2 + \varepsilon^{7/2}\}$$

and we conclude our proof. □

3 Acoustic Limit

3.1 Compressible Euler and Acoustic systems

We note that the acoustic system (1.8) is essentially linear wave equations. Thus the well-posedness follows from the linear theory of wave equations: for the given initial data $(\sigma^0,\, u^0,\, \theta^0) \in H^s$, there exist global-in-time classical solutions $(\sigma,\, u,\, \theta) \in C([0,\infty)\,;H^s)$ to the acoustic system (1.8). In particular, we have the following energy estimates: for each $s \geq 0$,

$$\|(\sigma,\, u\,, \sqrt{\tfrac{3}{2}}\theta)(t)\|^2_{H^s} = \|(\sigma^0,\, u^0,\, \sqrt{\tfrac{3}{2}}\theta^0)\|^2_{H^s}, \text{ for all } t \geq 0\,. \tag{3.1}$$

On the other hand, classical solutions to compressible Euler equations exist for only finite time [29]. Since the properties of solutions play an important role in our argument, we present the existence result of smooth solutions to compressible Euler system. Normalizing $R \equiv 1$ in the equation of state (1.5), we can write the invisid flow equations in variables ρ, $\mathfrak{u}$, T as follows:

$$\begin{aligned}
\partial_t\rho + (\mathfrak{u}\cdot\nabla)\rho + \rho\nabla\cdot\mathfrak{u} = 0\,,\\
\rho\partial_t\mathfrak{u} + \rho(\mathfrak{u}\cdot\nabla)\mathfrak{u} + \rho\nabla T + T\nabla\rho = 0\,,\\
\partial_t T + (\mathfrak{u}\cdot\nabla)T + \tfrac{2}{3}T\nabla\cdot\mathfrak{u} = 0\,.
\end{aligned} \tag{3.2}$$

It is a classical result from the theory of symmetric hyperbolic system that the lifespan of smooth solutions to 3D compressible Euler equations with smooth initial data, which are a small perturbation of amplitude δ from a constant state, is at least $O(\delta^{-1})$. We summarize the result in the following lemma:

Lemma 3.1. *Consider the compressible Euler system* (3.2) *with initial data:*

$$\rho^0 = 1 + \delta\sigma^0\,, \quad \mathfrak{u}^0 = \delta u^0\,, \quad T^0 = 1 + \delta\theta^0\,, \tag{3.3}$$

for any given $(\sigma^0, u^0, \theta^0) \in H^s$ with $s > \frac{5}{2}$. Choose $\delta_1 > 0$ so that for any $0 < \delta \leq \delta_1$, the positivity of ρ^0 and T^0 is guaranteed. Then, for each $0 < \delta \leq \delta_1$, there is a family of classical solutions $(\rho^\delta, \mathfrak{u}^\delta, T^\delta) \in C([0,\tau^\delta]\,;H^s) \cap C^1([0,\tau^\delta]\,;H^{s-1})$ of the Euler equations (3.2) *such that $\rho^\delta > 0$, $T^\delta > 0$, and the following estimates hold:*

$$\|(\rho^\delta, \mathfrak{u}^\delta, T^\delta) - (1,0,1)\|_{C([0,\tau^\delta];H^s)\cap C^1([0,\tau^\delta];H^{s-1})} \leq C_0\,. \tag{3.4}$$

Furthermore, the lifespans τ^δ have the following lower bound

$$\tau^\delta > \frac{C_1}{\delta}\,.$$

Here the constants $C_0\,, C_1$ are independent of δ, depend only on the H^s-norm of $(\sigma^0, u^0, \theta^0)$.

We omit the proof of Lemma 3.1 (see [9, 20, 25]).

Now, for any given $\tau > 0$ and given acoustic initial data $(\sigma^0, u^0, \theta^0) \in H^s$, we define

$$\delta_1 = \frac{C_1}{\tau}\,. \tag{3.5}$$

Thus the lifespan of the solutions $(\rho^\delta, \mathfrak{u}^\delta, T^\delta)$ of the compressible Euler equations constructed in Lemma 3.1 have a uniform lower bound

$$\tau^\delta > \frac{C_1}{\delta} > \frac{C_1}{\delta_1} = \tau\,.$$

From now on, we consider the solutions of the compressible Euler system on an arbitrary finite time interval $[0\,,\tau]$ and we fix $\delta_1 > 0$ as in (3.5).

Next, we derive a refined estimate of two solutions to compressible Euler and acoustic systems. In order to do so, we first introduce the following difference variables $(\sigma_d^\delta, u_d^\delta, \theta_d^\delta)$ that are given by the second order perturbation in δ of Euler solutions:

$$\delta^2\sigma_d^\delta \equiv \rho^\delta - 1 - \delta\sigma\,, \quad \delta^2 u_d^\delta \equiv \mathfrak{u}^\delta - \delta u\,, \quad \delta^2\theta_d^\delta \equiv T^\delta - 1 - \delta\theta\,. \tag{3.6}$$

Lemma 3.2. *Let $\tau > 0$. Let $(\sigma^0,\, u^0,\, \theta^0) \in H^s$ in (1.8) with corresponding acoustic solution $(\sigma,\, u,\, \theta)$. Let $(\rho^\delta, \mathfrak{u}^\delta, T^\delta)$ be the Euler solutions of (3.2) with the corresponding initial data (3.3) constructed in Lemma 3.1. Then for all $0 < \delta \le \delta_0$ and for $s \ge 3$, there exists a constant $C_2 > 0$ only depending on τ and H^{s+1}-norm of $(\sigma^0, u^0, \theta^0)$ such that*

$$\|(\sigma_d^\delta\,, u_d^\delta\,, \theta_d^\delta)\|_{H^s} \le C_2. \tag{3.7}$$

Lemma 3.2 verifies that the acoustic system is the linearization about the constant state of the compressible Euler system:

$$\sup_{0\le t\le\tau} \|(\rho^\delta - 1 - \delta\sigma,\ \mathfrak{u}^\delta - \delta u,\ T^\delta - 1 - \delta\theta)\|_{H^s} \le C_2\delta^2. \tag{3.8}$$

In addition, by the estimate (3.7) for $s \ge 3$ and Sobolev embedding theorem, we obtain the uniform point-wise estimates of the difference variables $(\sigma_d^\delta, u_d^\delta, \theta_d^\delta)$.

Proof. of Lemma 3.2: Rewrite (3.6) as

$$\rho^\delta = 1 + \delta\sigma + \delta^2\sigma_d^\delta\,, \quad \mathfrak{u}^\delta = \delta u + \delta^2 u_d^\delta\,, \quad T^\delta = 1 + \delta\theta + \delta^2\theta_d^\delta\,, \tag{3.9}$$

and plug into (3.2) to get:

$$\begin{aligned}
&\partial_t[\delta\sigma + \delta^2\sigma_d^\delta] + (\delta u + \delta^2 u_d^\delta)\cdot\nabla(\delta\sigma + \delta^2\sigma_d^\delta) + (1 + \delta\sigma + \delta^2\sigma_d^\delta)\nabla\cdot(\delta u + \delta^2 u_d^\delta) = 0\\
&(1 + \delta\sigma + \delta^2\sigma_d^\delta)\partial_t[\delta u + \delta^2 u_d^\delta] + (1 + \delta\sigma + \delta^2\sigma_d^\delta)[(\delta u + \delta^2 u_d^\delta)\cdot\nabla](\delta u + \delta^2 u_d^\delta)\\
&\qquad\qquad + (1 + \delta\sigma + \delta^2\sigma_d^\delta)\nabla(\delta\theta + \delta^2\theta_d^\delta) + (1 + \delta\theta + \delta^2\theta_d^\delta)\nabla(\delta\sigma + \delta^2\sigma_d^\delta) = 0\\
&\partial_t[\delta\theta + \delta^2\theta_d^\delta] + (\delta u + \delta^2 u_d^\delta)\cdot\nabla(\delta\theta + \delta^2\theta_d^\delta) + \tfrac{2}{3}(1 + \delta\theta + \delta^2\theta_d^\delta)\nabla\cdot(\delta u + \delta^2 u_d^\delta) = 0
\end{aligned}$$

Coefficients of δ in each equation form the acoustic system (1.8) in (σ, u, θ), which is indeed the acoustic solution by the assumption. Hence, the remaining terms are at least of order $O(\delta^2)$. For instance, the continuity equation reduces to

$$\delta^2[\partial_t\sigma_d^\delta + (u + \delta u_d^\delta)\cdot\nabla\sigma + \underbrace{(\delta u + \delta^2 u_d^\delta)}_{(a)}\cdot\nabla\sigma_d^\delta + (\sigma + \delta\sigma_d^\delta)\nabla\cdot u + \underbrace{(1 + \delta\sigma + \delta^2\sigma_d^\delta)}_{(b)}\nabla\cdot u_d^\delta] = 0\,.$$

Use (3.9) to replace (a) and (b) by $\mathfrak{u}^\delta$ and T^δ respectively. Then the equation can be written as

$$\partial_t\sigma_d^\delta + (\mathfrak{u}^\delta\cdot\nabla)\sigma_d^\delta + \rho^\delta\nabla\cdot u_d^\delta + \delta[\nabla\sigma\cdot u_d^\delta + (\nabla\cdot u)\sigma_d^\delta] + \nabla\cdot(\sigma u) = 0\,.$$

Similarly, one can deduce that $(\sigma_d^\delta, u_d^\delta, \theta_d^\delta)$ satisfies the following linear system of equations:

$$\begin{aligned}
&\partial_t\sigma_d^\delta + (\mathfrak{u}^\delta\cdot\nabla)\sigma_d^\delta + \rho^\delta\nabla\cdot u_d^\delta + \delta[\nabla\sigma\cdot u_d^\delta + (\nabla\cdot u)\sigma_d^\delta] = -\nabla\cdot(\sigma u)\\
&\rho^\delta\partial_t u_d^\delta + \rho^\delta(\mathfrak{u}^\delta\cdot\nabla)u_d^\delta + \rho^\delta\nabla\theta_d^\delta + T^\delta\nabla\sigma_d^\delta + \delta[(\partial_t u)\sigma_d^\delta + \rho^\delta(u_d^\delta\cdot\nabla)u + \theta_d^\delta\nabla\sigma + \sigma_d^\delta\nabla\theta]\\
&\quad = -\sigma\partial_t u - \rho^\delta(u\cdot\nabla)u - \nabla(\sigma\theta)\\
&\partial_t\theta_d^\delta + (\mathfrak{u}^\delta\cdot\nabla)\theta_d^\delta + \tfrac{2}{3}T^\delta\nabla\cdot u_d^\delta + \delta[\nabla\theta\cdot u_d^\delta + \tfrac{2}{3}(\nabla\cdot u)\theta_d^\delta] = -u\cdot\nabla\theta - \tfrac{2}{3}\theta\nabla\cdot u
\end{aligned} \tag{3.10}$$

The point here is that the above system is linear in $\sigma_d^\delta, u_d^\delta, \theta_d^\delta$, although the coefficients may depend on ρ^δ, $\mathfrak{u}^\delta$, T^δ of compressible Euler system and σ, u, θ of acoustic system. However, we already know that these coefficients are smooth at least up to time τ and they have Sobolev energy bounds (3.1) and (3.4). Note that the system (3.10) can be written as a symmetric system with the corresponding symmetrizer A_0:

$$A_0\partial_t U_d + \sum_{i=1}^{3} A_i\partial_i U_d + BU_d = F \tag{3.11}$$

where U_d, A_0, and A_i are given as follows:

$$U_d \equiv \begin{pmatrix} \sigma_d^\delta \\ (u_d^\delta)^t \\ \theta_d^\delta \end{pmatrix}, \; A_0 \equiv \begin{pmatrix} \frac{T^\delta}{\rho^\delta} & 0 & 0 \\ 0 & \rho^\delta\mathbb{I} & 0 \\ 0 & 0 & \frac{3\rho^\delta}{2T^\delta} \end{pmatrix}, \; A_i \equiv \begin{pmatrix} \frac{T^\delta}{\rho^\delta}(\mathfrak{u}^\delta)^i & T^\delta e_i & 0 \\ T^\delta(e_i)^t & \rho^\delta(\mathfrak{u}^\delta)^i\mathbb{I} & \rho^\delta(e_i)^t \\ 0 & \rho^\delta e_i & \frac{3\rho^\delta}{2T^\delta}(\mathfrak{u}^\delta)^i \end{pmatrix}.$$

$(\cdot)^t$ denotes the transpose of row vectors, e_i's for $i = 1, 2, 3$ are the standard unit (row) base vectors in $\mathbf{R}^3$, and $\mathbb{I}$ is the 3×3 identity matrix. B and F, which consist of ρ^δ, $\mathfrak{u}^\delta$, T^δ, σ, u, θ and first derivatives of σ, u, θ, can be easily written down. Note that since ρ^δ and T^δ have positive lower and upper bounds for $t \le \tau$, (3.11) is strictly hyperbolic and thus we can apply the standard energy method of the linear symmetric hyperbolic system to (3.11) to obtain the following energy inequality:

$$\frac{d}{dt}\|U_d\|_{H^s}^2 \le C_3\|U_d\|_{H^s}^2 + C_4\|U_d\|_{H^s}. \tag{3.12}$$

Here C_3, C_4 are constants depending on $\|(\rho^\delta, \mathfrak{u}^\delta, T^\delta)\|_{H^{s+1}}$ and $\|(\sigma, u, \theta)\|_{H^{s+1}}$. The second term in the right hand side comes from the forcing term F. By Gronwall inequality, we conclude that $\|(\sigma_d^\delta, u_d^\delta, \theta_d^\delta)\|_{H^s}$ is bounded by a constant depending on τ and H^{s+1}-norm of initial data $(\sigma^0, u^0, \theta^0)$ and this completes the proof of Lemma 3.2. □

3.2 Local Maxwellians μ^δ and Proof of Theorem 1.1

Now, from the refined estimate (3.8), we can choose δ_2 sufficiently small so that for each $0 < \delta \le \delta_2$, T^δ satisfies the following moderate temperature variation condition

$$T_M^\delta < T^\delta(t,x) < 2T_M^\delta \tag{3.13}$$

for some constant $T_M^\delta > 0$. We define

$$\delta_0 \equiv \min\{\delta_1, \delta_2\}. \tag{3.14}$$

We denote the local Mawellian, induced by compressible Euler solutions ρ^δ, $\mathfrak{u}^\delta$, T^δ with the initial data (3.3) as obtained in Lemma 3.1, by

$$\mu^\delta \equiv \mu^\delta(t,x,v) = \frac{\rho^\delta(t,x)}{[2\pi T^\delta(t,x)]^{3/2}} \exp\left\{ -\frac{[v-\mathfrak{u}^\delta(t,x)]^2}{2T^\delta(t,x)} \right\}. \tag{3.15}$$

For each $\delta < \delta_0$, we take the Hilbert expansion of the Boltzmann equation (1.1) around the local Maxwellian μ^δ of the form

$$F^\varepsilon = \sum_{n=0}^{6} \varepsilon^n F_n + \varepsilon^3 F_R^\varepsilon,$$

where $F_0, ..., F_6$ are the first 6 terms of the Hilbert expansion. Here we have set $F_0 = \mu^\delta$. Since $(\rho^\delta,\, \mathfrak{u}^\delta,\, T^\delta)$ is a smooth solution to the compressible Euler system satisfying the condition (3.13), from Theorem 1.2 on the compressible Euler limit, it follows that for sufficiently small $\varepsilon \le \varepsilon_0$,

$$\sup_{0\le t\le\tau} \|F^\varepsilon(t) - \mu^\delta(t)\|_\infty + \sup_{0\le t\le\tau} \|F^\varepsilon(t) - \mu^\delta(t)\|_2 \le C_\tau \varepsilon \tag{3.16}$$

where a constant C_τ depends on τ, μ^δ, $F_1, .., F_6$.

We now show that μ^δ stays close to $\mu^0 + \delta G$ where G is the acoustic perturbation defined in (1.11).

Lemma 3.3. *Consider smooth solutions* $(\rho^\delta,\, \mathfrak{u}^\delta,\, T^\delta)$ *and* $(\sigma\,, u\,, \theta)$ *so that* $(\sigma_d^\delta\,, u_d^\delta\,, \theta_d^\delta)$ *in* (3.9) *is smooth, for instance choose* $s \ge 3$ *in Lemma 3.2. Let* M^δ *be given as in* (3.15) *and* G *as in* (1.11). *Then there exists a small enough* $\delta_0 > 0$ *so that for each* $0 < \delta \le \delta_0$, *there exists a constant* C_5 *depending on* τ *and the initial data* $(\sigma^0,\, u^0,\, \theta^0)$ *such that*

$$\sup_{0\le t\le\tau} \|\mu^\delta(t) - \mu^0 - \delta G(t)\|_\infty + \sup_{0\le t\le\tau} \|\mu^\delta(t) - \mu^0 - \delta G(t)\|_2 \le C_5\, \delta^2. \tag{3.17}$$

Proof. We consider μ^δ as a function of δ and expand it around $\delta = 0$. Instead of directly expanding μ^δ, we first introduce auxiliary local Maxwellians

$$\mu(z) \equiv \mu^{\delta,z} = \frac{\rho^{\delta,z}(t,x)}{[2\pi T^{\delta,z}(t,x)]^{3/2}} \exp\left\{ -\frac{[v-\mathfrak{u}^{\delta,z}(t,x)]^2}{2T^{\delta,z}(t,x)} \right\}$$

induced by the following $\rho^{\delta,z}$, $\mathfrak{u}^{\delta,z}$, $T^{\delta,z}$:

$$\rho^{\delta,z} \equiv 1 + z\sigma + z^2\sigma_d^\delta\,, \quad \mathfrak{u}^{\delta,z} \equiv zu + z^2 u_d^\delta\,, \quad T^{\delta,z} \equiv 1 + z\theta + z^2\theta_d^\delta\,.$$

Compare with ρ^δ, $\mathfrak{u}^\delta$, T^δ in (3.9). Fix $\delta > 0$. Note that $\mu(z)$ is a smooth function of z, and moreover, $\mu(\delta) = \mu^\delta$, since $\rho^{\delta,\delta} = \rho^\delta$, $\mathfrak{u}^{\delta,\delta} = \mathfrak{u}^\delta$, $T^{\delta,\delta} = T^\delta$. Now we expand $\mu(z)$ as a function of z. By Taylor's formula, $\mu(z)$ can be written as

$$\mu(z) = \mu(0) + \mu'(0)z + \frac{\mu''(z_*)}{2} z^2 \tag{3.18}$$

for some $0 \le z_* \le z$ which may depend on (t,x,v) and δ. Note that $\mu(0) = \mu^0$. Denote $\frac{\partial}{\partial z}$

by $'$. $\mu'(z)$ is given by

$$\mu'(z) = \left\{ \frac{(\rho^{\delta,z})'}{\rho^{\delta,z}} - \frac{3(T^{\delta,z})'}{2T^{\delta,z}} + (v - \mathfrak{u}^{\delta,z}) \cdot \frac{(\mathfrak{u}^{\delta,z})'}{T^{\delta,z}} + \frac{|v-\mathfrak{u}^{\delta,z}|^2 (T^{\delta,z})'}{2(T^{\delta,z})^2} \right\} \mu^{\delta,z}$$
$$\equiv D^{\delta,z}\mu^{\delta,z},$$

where

$$(\rho^{\delta,z})' = \sigma + 2z\sigma_d^\delta\,, \quad (\mathfrak{u}^{\delta,z})' = u + 2zu_d^\delta\,, \quad (T^{\delta,z})' = \theta + 2z\theta_d^\delta\,.$$

Since $[(\rho^{\delta,z})',\, (\mathfrak{u}^{\delta,z})',\, (T^{\delta,z})'](0) = [\sigma\,, u\,, \theta]$, we obtain

$$\mu'(0) = \{\sigma + v\cdot u + (\frac{|v|^2-3}{2})\theta\}\mu^0 = G(t,x,v).$$

Take one more derivative to get

$$\mu''(z) = (D^{\delta,z})'\mu^{\delta,z} + (D^{\delta,z})^2\mu^{\delta,z}$$

where

$$(D^{\delta,z})' = \frac{(\rho^{\delta,z})''}{\rho^{\delta,z}} - \frac{((\rho^{\delta,z})')^2}{(\rho^{\delta,z})^2} - \frac{3(T^{\delta,z})''}{2T^{\delta,z}} + \frac{3((T^{\delta,z})')^2}{2(T^{\delta,z})^2} - \frac{|(\mathfrak{u}^{\delta,z})'|^2}{T^{\delta,z}}$$
$$+ (v-\mathfrak{u}^{\delta,z}) \cdot \{\frac{(\mathfrak{u}^{\delta,z})''}{T^{\delta,z}} - 2\frac{(T^{\delta,z})'(\mathfrak{u}^{\delta,z})'}{(T^{\delta,z})^2}\} + |v-\mathfrak{u}^{\delta,z}|^2\{\frac{(T^{\delta,z})''}{2(T^{\delta,z})^2} - \frac{((T^{\delta,z})')^2}{(T^{\delta,z})^3}\}\,.$$

And $(\rho^{\delta,z})''$, $(\mathfrak{u}^{\delta,z})''$, $(T^{\delta,z})''$ are given by

$$(\rho^{\delta,z})'' = 2\sigma_d^\delta\,, \quad (\mathfrak{u}^{\delta,z})'' = 2u_d^\delta\,, \quad (T^{\delta,z})'' = 2\theta_d^\delta\,.$$

Now take $z = \delta$ in (3.18) to obtain

$$\mu^\delta = \mu^0 + g\delta + \frac{\mu''(\delta_*)}{2}\delta^2, \text{ for some } 0 \le \delta_* \le \delta\,.$$

In order to prove (3.17), it now suffices to show that $\|\mu''(\delta_*)\|_\infty + \|\mu''(\delta_*)\|_2$ is uniformly bounded. This follows the uniform estimates of $\sigma_d^\delta\,, u_d^\delta\,, \theta_d^\delta$ in Lemma 3.2: For each $0 \le z = \delta_* \le \delta \le \delta_0$ and for $t \le \tau$, we have the uniform point-wise estimates of $\rho^{\delta,z}, \mathfrak{u}^{\delta,z}, T^{\delta,z}$, $(\rho^{\delta,z})', (\mathfrak{u}^{\delta,z})', (T^{\delta,z})'$, $(\rho^{\delta,z})'', (\mathfrak{u}^{\delta,z})'', (T^{\delta,z})''$ and moreover, $\rho^{\delta,z}$ and $T^{\delta,z}$ have uniform lower bounds for sufficiently small $\delta \le \delta_0$. This completes the proof of the lemma. □

Now (1.14) is an easy consequence of Lemma 3.3 within the compressible Euler limit regime.

Proof. of Theorem 1.1: From (1.9) and (1.11), we first get

$$\sup_{0\le t\le\tau} \|G^\varepsilon(t) - G(t)\|_\infty + \sup_{0\le t\le\tau} \|G^\varepsilon(t) - G(t)\|_2$$
$$= \sup_{0\le t\le\tau} \|\frac{F^\varepsilon(t) - \mu^0}{\delta} - G(t)\|_\infty + \sup_{0\le t\le\tau} \|\frac{F^\varepsilon(t)-\mu^0}{\delta} - G(t)\|_2\,.$$

Rewrite $\frac{F^\varepsilon(t)-\mu^0}{\delta} - G(t)$ as follows:

$$\frac{F^\varepsilon(t)-\mu^0}{\delta} - G(t) = \frac{F^\varepsilon(t)-\mu^\delta(t)}{\delta} + \frac{\mu^\delta(t)-\mu^0-\delta G(t)}{\delta}\,.$$

Therefore, we obtain

$$\begin{aligned}
&\sup_{0\le t\le\tau}\|G^\varepsilon(t)-G(t)\|_\infty + \sup_{0\le t\le\tau}\|G^\varepsilon(t)-G(t)\|_2 \\
&\le \sup_{0\le t\le\tau}\|\frac{F^\varepsilon(t)-\mu^\delta(t)}{\delta}\|_\infty + \sup_{0\le t\le\tau}\|\frac{F^\varepsilon(t)-\mu^\delta(t)}{\delta}\|_2 \\
&\quad + \sup_{0\le t\le\tau}\|\frac{\mu^\delta(t)-\mu^0-\delta G(t)}{\delta}\|_\infty + \sup_{0\le t\le\tau}\|\frac{\mu^\delta(t)-\mu^0-\delta G(t)}{\delta}\|_2\,.
\end{aligned}$$

By (3.16) and (3.17), the conclusion follows. □

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