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FROM PARTICLE TO KINETIC AND HYDRODYNAMIC DESCRIPTIONS OF FLOCKING

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Abstract. We discuss the Cucker-Smale's (C-S) particle model for flocking, deriving precise conditions for flocking to occur when pairwise interactions are sufficiently strong long range. We then derive a Vlasov-type kinetic model for the C-S particle model and prove it exhibits time-asymptotic flocking behavior for arbitrary compactly supported initial data. Finally, we introduce a hydrodynamic description of flocking based on the C-S Vlasov-type kinetic model and prove flocking behavior *without* closure of higher moments.

1. **Introduction.** Collective self-driven motion of self-propelled particles such as flocking of birds and mobile agents, schooling of fishes, swarming of bacteria, appears in many context, e.g., biological organism [3, 8, 9, 10, 18, 19, 22, 25, 26], mobile network [2, 5, 11, 12] appears in many contexts of biological system, mobile and human network [6, 7]. The flocking dynamics of self-propelled particles is important to understand the nature of the aforementioned self-propelled particles. The terminology *"flocking"* represents the phenomenon in which self-propelled individuals using only limited environmental information and simple rules, organize into an ordered motion (see [24], and it was a subject of biologists [3, 19]). The study of flocking mechanism based on mathematical models was first started from the work of Viscek et al [26], and was further motivated by the hydrodynamic approach [24].

Our starting point is a *particle description*, proposed recently by Cucker-Smale [6, 7], as a new simple dynamical system to explain the emergency of flocking mechanism with birds, with language evolution in primitive societies etc. The Cucker-Smale's system is different from previous flocking models, e.g., [26], in the sense

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that the collisionless momentum transfer between particles, $\{(x_i(t), v_i(t))\}_{i=1}^N$, is done through a long-range bi-particle interaction potential, $r(x,y) = r(|x-y|)$ depending on the distance $|x-y|$,

$$\frac{d}{dt}v_i(t) = \frac{\lambda}{N}\sum_{1\le j\le N} r(x_i(t), x_j(t))\big(v_j(t) - v_i(t)\big).$$

The Cucker-Smale's flocking system (in short C-S system) is reviewed in Section 2. Here we revisit the formation of flocking in C-S dynamics in terms of the *fluctuations* relative to the center of mass $x_c(t) := 1/N\sum x_i(t)$. The dynamics of fluctuations makes transparent the flocking dynamics. Our main result, summarized in theorem 2.4, improves [6] for slowly decaying interaction potential, $r(|x-y|) \sim |x-y|^{-2\beta}, 2\beta \le 1$. It is shown that flocking emerges in the sense that the following two main features occur: (i) the diameter $\max|x_i(t) - x_j(t)|$ remains uniformly bounded thus defining the "flock", and (ii) the "flock" is traveling with a bulk mean velocity which is asymptotically particle-independent, $v_i(t) \approx v_c := 1/N\sum v_i(0)$.

When the number of particles is sufficiently large, it is not economical to keep track of the motion of each particle through the Cucker-Smale's system. Instead, one is forced to study the mean field limit of C-S system and we introduce a *kinetic description* for flocking, in analogy with the Vlasov equation in plasma and astrophysics. In Section 3, we present a Vlasov type mean field model, which is derived from the C-S system using the BBGKY hierarchy in statistical mechanics. The formal derivation, carried in Section 3, follows by taking the limit of an N-particle interacting system consisting of self-propelled particles governed by C-S flocking dynamics. To this end, let $f = f(x,v,t)$ denote the one-particle distribution function of such particles positioned at $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$ with a velocity $v \in \mathbb{R}^d$: the dynamics of the distribution function f is determined by

$$\partial_t f + v\cdot\nabla_x f + \lambda\nabla_v \cdot Q(f,f) = 0,$$

where λ is a positive constant, and $Q(f,f)$ is the interaction term

$$Q(f,f)(x,v,t) := \int_{\mathbb{R}^{2d}} r(x,y)(v_* - v) f(x,v,t) f(y,v_*,t)dv_*dy,$$

dictated by a prescribed interparticle interaction kernel, $r = r(x,y)$. We refer to Degond and Motsch [8, 9, 10] for recent kinetic description of Vicsek type model of flocking. The dynamics of particle trajectories of the proposed kinetic description of flocking is analyzed in Section 3.2; in Section 3.3 we prove the global existence of smooth solutions to the kinetic model with arbitrary smooth compactly supported initial data. In Section 4, we show that the kinetic model reveals the time-asymptotic flocking behavior when the bounded interparticle interaction rate has a sufficiently strong long range. Our results are summarized in the main theorem 4.3, proving the decay of energy fluctuations, $\Lambda[f](t)$, around the mean bulk velocity, u_c,

$$\Lambda[f](t) := \int_{\mathbb{R}^{2d}} |v-u_c|^2 f(x,v,t)dvdx, \qquad u_c(t) = \frac{\int_{\mathbb{R}^{2d}} vf(x,v,t)dvdx}{\int_{\mathbb{R}^{2d}} f(x,v,t)dvdx} \equiv u_c(0).$$

Flocking is proved for the restricted range, $2\beta < 1/2$, realized by the asymptotic decay estimate, $\Lambda[f](t) \to 0$ as $t \to \infty$.

In Section 5 we turn our attention to the *hydrodynamic description* of flocking, furnished by moments of the kinetic distribution function. We study the dynamics of the resulting system of balanced laws related to the moments of Vlasov model.

Despite the lack of closure, we present a fundamental estimate which enables to conclude the flocking mechanism at the macroscopic hydrodynamic scales. Theorem 5.2 states that the energy-related functional, $\Gamma(t)$

$$\Gamma(t) := \int_{\mathbb{R}^{2d}} \Big(\frac{1}{2}|u(x)-u(y)|^2 + e(x)+e(y)\Big)\rho(x)\rho(y)dydx.$$

decays provided the interparticle interaction, $\varphi(s) = \inf r(x(s), y(s))$ decays *slowly* enough so that its primitive, $\Phi(t)$, diverges. This in turn in related to the *increase* of entropy

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}} f\log(f)dxdv \geq 0,$$

as particles with increasingly highly correlated velocities flock towards particle-independent bulk velocity.

2. A particle description of flocking.

2.1. The Cucker-Smale model. In this section, we briefly review the Cucker-Smale's flocking system in [6, 7, 23], which manifests the time-asymptotic flocking behavior of many particle systems. We reinterpret the C-S system in terms of *fluctuations* relative to the center of mass coordinates, which enables us to simplify and sharpen the derivation of sufficient conditions for flocking to occur.

Consider an N-particle interacting system consisting of identical particles with mass m to be assumed to be unity. Let $[x_i(t), v_i(t)] \in \mathbb{R}^{2d}$ be the phase space position of an i-particle. The Cucker-Smale dynamical system [6, 7] takes the form

$$\frac{d}{dt}x_i(t) = v_i(t), \qquad \frac{d}{dt}v_i(t) = \frac{\lambda}{N}\sum_{1\leq j\leq N} r(x_i(t), x_j(t))\big(v_j(t)-v_i(t)\big). \tag{2.1}$$

Here λ is a positive constant, and $r(x,y)$ is a symmetric, bi-particle *interaction kernel*,

$$r(x,y) = r(y,x) \leq A. \tag{2.2}$$

To discuss the time asymptotic flocking behavior, we will restrict our attention to interparticle interactions which are non-increasing functions of the distance[1],

$$r(x,y) = r(|x-y|), \qquad r(\cdot) \ \text{ is non} - \text{increasing}. \tag{2.3a}$$

A prototype example is the interaction kernel with a polynomial decay of order 2β, [6],

$$r(|x-y|) \geq \frac{A}{(1+|x-y|^2)^\beta}, \qquad \beta \geq 0. \tag{2.3b}$$

We note in passing that only a lower-bound of the interaction kernel matters.

For notational simplicity we often omit t-dependence from the particle identification, abbreviating $x_i \equiv x_i(t)$ and $v_i \equiv v_i(t)$.

Let $m_j(t),\ j = 0,1,2$ denote the moments.

$$m_0 := \sum_{i=1}^N 1 = N, \qquad m_1(t) := \sum_{i=1}^N v_i(t), \qquad m_2(t) := \sum_{i=1}^N |v_i(t)|^2.$$

Regarding the dynamics of these moments, we have the following estimate.

[1] To emphasize this point, we therefore continue to refer to general symmetric kernels, $r(x,y) = r(y,x)$, whenever translation invariance is not necessary.

Proposition 2.1. *Let $(x_i(t), v_i(t))$ be the solution to the C-S system (2.1),(2.2). Then the following estimates hold.*

$$\frac{d}{dt}m_1(t) = 0. \tag{2.4a}$$

$$\frac{d}{dt}m_2(t) = -\frac{\lambda}{N}\sum_{1\le i,j\le N} r(x_i,x_j)|v_j - v_i|^2. \tag{2.4b}$$

$$m_2(t) \ge m_2(0)e^{-2\lambda At} + \frac{|m_1(0)|^2}{m_0}\Big(1 - e^{-2\lambda At}\Big). \tag{2.4c}$$

Remark 2.1. Proposition 2.1 tells us that although the kinetic energy $m_2(t)$ is monotonically decreasing, (2.4b), it has the following nonzero lower bound if initial momentum $m_1(0) \neq 0$,

$$m_2(t) \ge \frac{|m_1(0)|^2}{m_0}. \tag{2.5}$$

Flocking occurs when equality takes place in the Cauchy-Schwarz inequality (2.5).

Proof. Conservation of momentum in (2.4a) follows from the symmetry $r(x_i, x_j) = r(x_j, x_i)$, for

$$\frac{d}{dt}\Big(\sum_{i=1}^N v_i\Big) = \frac{\lambda}{N}\sum_{1\le i,j\le N} r(x_i,x_j)(v_j - v_i) = 0.$$

Moreover, symmetry also implies

$$\sum_{1\le i,j\le N} r(x_i,x_j)v_i\cdot(v_i - v_j) = -\sum_{1\le i,j\le N} r(x_i,x_j)v_j\cdot(v_i - v_j),$$

and hence the energy dissipation (2.4b) follows

$$\begin{aligned}
&\frac{d}{dt}\Big(\sum_{i=1}^N |v_i|^2\Big)\\
= \quad & -\frac{2\lambda}{N}\sum_{1\le i,j\le N} r(x_i,x_j)v_i\cdot(v_i - v_j)\\
= \quad & \frac{2\lambda}{N}\sum_{1\le i,j\le N} r(x_i,x_j)v_j\cdot(v_i - v_j) = -\frac{\lambda}{N}\sum_{1\le i,j\le N} r(x_i,x_j)|v_i - v_j|^2.
\end{aligned}$$

Finally, to prove (2.4c), we use the energy dissipation in (2.4b), the fact that $r(x_i, x_j) \le A$ and the conservation of momentum in (2.4a) to find

$$\begin{aligned}
\frac{d}{dt}m_2(t) &= -\frac{\lambda}{N}\sum_{1\le i,j\le N} r(x_i,x_j)|v_i - v_j|^2 \ge -\frac{\lambda}{N}A\sum_{1\le i,j\le N}|v_i - v_j|^2\\
&= -2\lambda A\Big(m_2(t) - \frac{|m_1(t)|^2}{N}\Big) = -2\lambda A\Big(m_2(t) - \frac{|m_1(0)|^2}{m_0}\Big).
\end{aligned}$$

Gronwall's lemma yields (2.4c). ∎

2.2. **Asymptotic behavior of fluctuations — flocking.** We now turn to study the time-asymptotic behavior of solutions to C-S system (2.1),(2.3a). To this end, we introduce a center of mass system $(x_c(t), v_c(t))$,

$$x_c(t) := \frac{1}{N}\sum_{i=1}^{N} x_i(t), \qquad v_c(t) := \frac{1}{N}\sum_{i=1}^{N} v_i(t).$$

Then, thanks to the conservation of momentum, the velocity v_c is constant in t, and the trajectory of center of mass x_c is a straight line:

$$v_c(t) = v_c(0), \qquad x_c(t) = x_c(0) + tv_c(0).$$

Observe that the fluctuations around the center of mass,

$$x_i(t) \mapsto x_i(t) - x_c(t), \qquad v_i(t) \mapsto v_i(t) - v_c(t),$$

satisfy the same C-S system (2.1),(2.3a): it is here that we take into account the fact that the interparticle kernel depends on the distance, $r(x, y) = r(|x - y|)$. We shall show that under appropriate conditions, flocking occurs in the sense that these *fluctuations* decay in time. Thus, the time-asymptotic dynamics of C-S solutions emerges as a linear movement with a fixed velocity dictated by the coordinates of center of mass.

To proceed, we introduce the two auxiliary functions which measure the fluctuations of the solution around their center of mass,

$$X(t) := \sum_{1\le i\le N} |x_i(t) - x_c(t)|^2, \qquad V(t) := \sum_{1\le i\le N} |v_i(t) - v_c(0)|^2,$$

subject to initial conditions $(X_0, V_0) = (X(0), V(0))$. The flocking behavior will depend in an essential way on the behavior of the minimal value of the interparticle interaction at time t,

$$\varphi(t) := \min_{1\le i,j\le N} r(x_i(t), x_j(t)). \tag{2.6}$$

We begin with the fluctuations of velocities.

Lemma 2.2. *[Fluctuations of velocities]. Let $(x_i(t), v_i(t))$ be the solution of the system* (2.1),(2.3a)*. Then we have*

$$V(t) \le V_0 e^{-2\lambda\Phi(t)}, \quad \Phi(t) := \int_0^t \varphi(\tau)d\tau.$$

Proof. We invoke (2.4b) with $v_i(t) - v_c(t)$ replacing $v_i(t)$ to find that

$$\frac{d}{dt}\sum_{i=1}^{N} |v_i - v_c|^2 = -\frac{\lambda}{N}\sum_{1\le i,j\le N} r(x_i, x_j)|v_j - v_i|^2. \tag{2.7}$$

Since $\sum_i (v_i - v_c) = 0$, we have $\sum_{1\le i,j\le N} |v_i - v_j|^2 = 2NV(t)$, and the result follows from Gronwall's integration of

$$\frac{d}{dt}V(t) = -\frac{\lambda}{N}\sum_{1\le i,j\le N} r(x_i, x_j)|v_j - v_i|^2 \le -2\lambda\varphi(t)V(t).$$

$\square$

Remark 2.2. Lemma 2.2 implies the sufficient condition for flocking is that the interparticle interaction potential decays *sufficiently slow*, so that its primitive, $\Phi(t)$, diverges, i.e.,

$$\text{if } \lim_{t\to\infty} \Phi(t) \equiv \int_0^t \varphi(\tau)d\tau = \infty \quad \text{then } \lim_{t\to\infty} |v_i(t)-v_c| = 0, \qquad i = 1,\cdots,N. \tag{2.8}$$

The answer whether $\varphi(t)$ decays sufficiently slow to enforce flocking depends on the variance of positions $x_i(t)$.

Lemma 2.3. *[Fluctuations of positions]. Let $(x_i(t), v_i(t))$ be the solution of the system (2.1),(2.3a). Then we have*

$$X(t) \le 2X_0 + V_0\frac{t^2}{2}, \qquad t \ge 0.$$

Proof. We use Cauchy-Schwartz's inequality to see

$$\begin{aligned} \frac{d}{dt}\sum_{i=1}^N |x_i - x_c|^2 &= \sum_{i=1}^N (x_i - x_c)\cdot\left(\frac{dx_i}{dt} - \frac{dx_c}{dt}\right) \\ &= \sum_{i=1}^N (x_i - x_c)\cdot(v_i - v_c) \le \sqrt{\sum_{i=1}^N |x_i - x_c|^2}\sqrt{\sum_{i=1}^N |v_i - v_c|^2}. \end{aligned}$$

Using Lemma 2.2 we obtain, $\dfrac{d}{dt}X(t) \le \sqrt{V(t)}\sqrt{X(t)} \le \sqrt{V_0}e^{-\lambda\Phi(t)}\sqrt{X(t)}$, and the solution of this differential inequality yields

$$X(t) \le 2X_0 + \frac{V_0}{2}\Big[\int_0^t e^{-\lambda\Phi(\tau)}d\tau\Big]^2 \le 2X_0 + V_0\frac{t^2}{2}. \tag{2.9}$$

$\square$

As a corollary of Lemma 2.3 we now obtain the desired lower bound for $\varphi(t)$.

Corollary 1. *Let $(x_i(t), v_i(t))$ be the solutions to (2.1),(2.3a). Then $\varphi(t)$ satisfies*

$$\varphi(t) \ge r\big(\sqrt{2X(t)}\big) \ge r\big(\sqrt{4X_0 + V_0t^2}\big).$$

Proof. We use Lemma 2.3 to see that for each $i, j \in \{1,\cdots,N\}$,

$$|x_i - x_j|^2 \le 2(|x_i - x_c|^2 + |x_j - x_c|^2) \le 2X(t) \le 4X_0 + V_0t^2.$$

Since $r(\cdot)$ is non-increasing, we have $\varphi(t) = \min\limits_{1\le i,j\le N} r(x_i(t), x_j(t)) \ge r\big(\sqrt{4X_0 + V_0t^2}\big)$.

$\square$

The asymptotic flocking now depends on the specific decay of the interparticle interaction $r(\cdot)$. As an example, consider the C-S system with the 2β interaction (2.1),(2.3b), where

$$\varphi(t) \ge \frac{A}{(1 + 4X_0 + V_0t^2)^\beta} \ge \frac{A\kappa_1}{(1+t)^{2\beta}}, \qquad \kappa_1 := \Big(\max\{1 + 4X_0, V_0\}\Big)^{-\beta}. \tag{2.10}$$

We conclude that the divergence of $\Phi(t) = \int^t \varphi(\tau)d\tau$ and hence, by (2.8) that flocking occurs, for $2\beta < 1$. This recovers the Cucker-Smale result [6, 7]. Below we improve the Cucker-Smale result proving *unconditional* flocking result for $\beta = 1/2$.

Theorem 2.4. *Let $(x_i(t), v_i(t))$ be the solutions to (2.1),(2.3b) with $V_0 > 0$. Then the following holds.*

(i) There exist a positive constant, C_2 (depending only on κ_1, A and β as specified in (2.12) below), such that

$$|x_i(t) - x_c| \lesssim |x_i(0) - x_c| + C_2. \tag{2.11a}$$

(ii) There exists constants, $\kappa_i > 0, i = 1, 2$, such that

$$|v_i(t) - v_c| \lesssim \sqrt{V_0} \times \begin{cases} e^{-\lambda A\kappa_2 t}, & \beta \in [0, \frac{1}{2}), \qquad \kappa_2 := (1 + 4X_0 + 8C_2^2)^{-\beta}, \\ (1+t)^{-\lambda A\kappa_1}, & \beta = \frac{1}{2}, \qquad \kappa_1 = \Big(\max\{1 + 4X_0, V_0\}\Big)^{-\beta}. \end{cases} \tag{2.11b}$$

Remark 2.3. Theorem 2.4 shows the two main features of flocking occur with the 2β-interaction potential, $2\beta \leq 1$, namely, the diameter $\max|x_i(t) - x_j(t)|$ remains uniformly bounded thus defining the traveling "flock" with velocity which is asymptotically particle-independent, $v_i(t) \approx v_c$.

Proof. We begin with the case $0 \leq \beta < \frac{1}{2}$. To get the optimal exponential convergence rate, we employ a bootstrapping argument in three steps.

Step 1. We first obtain a weak integrable decay rate for $|v_i - v_c|$. Using (2.10) we find

$$-\int_0^t \varphi(\tau)d\tau \leq -A\kappa_1 \int_0^t (1+\tau)^{-2\beta} d\tau \lesssim C_1\Big(1 - (1+t)^{1-2\beta}\Big), \qquad 0 \leq \beta < \frac{1}{2}. \tag{2.12a}$$

The above estimate together with Lemma 2.2 yield $V(t) \lesssim V_0 e^{-2C_1(1+t)^{1-2\beta}}$.

Step 2. Next, we improve lemma 2.3, observing that for $\beta < 1/2$, the position $X(t)$ remains uniformly bounded in time. Indeed, we have,

$$x_i(t) = x_i(0) + \int_0^t v_i(\tau)d\tau, \qquad x_c(t) = x_c(0) + \int_0^t v_c d\tau.$$

Time integrability of $|v_i(t) - v_c(t)|$ then yields (2.11a),

$$\begin{aligned} |x_i(t) - x_c(t)| &\leq |x_i(0) - x_c(0)| + \int_0^t |v_i(\tau) - v_c| d\tau \\ &\lesssim |x_i(0) - x_c(0)| + \int_0^\infty e^{-C_1(1+t)^{1-2\beta}} dt \leq |x_i(0) - x_c(0)| + C_2, \end{aligned}$$

where

$$C_2 \lesssim \int_0^\infty e^{-C_1(1+t)^{1-2\beta}} dt < \infty, \quad 0 < \beta < \frac{1}{2}. \tag{2.12b}$$

Step 3. The uniform bound of $X(t)$ implies an improved estimate for the interparticle interaction $\varphi(t)$: corollary 1 implies

$$\varphi(t) \geq r(\sqrt{2X_0}) \geq A(1 + 4X_0 + 8C_2^2)^{-\beta} = A\kappa_2,$$

which in turn, using lemma 2.2, yields the optimal exponential convergence rate (2.11b)

$$|v_i(t) - v_c(t)|^2 < V(t) \leq V_0 e^{-2\lambda\Phi(t)} \lesssim e^{-2\lambda A\kappa_2 t}.$$

It remains to deal with the case $\beta = \frac{1}{2}$. Here, we have

$$-2\lambda\Phi(t) \leq -2\lambda A\Big(\max\{1+4X_0, V_0\}\Big)^{-\frac{1}{2}} \int_0^t (1+\tau)^{-1} d\tau = -2\lambda A\kappa_1 \ln(1+t),$$

which in turn implies (2.11b), $V(t) \leq V_0 e^{-2\lambda\Phi(t)} \leq V_0(1+t)^{-2\lambda A\kappa_1}$. □

Remarks.

1. Consider the borderline case $\beta = \frac{1}{2}$ with initial configuration satisfying

$$\lambda\kappa_1 > 1, \quad \text{i.e.,} \quad \sqrt{\max\{1+4X_0, V_0\}} < \lambda;$$

then the same bootstrapping argument used for $\beta \in [0, \frac{1}{2})$ gives the exponential convergence:

$$|v_i(t) - v_c| \leq \sqrt{V_0} e^{-A\tilde{\kappa}_2 t}.$$

2. Flocking occurs even if $\beta > \frac{1}{2}$, but only for special initial configurations. Sufficient flocking conditions for such initial profiles is presented in [6].

3. From particle to kinetic description of flocking.

3.1. **Derivation of a mean-field model.** We assume that the number of particles involved in the C-S model (2.1),(2.2) is large enough that it becomes meaningful to observe the N-particle distribution function,

$$f^N = f^N(x_1, v_1, \ldots, x_N, v_N, t), \quad (x_i, v_i) \in \mathbb{R}^d \times \mathbb{R}^d, \quad i = 1, 2, \ldots, N. \tag{3.1a}$$

Since particles are indistinguishable, the probability density $f^N = f^N(\cdot)$ is symmetric in its phase-space arguments,

$$f^N(\cdots, x_i, v_i, \cdots x_j, v_j, \cdots, t) = f^N(\cdots, x_j, v_j, \cdots x_i, v_i, \cdots, t), \tag{3.1b}$$

so we can 'probe' f^N by any of its N pairs of phase-variables. Let $f^N(\cdot, \cdot, t)$ denote the marginal distribution[2]

$$\begin{aligned} f^N(x_1, v_1, t) \quad &:= \quad \int_{\mathbb{R}^{2d(N-1)}} f^N(x_1, v_1, x_-, v_-, t) dx_- dv_-, \\ &\qquad\qquad (x_-, v_-) := (x_2, v_2, \cdots, x_N, v_N). \end{aligned}$$

The formal derivation of a kinetic description for the C-S particle system (2.1),(2.2) is carried out below using the BBGKY hierarchy, e.g., [4, 20, 21], based on the Liouville equation, [16]

$$\partial_t f^N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f^N + \frac{\lambda}{N} \sum_{i=1}^N \nabla_{v_i} \cdot \Big(\sum_{j=1}^N r(x_i, x_j)(v_j - v_i) f^N\Big) = 0. \tag{3.2}$$

To this end, one study the marginal distribution $f^N(x_1, v_1, t)$ by integration of (3.2) with respect to $dx_- dv_- = dv_2 dx_2 \cdots dv_N dx_N$ (to simplify the notations, we now suppress the time-dependence whenever it is clear by the context, denoting $f^N(x_1, v_1, \cdots, x_N, v_N, t) = f^N(x_1, v_1, \cdots, x_N, v_N)$). Since $f^N(\cdot, \cdot)$ is rapidly decaying at infinity, the transport term in (3.2) amounts to

$$\int_{\mathbb{R}^{2d(N-1)}} \sum_{i=1}^N v_i \cdot \nabla_{x_i} f^N dx_- dv_- = v_1 \cdot \nabla_{x_1} f^N(x_1, v_1). \tag{3.3}$$

[2]To simplify notations, we use the same f^N to denote the marginal distribution of $2d+1$ varibales which is distinguished from its underlying distribution f^N depending on $2Nd+1$ variables.

The corresponding integration of the forcing term in (3.2), yields

$$\begin{aligned}&\frac{\lambda}{N}\sum_{i=1}^{N}\int_{\mathbb{R}^{2d(N-1)}}\sum_{j=1}^{N}\nabla_{v_i}\cdot\Big(r(x_i,x_j)(v_j-v_i)f^N\Big)dx_-dv_-\\&\quad=\frac{\lambda}{N}\int_{\mathbb{R}^{2d(N-1)}}\sum_{2\le j\le N}\nabla_{v_1}\cdot\Big(r(x_1,x_j)(v_j-v_1)f^N\Big)dx_-dv_-.\end{aligned}$$

But the symmetry of f^N, (3.1b), implies that the integrals being summed above are the same for $j=2,3\ldots,N$. Consequently, it will suffice to consider $j=2$:

$$\begin{aligned}&\frac{\lambda}{N}\sum_{i=1}^{N}\int_{\mathbb{R}^{2d(N-1)}}\sum_{j=1}^{N}\nabla_{v_i}\cdot\Big(r(x_i,x_j)(v_j-v_i)f^N\Big)dx_-dv_-\\&\quad=\frac{\lambda}{N}(N-1)\int_{\mathbb{R}^{2d(N-1)}}r(x_1,x_2)\nabla_{v_1}\cdot\Big((v_2-v_1)f^N\Big)dx_2dv_2\cdots dx_Ndv_N\\&\quad=\left(\lambda-\frac{\lambda}{N}\right)\nabla_{v_1}\cdot\Big(\int_{\mathbb{R}^{2d}}r(x_1,x_2)(v_2-v_1)g^Ndx_2dv_2\Big).\end{aligned}\tag{3.4}$$

Here g^N is the two-particle marginal function

$$g^N(x_1,v_1,x_2,v_2,t):=\int_{\mathbb{R}^{2d(N-2)}}f^Ndx_3dv_3\cdots dx_Ndv_N.$$

Thus, in view of (3.3) and (3.4), marginal integration of (3.2) over (x_-,v_-) implies that the one-particle density function, $f^N(x_1,v_1,t)$, satisfies

$$\partial_tf^N+v_1\cdot\nabla_{x_1}f^N+\left(\lambda-\frac{\lambda}{N}\right)\nabla_{v_1}\cdot\Big(\int_{\mathbb{R}^{2d}}r(x_1,x_2)(v_2-v_1)g^Ndx_2dv_2\Big)=0.$$

We now take the mean-field limit $N\to\infty$: we end up with the one- and two-particle limiting densities, $f:=\lim_{N\to\infty}f^N(x_1,v_1)$ and $g:=\lim_{N\to\infty}g^N(x_1,v_1,x_2,v_2)$, which satisfy

$$\partial_tf+v_1\cdot\nabla_{x_1}f+\lambda\nabla_{v_1}\cdot\Big(\int_{\mathbb{R}^{2d}}r(x_1,x_2)(v_2-v_1)gdx_2dv_2=0\Big).\tag{3.5}$$

To close the above equation we make the "molecular chaos" assumption about the independence of the two-point particle distribution,

$$g(x_1,v_1,x_2,v_2,t)=f(x_1,v_1,t)f(x_2,v_2,t);$$

Relabel, $(x_1,v_1)\mapsto(x,v)$ and $(x_2,v_2)\mapsto(y,v_*)$. We conclude that the one-particle distribution function $f(x,v,t)$ satisfies the Vlasov-type mean-field model,

$$\partial_tf+v\cdot\nabla_xf+\lambda\nabla_v\cdot Q(f,f)=0,\tag{3.6a}$$

$$Q(f,f)(x,v,t):=\int_{\mathbb{R}^{2d}}r(x,y)(v_*-v)f(x,v,t)f(y,v_*,t)dv_*dy.\tag{3.6b}$$

Here, $Q(f,f)$ is the quadratic interaction which can be expressed in the equivalent form

$$Q(f,f)(x,v,t)=fL[f],\quad L[f](x,v,t):=\int_{\mathbb{R}^{2d}}r(x,y)(v_*-v)f(y,v_*,t)dv_*dy.\tag{3.6c}$$

3.2. **A priori estimates.** We begin our study with a series of a priori estimates on the solution of the mean-field model (3.6), and the growth rate of the x and v-support of f. We first set the linear and quadratic functions

$$\psi_0(\xi) := \xi, \quad \psi_i(\xi) := \xi_i \ \ i = 1, \ldots, d, \quad \text{and} \quad \psi_{d+1}(\xi) := |\xi|^2.$$

Let f be a classical solution to (3.6) with a rapid decay in phase space $\mathbb{R}^{2d}$. A straightforward integration of (3.6) yields

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}} \psi_i(v) f(x,v)dvdx = \lambda \int_{\mathbb{R}^{2d}} \nabla_v \psi_i(v) \cdot Q(f,f)dvdx, \tag{3.7a}$$

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}} \psi_i(x) f(x,v)dvdx = \int_{\mathbb{R}^{2d}} \nabla_x \big(\psi_i(x)\cdot v\big) f(x,v)dvdx. \tag{3.7b}$$

Using (3.7) we obtain

Proposition 3.1. *Let f be a classical solutions decaying fast enough at infinity in phase space. Then following macroscopic quantities associated with f, satisfy*

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}} v f(x,v)dxdv = 0; \tag{3.8a}$$

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}} |v|^2 f(x,v)dxdv = -\lambda \int_{\mathbb{R}^{4d}} r(x,y)|v-v_*|^2 f(x,v)f(y,v_*)dv_*dydvdx; \tag{3.8b}$$

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}} f^p(x,v)dvdx = -d\lambda(p-1)\int_{\mathbb{R}^{4d}} r(x,y)f(y,v_*)f^p(x,v)dv_*dvdydx. \tag{3.8c}$$

Proof. Equality (3.8a) follows from (3.7a) with $\psi_i(v) = v_i$,

$$\int_{\mathbb{R}^{2d}} \nabla_v \psi_i(v)\cdot Q(f,f)dvdx = \int_{\mathbb{R}^{4d}} r(x,y)(v_{*i}-v_i)f(y,v_*)f(x,v)dv_*dydvdx = 0.$$

The last integral vanishes due to antisymmetry of the integrand, realized by the interchange of variables $(x,v) \leftrightarrow (y,v_*)$. The statement of (3.8b) follows from (3.7a) with $\psi_{d+1}(v) = |v|^2$, and observing that

$$\begin{aligned}
\int_{\mathbb{R}^{2d}} \nabla_v \psi_{d+1}(v)\cdot Q(f,f)dvdx &= 2\int_{\mathbb{R}^{2d}} v\cdot Q(f,f)dvdx \\
&= 2\int_{\mathbb{R}^{4d}} r(x,y)v\cdot(v_*-v)f(y,v_*)f(x,v)dv_*dydvdx \\
&= -2\int_{\mathbb{R}^{4d}} r(x,y)v_*\cdot(v_*-v)f(y,v_*)f(x,v)dv_*dydvdx \\
&= -\int_{\mathbb{R}^{4d}} r(x,y)|v-v_*|^2 f(x,v)f(y,v_*)dv_*dydvdx.
\end{aligned}$$

Finally, we note the two identities, $f^{p-1}v\cdot\nabla_x f \equiv \frac{1}{p}v\cdot\nabla_x f^p$, and

$$f^{p-1}\nabla_v\cdot Q(f,f) \equiv \nabla_v\cdot\Big(\frac{L[f]f^p}{p}\Big) + \Big(1-\frac{1}{p}\Big)\Big(\nabla_v\cdot L[f]\Big)f^p, \qquad Q(f,f) = fL[f].$$

Integration of (3.6) against f^{p-1} then yields

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^{2d}} f^p dvdx &= -p\int_{\mathbb{R}^{2d}} f^{p-1}\Big(v\cdot\nabla_x f + \lambda\nabla_v\cdot Q(f,f)\Big)dvdx \\
&= -\lambda(p-1)\int_{\mathbb{R}^{2d}}\Big(\nabla_v\cdot L[f]\Big)f^p dvdx \\
&= -d\lambda(p-1)\int_{\mathbb{R}^{4d}} r(x,y)f(y,v_*)f^p(x,v)dv_*dvdydx.
\end{aligned}$$

□

Let f be a classical kinetic solution of (3.6). The statement of (3.8c) shows that its $L^1(dxdv)$norm, the total macroscopic mass is conserved in time (while according to (3.8c), higher $L^p(dxdv)$-norms of f decay in time),

$$\mathcal{M}_0(t) := \int_{\mathbb{R}^{2d}} f(x,v,t)dxdv \equiv \mathcal{M}_0. \tag{3.9a}$$

Similarly, (3.8a) tells us that the total macroscopic momentum is conserved in time,

$$\mathcal{M}_1(t) := \int_{\mathbb{R}^{2d}} vf(x,v,t)dxdv \equiv \mathcal{M}_1. \tag{3.9b}$$

Finally, (3.8b) tells us that the total amount of macroscopic energy is non-increasing in time,

$$\mathcal{M}_2(t) := \int_{\mathbb{R}^{2d}} |v|^2 f(x,v,t)dvdx \le \mathcal{M}_2(0). \tag{3.9c}$$

Here, $\mathcal{M}_0 := \mathcal{M}_0(0)$, $\mathcal{M}_1 := \mathcal{M}_1(0)$ and $\frac{1}{2}\mathcal{M}_2(0)$ denote, respectively, the initial amounts of mass, momentum and energy at $t = 0$. Next, we turn to the following a priori bound on the kinetic velocity.

Lemma 3.2. *Let $[x(t), v(t)]$ be the particle trajectory issued from $(x,v) \in supp_{(x,v)} f_0$ at time 0. Then the i-component of velocity trajectory, $v_i(t) = v_i(t; 0, x, v)$, $i = 1, \cdots, d$, satisfies $v_i(t) \in (v_\ell, v_r)$ where,*

$$\begin{aligned} v_\ell &:= v_i(0)e^{-\lambda A\mathcal{M}_0 t} - \frac{J_0}{\mathcal{M}_0}(1 - e^{-\lambda A\mathcal{M}_0 t}), \\ v_r &:= v_i(0)e^{-\lambda\mathcal{M}_0\Phi(t)} + \lambda AJ_0 \int_0^t e^{-\lambda\mathcal{M}_0(\Phi(t)-\Phi(s))}ds. \end{aligned}$$

Here, $\Phi(t) := \int_0^t \varphi(s)ds$, $\mathcal{M}_0 = \|f_0\|_{L^1_{x,v}}$ is the initial total mass and $J_0 := \sqrt{\mathcal{M}_0\mathcal{M}_2}$.

Proof. For given $(x,v) \in \mathrm{supp}_{x,v} f_0$, we set $x(s) \equiv x(s; 0, x, v)$ and $v(s) \equiv v(s; 0, x, v)$. Note that for each $i = 1, \cdots, d$, we have

$$\begin{aligned} L_i[f(x(t), v(t), t)] &= \int_{\mathbb{R}^{2d}} r(x(t), y)(v_{*i}(t) - v_i)f(y, v_*)dv_*dy \\ &= v_i(t) + \int_{\mathbb{R}^{2d}} r(x(t), y)v_{*i} f(y, v_*)dv_*dy - \Big(\int_{\mathbb{R}^{2d}} r(x(t), y)f(y, v_*)dv_*dy\Big). \end{aligned} \tag{3.10}$$

Lower and upper bounds for the kinetic velocities are obtained in terms of the estimates,

$$\begin{aligned} &\varphi(t)\mathcal{M}_0 \le \int_{\mathbb{R}^{2d}} r(x(t), y)f(y, v_*)dv_*dy \le A\mathcal{M}_0, \\ &\Big|\int_{\mathbb{R}^2} r(x,y)v_* f(y, v_*)dv_*dy\Big| \le A\sqrt{\|f_0\|_{L^1_{x,v}}}\sqrt{\||v|^2 f_0\|_{L^1_{x,v}}} = AJ_0. \end{aligned}$$

It then follows from (3.10) that

$$-\lambda A\mathcal{M}_0 v_i(s) - \lambda AJ_0 \le \frac{d}{dt}v_i(t) = \lambda L_i[f(x(t), v(t), t)] \le -\lambda\varphi(t)\mathcal{M}_0 v_i(t) + \lambda AJ_0,$$

and the desired result follows by Gronwall's integration. □

Remark 3.1. Let $\Omega(t)$ denote the v-projection of supp$f(\cdot,t)$,

$$\Omega(t) := \{v \in \mathbb{R}^d : \exists\ (x,v) \in \mathbb{R}^{2d} \quad \text{such that} \quad f(x,v,t) \neq 0\}. \tag{3.11}$$

Lemma 3.2 shows that if $f_0(x,\cdot)$ is compactly supported, then supp$f(x,\cdot,t)$ remains finite, with a weak growth estimate for the velocity trajectory,

$$|v_i(t)| \le \max\left\{\eta_0 + \frac{J_0}{\mathcal{M}_0}, \eta_0 + \lambda J_0 t\right\} \le \eta_0 + \frac{J_0}{\mathcal{M}_0} + \lambda J_0 t, \quad \eta_0 := \max_{v\in\Omega(0)} |v|. \tag{3.12}$$

3.3. **Global existence of classical solutions.** In this section we develop a global existence theory for classical solutions of the Vlasov-type flocking equation,

$$\partial_t f + v\cdot\nabla_x f + \lambda\nabla_v\cdot\big(fL[f]\big) = 0, \quad x,v\in\mathbb{R}^d, t>0, \tag{3.13a}$$

$$L[f](x,v,t) = \int_{\mathbb{R}^{2d}} r(x,y)(v_*-v)f(y,v_*,t)dv_*dy, \quad r(x,y) = \frac{A}{(1+|x-y|^2)^\beta}, \tag{3.13b}$$

subject to initial datum

$$f(x,v,0) = f_0(x,v). \tag{3.13c}$$

We begin by noting that the kinetic solution f remains *uniformly bounded*. To this end, rewrite the mean-field model (3.13) in a 'non-conservative' form,

$$\partial_t f + v\cdot\nabla_x f + \lambda L[f]\cdot\nabla_v f = -\lambda f\nabla_v\cdot L[f], \quad x,v\in\mathbb{R}^d, t>0. \tag{3.14}$$

Consider the particle trajectories, $[x(t),v(t)] \equiv [x(t;t_0,x_0,v_0), v(t;t_0,x_0,v_0)]$, passing through $(x_0,v_0)\in\mathbb{R}^d\times\mathbb{R}^d$ at time $t_0\in\mathbb{R}_+$,

$$\frac{d}{dt}x(t) = v(t), \qquad \frac{d}{dt}v(t) = \lambda L[f(x(t),v(t),t)]. \tag{3.15}$$

Noting that $-\nabla_v\cdot L[f] = d\int_{\mathbb{R}^{2d}} r(x,y)f(y,v_*,t)dv_*dy$, we find

$$\|\nabla_v\cdot L[f]\|_{L^\infty_{x,v}} \le dA\|f\|_{L^1_{x,v}} = dA\mathcal{M}_0,$$

which implies that the following inequality holds along the particle trajectories,

$$\frac{d}{dt}f(x(t),v(t),t) \le \lambda dA\mathcal{M}_0 f(x(t),v(t),t).$$

It follows that as long as initial data f_0 has a finite mass, there will be no finite time blow-up for $f(\cdot,t)$,

$$\|f(t)\|_{L^\infty_{x,v}} \le e^{\lambda dA\mathcal{M}_0 t}\|f_0\|_{L^\infty_{x,v}}. \tag{3.16}$$

Next, we turn to study the *smoothness* of $f(\cdot,t)$. Since the local existence theory will be followed from the standard fixed point argument, e.g., [4], we only obtain a priori C^1-norm bound of f to conclude a global existence of classical solutions.

Theorem 3.3. *Consider the flocking kinetic model (3.13). Suppose that the initial datum $f_0\in(C^1\cap W^{1,\infty})(\mathbb{R}^{2d})$ satisfies*

1. *Initial datum is compactly supported in the phase space, supp$_{(x,v)}f_0(\cdot)$ is bounded, and in particular, $\Omega(0)\subset B_{\eta_0}(0)$.*
2. *Initial datum is C^1-regular and bounded:*

$$\sum_{0\le|\alpha|+|\beta|\le1} \|\nabla_x^\alpha\nabla_v^\beta f_0\|_{L^\infty_{x,v}} < \infty.$$

Then, for any $T \in (0,\infty)$, there exists a unique classical solution $f \in C^1(\mathbb{R}^{2d} \times [0,T))$.

Proof. We express the non-conservative kinetic model (3.14) in terms of the non-linear transport operator $\partial_t + v\cdot\nabla_x + \lambda L[f]\cdot\nabla_v$,

$$\mathcal{T} f = -\lambda f \nabla_v \cdot L[f], \qquad \mathcal{T} := \partial_t + v\cdot\nabla_x + \lambda L[f]\cdot\nabla_v. \tag{3.18}$$

We claim that there exist (possibly different) positive constants, $C = C(d,\lambda,\mathcal{M}_0,J_0) > 0$, such that

$$|\mathcal{T}(f)| \leq C|f|, \tag{3.19a}$$

$$|\mathcal{T}(\partial_{x_i} f)| \leq C\Big(|f| + (\eta(t)+1)|\nabla_v f| + |\partial_{x_i} f|\Big), \quad \eta(t) := \max_{v\in\Omega(t)} |v|, \tag{3.19b}$$

$$|\mathcal{T}(\partial_{v_i} f)| \leq C\Big(|\partial_{x_i} f| + |\nabla_v f|\Big). \tag{3.19c}$$

To verify these inequalities, observe that by (3.18)

$$\mathcal{T}(f) = -\lambda f \nabla_v \cdot L[f] = \lambda d \int_{\mathbb{R}^{2d}} r(x,y) f(y,v_*,t)\, dv_* dy$$

and (3.19a) follows with $C := \lambda d A \mathcal{M}_0 \geq \lambda \|\nabla_v \cdot L[f](t)\|_{L^\infty_{x,v}}$.

Next, differentiating (3.18) we obtain

$$\mathcal{T}(\partial_{x_i} f) = -\lambda(\partial_{x_i} L[f])\cdot\nabla_v f - \lambda(\partial_{x_i}\nabla_v\cdot L[f]) f - \lambda(\nabla_v\cdot L[f])\partial_{x_i} f.$$

Straightforward calculation yields

$$\partial_{x_i} L[f] = -\int_{\mathbb{R}^{2d}} \frac{2\beta A(x_i-y_i)}{(1+|x-y|^2)^{\beta+1}}(v_*-v) f(y,v_*,t)\, dv_* dy,$$

and since the variation of the relevant kinetic velocities at time t does not exceed $|v-v_*| \leq 2\eta(t)$, we find $\|\partial_{x_i} L[f](t)\| \leq 4\beta A \eta(t)\mathcal{M}_0$; similarly, for

$$\partial_{x_i}\nabla_v\cdot L[f] = \int_{\mathbb{R}^{2d}} \frac{2\beta A d(x_i-y_i)}{(1+|x-y|^2)^{\beta+1}} f(y,v_*,t)\, dv_* dy,$$

we have

$$\|\partial_{x_i}\nabla_v\cdot L[f](t)\|_{L^\infty_{x,v}} \leq 2\beta d A \mathcal{M}_0.$$

We conclude that (3.19b) holds with, say, $C = \lambda d A \mathcal{M}_0(1 + 2\beta + 4\beta\eta(t))$.

Finally, we differentiate (3.18) with respect to v_i (noting that $\partial_{v_i}\nabla_v\cdot L[f] = 0$)

$$\mathcal{T}(\partial_{v_i} f) = -\lambda\partial_{x_i} f - \lambda(\partial_{v_i} L[f])\cdot\nabla_v f - \lambda(\nabla_v\cdot L[f])\partial_{v_i} f;$$

Straightforward calculation then yields,

$$\partial_{v_i} L[f] = -\int_{\mathbb{R}^{2d}} r(x,y) f(y,v_*,t)\, dv_* dy \quad \mapsto \quad \|\partial_{v_i} L[f](t)\|_{L^\infty_{x,v}} \leq \mathcal{M}_0,$$

and (3.19c) follows with $C = \lambda + \lambda(d+1)A\mathcal{M}_0$.

Now, let $\mathcal{F}(t)$ measure the $W^{1,\infty}$-norm of $f(\cdot,t)$

$$\mathcal{F}(t) := \sum_{0\leq|\alpha|+|\beta|\leq 1} \|\nabla_x^\alpha \nabla_v^\beta f(t)\|_{L^\infty_{x,v}}.$$

The inequalities (3.19) imply

$$\frac{d}{dt}\mathcal{F}(t) \lesssim \big(\eta(t)+1\big)\mathcal{F}(t).$$

Lemma 3.2 (see remark 3.1), tells us that $\eta(t) \lesssim \eta_0 + t$, and we end up with the energy bound

$$\mathcal{F}(t) \le \mathcal{F}(0)e^{C(t+t^2)}, \qquad C = C(\eta_0, \mathcal{M}_0, J_0, \beta, d, A).$$

Equipped with this a priori $W^{1,\infty}$ estimate, standard continuation principle yields a global extension of local classical solutions. □

Remarks.

1. The above a priori estimate need not be optimal. Since we used a rough estimate (3.12) for the size of $\Omega(t)$,

$$\max_{v\in\Omega(t)} |v| \lesssim \eta_0 + t,$$

we end with the quadratic exponential growth, $e^{C(t+t^2)}$. An optimal bound, however, could be e^{Ct}. Of course, one cannot expect a uniform bound for C^1-norm, because the one-particle distribution function may grow exponentially along the particle trajectory (see (3.16)).
2. The global existence of classical solution can be improved for more general kernels.
3. For related works on kinetic granular type dissipative systems, we refer to [13, 14, 15].

4. **Time-asymptotic behavior of kinetic flocking.** In this section, we present the time-asymptotic flocking behavior of the kinetic model for flocking (3.13). As in the case with particle description discussed in Section 2, we will show that the velocity of particles will be contracted to the mean bulk velocity u_c, which corresponds to the velocity at the center of mass:

$$u_c(t) := \frac{1}{\mathcal{M}_0}\int_{\mathbb{R}^{2d}} vf(x,v,t)dvdx, \qquad u_c(t) \equiv u_c(0).$$

We recall that the energy decay in (3.8b)

$$\frac{d}{dt}\mathcal{M}_2(t) \le 0. \tag{4.1}$$

We note that unlike granular flows, for example, e.g. [4], the energy decay (4.1) does *not* drive the energy to zero: if the initial momentum $\mathcal{M}_1 \neq 0$, then the kinetic energy $\mathcal{M}_2(t)$ has a nonzero lower bound, in analogy with the discrete case, consult remark 2.1. Indeed, since the total mass and momentum are conserved, $\mathcal{M}_i(t) \equiv \mathcal{M}_i$, $i = 0, 1$, (3.8b) implies

$$\begin{aligned} &\frac{d}{dt}\mathcal{M}_2(t) \\ \ge\ & -A\int_{\mathbb{R}^{2d}} |v-v_*|^2 f(x,v)f(y,v_*)dv_*dydvdx = -2A\mathcal{M}_0\mathcal{M}_2(t) + 2A|\mathcal{M}_1|^2, \end{aligned}$$

and Gronwall's lemma yields the following kinetic analog of (2.4c),(2.5)

$$\mathcal{M}_2(t) \ge \mathcal{M}_2(0)e^{-2A\mathcal{M}_0 t} + \frac{|\mathcal{M}_1|^2}{\mathcal{M}_0}(1-e^{-2A\mathcal{M}_0 t}) \ge \frac{|\mathcal{M}_1|^2}{\mathcal{M}_0}. \tag{4.2}$$

Thus, energy decay by itself does not assert flocking. As in the particle description, the emergence of the time-asymptotic flocking behavior depends on the sufficiently slow decay rate of the interparticle interaction $\varphi(s) = r(x(s), y(s))$. To this end, we let Σ denote the x-projection of supp$f(\cdot, t)$

$$\Sigma(t) := \{x \in \mathbb{R}^d : \exists\ (x, v) \in \mathbb{R}^{2d} \quad \text{such that} \quad f(x, v, t) \neq 0\}, \tag{4.3}$$

and denote its initial size, $\zeta_0 := \max_{x\in\Sigma_0} |x|$.

Lemma 4.1. *Let f be a global classical solution to (3.13). Then, there exists a $\kappa_3 > 0$ such that $\varphi(t) = \inf_{(x,y)\in\Sigma(t)} r(x, y)$ satisfies*

$$\varphi(t) \geq A\kappa_3^{-\beta}(1 + t^2 + t^4)^{-\beta}. \tag{4.4}$$

The constant κ_3 is given by

$$\kappa_3 := \max\Big\{1 + 12\zeta_0^2, 12\Big(\eta_0 + \frac{J_0}{\mathcal{M}_0}\Big)^2, 3\lambda^2 J_0^2\Big\}.$$

Proof. Let $(x, v) \in \text{supp}_{x,v} f_0$. It follows from remark 3.1 that

$$|v_i(t)| \leq \eta_0 + \frac{J_0}{\mathcal{M}_0} + \lambda J_0 t, \tag{4.5}$$

and hence

$$|x_i(t; 0, x, v)| \quad \leq \quad |x_i| + \int_0^t |v_i(s; 0, x, v)|ds \leq \zeta_0 + \Big(\eta_0 + \frac{J_0}{\mathcal{M}_0}\Big)t + \frac{\lambda J_0 t^2}{2}.$$

This gives an estimate on the size of x-support $\Sigma(t)$ of f. For $x, y \in \Sigma(t)$,

$$\begin{aligned}
\zeta(t) \leq 1 + |x - y|^2 \quad &\leq \quad 1 + 2(|x|^2 + |y|^2) \\
&\leq \quad 1 + 4\Big[\zeta_0 + \Big(\eta_0 + \frac{J_0}{\mathcal{M}_0}\Big)t + \frac{\lambda J_0 t^2}{2}\Big]^2 \\
&\leq \quad 1 + 12\zeta_0^2 + 12\Big(\eta_0 + \frac{J_0}{\mathcal{M}_0}\Big)^2 t^2 + 3\lambda^2 J_0^2 t^4 \\
&\leq \quad \kappa_3(1 + t^2 + t^4),
\end{aligned}$$

and (4.4) follows, $\varphi(t) \geq r(\zeta(t))$. $\square$

Let $v - u_c$ denote the fluctuation (or peculiar) kinetic velocity. We will quantify the emergence of the time-asymptotic flocking behavior in term of the corresponding energy fluctuation

$$\Lambda[f(t)] := \int_{\mathbb{R}^{4d}} |v - u_c|^2 f(x, v, t)dvdx. \tag{4.6}$$

The time-evolution estimate of $\Lambda[f(t)]$, will depends on the decay rate of $\varphi(t)$.

Let f be a classical solution of (3.13) with compact support in x and v. Direct calculation implies

$$\begin{aligned}
\frac{d}{dt}\Lambda[f(t)] &= \int_{\mathbb{R}^{2d}} |v - u_c|^2 \partial_t f(x, v)dvdx \\
&= -\int_{\mathbb{R}^{2d}} |v - u_c|^2 v \cdot \nabla_x f dvdx - \lambda\int_{\mathbb{R}^{2d}} |v - u_c|^2 \nabla_v \cdot Q(f, f)dvdx \\
&=: \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}$$

The first term on the right vanishes by the divergence theorem

$$\mathcal{I}_1 = -\int_{\mathbb{R}^{2d}} |v - u_c|^2 v \cdot \nabla_x f dvdx = -\int_{\mathbb{R}^{2d}} \text{div}_x\Big(|v - u_c|^2 v f\Big)dvdx = 0.$$

The second term is simplified as follows.

$$
\begin{aligned}
\mathcal{I}_2 = 2\lambda \int_{\mathbb{R}^{2d}} (v-u_c)\cdot Q(f,f)dvdx \\
&= -2\lambda \int r(x,y)(v-u_c)\cdot(v-v_*)f(y,v_*)f(x,v)dv_*dvdydx \\
&= -2\lambda \int r(x,y)v\cdot(v-v_*)f(y,v_*)f(x,v)dv_*dvdydx \\
&= -\lambda \int r(x,y)|v-v_*|^2 f(y,v_*)f(x,v)dv_*dvdydx.
\end{aligned}
$$

We summarize the last three equalities in the following lemma.

Lemma 4.2. *Let f be a classical solution of (3.13) subject to compactly supported initial conditions f_0. Then, the decay of the energy functional $\Lambda[f(t)]$ in (4.6) is governed by*

$$
\frac{d}{dt}\Lambda[f(t)] = -\lambda \int r(x,y)|v-v_*|^2 f(y,v_*)f(x,v)dv_*dvdydx. \tag{4.7}
$$

Equipped with lemma 4.2 we can state the main result of this section.

Theorem 4.3. *Let f be the classical kinetic solution constructed in Theorem 3.3. Then, the decay of its energy fluctuation around the mean bulk velocity u_c, is given by*

$$
\Lambda[f(t)] \lesssim \Lambda[f_0] \times \begin{cases} C_3 e^{-\kappa_4 t^{1-4\beta}}, & 0 \le \beta < \frac{1}{4}, \\ (1+t)^{-\kappa_5}, & \beta = \frac{1}{4}. \end{cases} \tag{4.8}
$$

The constants involved are $\kappa_4 = 1/(3\kappa_3)^\beta(1-4\beta) > 0$ and $\kappa_5 = 2\lambda A/\sqrt[4]{3\kappa_3} > 0$.

Proof. Lemma 4.2 implies

$$
\begin{aligned}
\frac{d}{dt}\Lambda[f(t)] &= -\lambda \int_{\mathbb{R}^{4d}} r(x,y)|v-v_*|^2 f(y,v_*)f(x,v)dv_*dvdydx \\
&\le -\lambda\varphi(t) \int_{\mathbb{R}^{4d}} |v-v_*|^2 f(y,v_*)f(x,v)dv_*dvdydx. = -2\lambda\varphi(t)\mathcal{M}_0\Lambda[f(t)].
\end{aligned}
$$

As before, the identity $|v-v_*|^2 = |v-u_c|^2 + |v_*-u_c|^2 + 2(v-u_c)\cdot(v_*-u_c)$ induces the corresponding decomposition of the integrand on the right. Noting that

$$
\int_{\mathbb{R}^{2d}} (v-u_c)f(x,v,t)dvdx = 0.
$$

We conclude

$$
\frac{d}{dt}\Lambda[f(t)] \le -2\lambda\varphi(t)\mathcal{M}_0\Lambda[f(t)]
$$

and Gronwall's integration yields

$$
\Lambda[f(t)] \le \Lambda[f_0]e^{-2\lambda\Phi(t)}, \qquad \Phi(t) = \int_0^t \varphi(s)ds. \tag{4.9}
$$

We distinguish between two cases.

Case 1 $[0 \le \beta < \frac{1}{4}]$. According to Lemma 4.1

$$
\varphi(t) \ge A\kappa_3^{-\beta}(1+t^2+t^4)^{-\beta} \ge A(3\kappa_4)^{-\beta}t^{-4\beta}, \qquad t \ge 1.
$$

We compute for $t \geq 1$,

$$-\Phi(t) \lesssim -\int_1^t \varphi(\tau)d\tau \leq -\frac{A}{(3\kappa_4)^\beta(1-4\beta)}(t^{1-4\beta}-1)$$

which yields the first part of (4.8).

Case 2 $[\beta = \frac{1}{4}]$. For $t \geq 1$ we have

$$\exp\Big(-2\lambda A\Phi(t)\Big) \leq \exp\Big(-\frac{2\lambda \ln t}{(3\kappa_4)^{1/4}}\Big) \lesssim (1+t)^{-\frac{2\lambda A}{\sqrt[4]{3\kappa_4}}},$$

and the second part of (4.8) follows. □

5. **From kinetic to hydrodynamic description of flocking.** In this section we discuss the hydrodynamic description for flocking, which is formally obtained by taking moments of the kinetic model (3.13) .

$$\partial_t f + v\cdot\nabla_x f) + \lambda\nabla_v \cdot Q(f,f) = 0, \quad x,v \in \mathbb{R}^d,\ t>0, \tag{5.1a}$$

$$Q(f,f)(x,v,t) = \int_{\mathbb{R}^{2d}} r(x,y)(v_*-v)f(x,v,t)f(y,v_*,t)dv_*dy. \tag{5.1b}$$

We first set hydrodynamic variables: the mass $\rho := \int_{\mathbb{R}^d} f dv$, the momentum, $\rho u := \int_{\mathbb{R}^d} v f dv$, and the energy, $\rho E := 1/2\int_{\mathbb{R}^d}|v|^2 f dv$, which is the sum of kinetic and internal energies (corresponding to the first two terms in the decomposition of kinetic velocities $|v|^2 = |u|^2 + |v-u|^2 + 2(v-u)\cdot u$)

$$\rho E = \rho e + \frac{1}{2}\rho|u|^2, \qquad \rho e := \frac{1}{2}\int |v-u(x)|^2 f(x,v,t)dv. \tag{5.2}$$

For notational simplicity, we suppress time-dependence, denoting $\rho(x) \equiv \rho(x,t)$, $u(x) \equiv u(x,t)$ and $E(x) = E(x,t)$ when the context is clear.

We compute the v-moments of (5.1): multiply (5.1) against $1, v$ and $|v|^2/2$ and integrate over the velocity space $\mathbb{R}^d$. We end up with the system of equations,

$$\partial_t \rho + \nabla_x\cdot(\rho u) = 0, \tag{5.3a}$$

$$\partial_t(\rho u) + \nabla_x \cdot \Big(\rho u\otimes u + P\Big) = \mathcal{S}^{(1)}, \tag{5.3b}$$

$$\partial_t(\rho E) + \nabla_x\cdot\Big(\rho E u + Pu + q\Big) = \mathcal{S}^{(2)}. \tag{5.3c}$$

Here, $\mathcal{S}^{(j)}, j=1,2$, are the nonlocal source terms given by

$$\mathcal{S}^{(1)}(x,t) \ := \ -\lambda\int_{\mathbb{R}^d} r(x,y)(u(x)-u(y))\rho(x)\rho(y)dy, \tag{5.3d}$$

$$\mathcal{S}^{(2)}(x,t) \ := \ -\lambda\int_{\mathbb{R}^d} r(x,y)\Big[E(x)+E(y)-u(x)\cdot u(y)\Big]\rho(x)\rho(y)dy, \tag{5.3e}$$

and $P=(p_{ij}), q=(q_i)$ denote, respectively, the stress tensor and heat flux vector,

$$p_{ij} := \int_{\mathbb{R}^d}(v_i-u_i)(v_j-u_j)f dv, \quad q_i := \int_{\mathbb{R}^d}(v_i-u_i)|v-u|^2 f dv. \tag{5.3f}$$

Remark 5.1. The total mass of the source term $\mathcal{S}^{(1)}$ vanishes: exchange of variables $x \leftrightarrow y$ yields

$$\int_{\mathbb{R}^d}\mathcal{S}^{(1)}(x,t)dx = -\lambda\int_{\mathbb{R}^{2d}} r(x,y)(u(x)-u(y))\rho(x)\rho(y)dxdy = 0. \tag{5.4a}$$

The source term $\mathcal{S}^{(2)}$ is non-positive: using (5.2) we find

$$\mathcal{S}^{(2)}(x,t) = -\lambda \int_{\mathbb{R}^d} r(x,y)\Big[\frac{1}{2}|u(x)-u(y)|^2 + e(x) + e(y)\Big]\rho(x)\rho(y)dy \le 0. \quad (5.4b)$$

We conclude that the total mass and momentum, $\int \rho(x,t)dx$ and $\int \rho(x,t)u(x,t)dx$, are conserved in time. The total energy, however,

$$\mathcal{E}(t) = \int_{\mathbb{R}^d} \rho(x,t)E(x,t)dc = \frac{1}{2}\int_{\mathbb{R}^{2d}} |v|^2 f(x,v,t)dvdx,$$

is dissipating, which is responsible for the formation of time-asymptotic flocking behavior. We turn to quantify this decay. We first write the total energy as the sum of total kinetic and potential energies corresponding to (5.2), $\mathcal{E}(t) = \mathcal{E}_{\mathrm{k}}(t) + \mathcal{E}_{\mathrm{p}}(t)$, where

$$\mathcal{E}_{\mathrm{k}}(t) := \frac{1}{2}\int_{\mathbb{R}^d} \rho(x,t)|u(x,t)|^2 dx, \quad \mathcal{E}_{\mathrm{p}}(t) := \frac{1}{2}\int_{\mathbb{R}^{2d}} |v-u(x,t)|^2 f dvdx.$$

Lemma 5.1. *The time evolution of the total, kinetic and internal energies is governed by*

$$\frac{d}{dt}\mathcal{E}(t) = -\lambda \int_{\mathbb{R}^{2d}} r(x,y)\Big[\frac{1}{2}|u(x)-u(y)|^2 + e(x) + e(y)\Big]\rho(x)\rho(y)dydx; \quad (5.5a)$$

$$\frac{d}{dt}\mathcal{E}_p(t) = -\lambda \int_{\mathbb{R}^{2d}} r(x,y)\big(e(x)+e(y) - u(x)\cdot u(y)\big)\rho(x)\rho(y)dydx$$
$$-2\int_{\mathbb{R}^d} (\nabla_x \cdot u)\rho e dx; \quad (5.5b)$$

$$\frac{d}{dt}\mathcal{E}_k(t) = -\frac{\lambda}{2}\int_{\mathbb{R}^{2d}} r(x,y)\Big(|u(x)|^2 + |u(y)|^2\Big)\rho(x)\rho(y)dxdy$$
$$+2\int_{\mathbb{R}^d} (\nabla_x \cdot u)\rho e dx. \quad (5.5c)$$

Proof. The equality (5.5a) follows from integration of (5.3c) and invoking (5.4b). For the decay rate of the total internal energy $\mathcal{E}_{\mathrm{p}}(t)$ in (5.5b), we calculate

$$\frac{d}{dt}\mathcal{E}_{\mathrm{p}}(t) = \int_{\mathbb{R}^{2d}} \partial_t\Big(\frac{|v-u|^2}{2}\Big) f dvdx + \int_{\mathbb{R}^{2d}} \frac{|v-u|^2}{2}\partial_t f dvdx =: \mathcal{K}_1 + \mathcal{K}_2.$$

We estimate $\mathcal{K}_i$ separately. The first term, $\mathcal{K}_1$ vanishes, for

$$\begin{aligned}\mathcal{K}_1 &= \int_{\mathbb{R}^{2d}} (u-v)\cdot\partial_t u f dvdx = \int_{\mathbb{R}^{2d}} u\cdot\partial_t u f dvdx - \int_{\mathbb{R}^{2d}} \partial_t u\cdot(vf)dvdx \\ &= \int_{\mathbb{R}^d} u\cdot\partial_t u\rho dx - \int_{\mathbb{R}^d} \partial_t u\cdot(\rho u)dx = 0.\end{aligned}$$

For the second term, $\mathcal{K}_2$, we use the kinetic model to find

$$\begin{aligned}\mathcal{K}_2 &= \frac{1}{2}\int_{\mathbb{R}^{2d}} |v-u|^2\partial_t f dvdx = -\frac{1}{2}\int_{\mathbb{R}^{2d}} |v-u|^2\Big(v\cdot\nabla_x f + \lambda\nabla_v\cdot Q(f,f)\Big)dvdx \\ &= -\frac{1}{2}\int_{\mathbb{R}^{2d}} |v-u|^2 v\cdot\nabla_x f dvdx + \lambda\int_{\mathbb{R}^{2d}} (v-u)\cdot Q(f,f)dvdx =: \mathcal{K}_{21} + \lambda\mathcal{K}_{22}.\end{aligned}$$

The term $\mathcal{K}_{21}$ amounts to

$$\begin{aligned}\mathcal{K}_{21} &= -\frac{1}{2}\int_{\mathbb{R}^{2d}} |v-u|^2 v\cdot\nabla_x f dvdx = \frac{1}{2}\int_{\mathbb{R}^{2d}} \Big(\nabla_x|v-u|^2\Big)\cdot(vf)dvdx \\ &= -\int_{\mathbb{R}^{2d}} (v-u)\cdot\Big((\nabla_x\cdot u)vf\Big)dvdx = -2\int_{\mathbb{R}^d} (\nabla_x\cdot u)(\rho e)dx.\end{aligned}$$

A lengthy calculation shows that the remaining term, $\mathcal{K}_{22}$, equals

$$\begin{aligned}
\mathcal{K}_{22} &= \int_{\mathbb{R}^{4d}} r(x,y)(v-u(x))\cdot(v_*-v)f(x,v)f(y,v_*)dv_*dvdydx\\
&= \int_{\mathbb{R}^{4d}} r(x,y)v\cdot(v_*-v)f(x,v)f(y,v_*)dv_*dvdydx\\
&\quad -\int_{\mathbb{R}^{4d}} r(x,y)u(x)\cdot(v_*-v)f(x,v)f(y,v_*)dv_*dvdydx\\
&= -\frac{1}{2}\int_{\mathbb{R}^{4d}} r(x,y)|v-v_*|^2 f(x,v)f(y,v_*)dv_*dvdydx\\
&\quad +\frac{1}{2}\int_{\mathbb{R}^{2d}} r(x,y)\Big(|u(x)-u(y)|^2\Big)\rho(x)\rho(y)dydx\\
&= -\int_{\mathbb{R}^{2d}} r(x,y)\Big(E(x)+E(y)-\frac{1}{2}|u(x)-u(y)|^2\Big)\rho(x)\rho(y)dydx\\
&= -\int_{\mathbb{R}^{2d}} r(x,y)\Big(e(x)+e(y)+u(x)\cdot u(y)\Big)\rho(x)\rho(y)dydx
\end{aligned}$$

Finally (5.5c) follows by subtracting (5.5b) from (5.5a). □

Next, we present a fundamental estimate for the flocking behavior to the system (5.3). We set

$$\Gamma(t) := \int_{\mathbb{R}^{2d}} \Big(\frac{1}{2}|u(x)-u(y)|^2 + e(x)+e(y)\Big)\rho(x)\rho(y)dydx. \tag{5.6}$$

The functional $\Gamma(t)$ can be expressed in terms of the moments $\mathcal{M}_i$ in (3.9) (corresponding to the splitting of its integrand $\frac{1}{2}|u(x)-u(y)|^2+e(x)+e(y) \equiv \rho(x)E(x)+\rho(y)E(y)-u(x)\cdot u(y)$),

$$\Gamma(t) \equiv 2\mathcal{E}(t)\mathcal{M}_0 - |\mathcal{M}_1|^2, \qquad \mathcal{E}(t) = \frac{1}{2}\mathcal{M}_2(t).$$

Since $\mathcal{M}_i, i=0,1$ are constants, this reveals that $\Gamma(t)$ is essentially the total energy. We arrive at the main theorem of this section.

Theorem 5.2. *Assume (ρ,u,e) is a smooth solution of the system (5.3), $(\rho,u,e)\in C^1(\mathbb{R}^d\times[0,T))$. Then we have*

$$\Gamma(t) \le \Gamma(0)e^{-2\mathcal{M}_0\lambda\Phi(t)}, \qquad \Phi(t) = \int_0^t \varphi(s)ds, \quad \varphi(s) := \inf_{(x_0,y_0)} r\big(x(s),y(s)\big)ds.$$

Here, $\mathcal{M}_0$ is the initial total mass, $\mathcal{M}_0 = \|\rho_0\|_{L^1}$ and the infimum is taken over all particle trajectories, $(x_0,y_0)\mapsto(x(s),y(s))$.

Proof. We use (5.5a) and the relation $\Gamma(t) = 2\mathcal{E}(t)\mathcal{M}_0 - |\mathcal{M}_1|^2$ to find

$$\begin{aligned}
\frac{d}{dt}\Gamma(t) &= -\lambda\mathcal{M}_0\textstyle\int_{\mathbb{R}^{2d}} r(x,y)\Big(|u(x)-u(y)|^2+2e(x)+2e(y)\Big)\rho(x)\rho(y)dydx \quad (5.7)\\
&\le -\lambda\mathcal{M}_0\varphi(t)\textstyle\int_{\mathbb{R}^{2d}}\Big(|u(x)-u(y)|^2+2e(x)+2e(y)\Big)\rho(x)\rho(y)dydx\\
&= -2\lambda\mathcal{M}_0\varphi(t)\Gamma(t).
\end{aligned}$$

Gronwall's inequality then yields the desired result. □

We conclude that whenever the interparticle interaction, $\varphi(s) = \inf r(x(s),y(s))$ decays *slowly* enough so that its primitive, $\Phi(t)$, diverges, then flocking occurs, $\rho(x)\rho(y)|u(x,t)-u(y,t)|\to 0$, in agreement with the flocking behavior of the C-S particle model, consult remark 2.2. It is remarkable that the emergence of flocking is deduced here *independently* of the constitutive relation for P. In this context we observe that energy dissipation, driven by the negative source term $\mathcal{S}^{(2)}$ in (5.3c) vanishes as $t \to \infty$. Indeed, theorem 5.2 tells us that by (5.4b),

$$|\mathcal{S}^{(2)}| \le A\Gamma(t) \to 0.$$

6. **Epilogue: flocking dissipation and entropy.** We have seen that the self-propelled flocking dynamics is driven by energy dissipation. The dissipation mechanism reveals itself through energy decay in the particle description (2.7), in the kinetic description (3.8b) and equivalently, in the hydrodynamic description (5.7). Observe that,

$$\Gamma(t) = \mathcal{M}_2\mathcal{M}_0 - |\mathcal{M}_1|^2 \ge 0. \tag{6.8}$$

The right of (6.8) is the usual Cauchy-Schwartz inequality

$$\Big|\int_{\mathbb{R}^{2d}} vf(x,v)dvdx\Big|^2 \le \int_{\mathbb{R}^{2d}} |v|^2 f(x,v)dvdx \times \int_{\mathbb{R}^{2d}} f(x,v)dvdx.$$

Thus, by theorem 5.2, $\Phi(t) \to \infty$ implies time asymptotic flocking by letting $\Gamma(t) \to 0$ which in turn, enforces an approximate Cauchy-Schwartz *equality*. It then follows that v approaches the bulk velocity, $v \to u_c$ as $t \to \infty$. We refer to this mechanism as flocking dissipation. It is intimately related to the entropy *increase* in the kinetic model (3.6). To this end we compute

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^{2d}} f\log(f)dxdv &= -\lambda\int_{\mathbb{R}^{4d}} r(x,y)\nabla_v \log(f)\cdot(v-v_*)ff_*dv_*dvdydx \\
&= -\lambda\int_{\mathbb{R}^{4d}} r(x,y)\nabla_v f\cdot(v-v_*)f_*dv_*dvdydx \\
&= \lambda\int_{\mathbb{R}^{4d}} r(x,y)ff_*dv_*dvdydx = \lambda\int_{\mathbb{R}^{4d}} r(x,y)\rho(x)\rho(y)dydx.
\end{aligned}$$

This is a reversed H-theorem. Entropy increases due to the "improbable" statistical behavior of particles with increasingly highly correlated velocities, as they flock towards particle-independent bulk velocity.

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