

# Linearized model Fokker–Planck collision operators for gyrokinetic simulations.

## I. Theory

I. G. Abel,<sup>1,2,\*</sup> M. Barnes,<sup>3,†</sup> S. C. Cowley,<sup>1,2</sup> W. Dorland,<sup>3</sup> G. W. Hammett,<sup>4</sup> and A. A. Schekochihin<sup>1</sup>

<sup>1</sup>*Plasma Physics Group, Blackett Laboratory, Imperial College, London SW7 2AZ, UK*

<sup>2</sup>*Euratom/UKAEA Fusion Association, Culham Science Centre, Abingdon OX14 3DB, UK*

<sup>3</sup>*Department of Physics, IREAP and CSCAMM,*

*University of Maryland, College Park, Maryland 20742-3511*

<sup>4</sup>*Princeton Plasma Physics Laboratory, Princeton, New Jersey 08543-0451*

(Dated: August 23, 2008)

A new analytically and numerically manageable model collision operator is developed specifically for turbulence simulations. The like-particle collision operator includes both pitch-angle scattering and energy diffusion and satisfies the physical constraints required for collision operators: it conserves particles, momentum and energy, obeys Boltzmann’s  $H$ -theorem (collisions cannot decrease entropy), vanishes on a Maxwellian, and efficiently dissipates small-scale structure in the velocity space. The process of transforming this collision operator into the gyroaveraged form for use in gyrokinetic simulations is detailed. The gyroaveraged model operator is shown to have more suitable behavior at small scales in phase space than previously suggested models. A model operator for electron-ion collisions is also presented.

### I. INTRODUCTION

It has long been known that in many turbulent systems the differences between vanishingly small dissipation and neglecting dissipation completely are striking, and that this can be linked theoretically to the non-interchangeability of limits  $t \rightarrow \infty$  and  $\nu \rightarrow 0$ , where  $\nu$  is e.g. viscosity, resistivity or collision frequency. Physically, the dissipation is important in turbulence for the following reason. The fundamental property of turbulence is to transfer energy from scales at which it is injected into the system to scales where it is dissipated, leading to heating. When the dissipation coefficients are small, the system has to generate very fine-scale fluctuations in order to transfer the energy to scales at which dissipation becomes efficient.

In plasma turbulence, all dissipation (meaning any effect that leads to irreversible heating) is ultimately collisional, so the transfer of energy generally occurs in phase space — i.e., both in the position and velocity space (see extended discussion of energy cascade in plasma turbulence in Ref. 1 and references therein). There are a number of specific mechanisms, both linear and nonlinear, that give rise to phase-space mixing.<sup>1–7</sup> It is the resulting large gradients in the velocity space that eventually bring collisions into play however small the collision frequency is (such small-scale velocity-space structure has, e.g., been found and explicitly measured in gyrokinetic simulations<sup>6,8–10</sup>). Thus, in any plasma turbulence simulation, some effective collisionality has to be present in order to smooth the the small-scale structure in the velocity.

While one may take the view that the numerical grid can play the role of effective collisions,<sup>11</sup> we consider it to be a safer course of action to model collisional physics in a controlled fashion. In order to explain why, we would

like to emphasize that, besides velocity-space smoothing, there is another key reason why collisions must be included. Collisions, through the dissipation of small-scale fluctuations in phase space, provide the link between irreversible plasma heating (macroscopic transport) and turbulence, so they are necessary in order for the system to converge to a statistically steady state. We shall now explain this statement.

Consider the “ $\delta f$  kinetics,” i.e., assume that it is physically reasonable to split the distribution function into a slowly (both spatially and temporally) varying equilibrium part and a rapidly varying fluctuating part:  $f = F_0 + \delta f$ . We further assume that  $F_0$  is a Maxwellian distribution,  $F_0 = (n_0/\pi^{3/2}v_{\text{th}}^3)\exp(-v^2/v_{\text{th}}^2)$ , where  $n_0$  is density,  $v_{\text{th}} = (2T_0/m)^{1/2}$  the thermal speed,  $T_0$  temperature and  $m$  the particle mass. This will be the case if collisions are not extremely weak (for the weakly collisional formulation of  $\delta f$  gyrokinetics, see Ref. 12). One can show that the fundamental energy balance governing the evolution of the turbulent fluctuations is<sup>1,4,6,7,12–15</sup>

$$\frac{d}{dt} \left( - \sum_s T_{0s} \delta S_s + U \right) = P + \sum_s \iint \frac{T_{0s} \delta f_s}{F_{0s}} C[\delta f_s] d\mathbf{v} d\mathbf{r}, \quad (1)$$

where  $s$  is the species index,  $\delta S = - \iint d\mathbf{r} d\mathbf{v} \delta f^2 / 2F_0$  is the entropy of the fluctuations,  $U = \int d\mathbf{r} (E^2 + B^2) / 8\pi$  is the energy of the (fluctuating) electromagnetic field,  $P$  is the input power (energy source of the turbulence), and  $C[\delta f]$  is the linearized collision operator. In many types of plasma turbulence studied in fusion contexts, the input power  $P$  is proportional to the heat flux and it is the parameter dependence of the mean value of this quantity in the statistically stationary state that is sought as the

principal outcome of the simulations. We can see immediately from the above equation that collisions are required to achieve such a steady state (as has been shown in numerical simulations<sup>5,6,11,16</sup>) and that in this steady state,  $P$  must be balanced on the average by the collisional dissipation term.

The key property of the collision operator required for this transfer of energy from turbulence to the equilibrium distribution to work correctly and, therefore, for the heat fluxes to converge to correct steady-state values, is that the collision term in Eq. (1) must be negative-definite:

$$\iint \frac{\delta f}{F_0} C[\delta f] dr dv \leq 0. \quad (2)$$

This ensures that heating is irreversible and that collisions cannot decrease entropy, the latter being the statement of Boltzmann's  $H$ -theorem.<sup>17</sup> Any spurious sink of entropy will adversely affect the balance between turbulent fluxes and dissipation, so it is clear that any model for the collisional dissipation must respect the  $H$ -theorem.

In view of the above discussion, we can formulate a reasonably restrictive set of criteria for any model collision operator: providing dissipation at small scales, obeying the  $H$ -theorem [Eq. (2)], and also, obviously, conserving particle number, momentum, energy, and vanishing on a (local, perturbed) Maxwellian distribution. Whilst these properties are analytically convenient, for numerical simulations the operator should also be efficiently implementable and carry these properties (at least approximately) over to the numerical scheme.

The effect of small angle Coulomb collisions on an arbitrary distribution function was originally calculated by Landau.<sup>18</sup> In the  $\delta f$  kinetics we would naturally consider the linearized Landau operator<sup>19</sup>. However, it is sufficiently complex that it would exceed the limits on numerical resources that can be realistically expended on modelling the collisional physics. Consequently several simplified model collision operators have been developed, both for analytical and computational convenience, that try to capture the qualitative essence, if not the quantitative detail, of the physics involved.<sup>20-22</sup> This course of action is, indeed, eminently sensible: from Eq. (1), it seems plausible that, at least as far as calculating integral characteristics such as the turbulent fluxes is concerned, neither the exact functional form of the collision operator (provided it satisfies the criteria discussed above) nor the exact value of the collision frequency (provided it is sufficiently small) should be important. All we need is a physically reasonable dissipation mechanism.

For these purposes, it has often been deemed sufficient to use the pitch-angle-scattering (Lorentz) operator, sometimes adjusted for momentum conservation.<sup>19,20</sup> However, in kinetic turbulence, there is no reason that small-scale velocity-space structure should be restricted to pitch angles. In fact, standard phase-mixing mechanisms applied to gyrokinetics produce structure in  $v_{\parallel}$ ,<sup>2,6</sup> and there is also a nonlinear gyrokinetic phase mixing

that gives rise to structure in  $v_{\perp}$ , which may be an even faster and more efficient process.<sup>1,3,7</sup> Thus, *a priori* one expects to see small scales both in pitch angle and in the energy variable ( $\xi$  and  $v$ ). It has, indeed, been confirmed in simulations<sup>9</sup> that with only Lorentz scattering structure rapidly forms at the grid scale in energy. Thus, a numerically suitable model collision operator must include energy diffusion.

In this paper, we propose such an operator (other operators including energy diffusion have been previously suggested;<sup>21,22</sup> we include a detailed comparison of our operator with these in Appendix C). Our model operator for like-particle collisions, including both pitch-angle scattering and energy diffusion and satisfying all of the physical constraints discussed above, is given in Sec. II (the proof of the  $H$ -theorem for it is presented in Appendix A). In Sec. III, it is converted (gyroaveraged) into the form suitable for use in gyrokinetic simulations — a procedure that produces some nontrivial modifications. In Sec. IV, we explain how interspecies (and, in particular, electron-ion) collisions can be modelled in gyrokinetic simulations to ensure that such effects as resistivity are correctly captured. Section V contains a short summary and a discussion of the consequences of the work presented here.

The analytical developments presented in this paper form the basis for the numerical implementation of collisions in the publicly available gyrokinetic code GS2. This numerical implementation, as well as a suite of numerical tests are presented in the companion paper, Ref. 9 (henceforth Paper II).

## II. A NEW MODEL COLLISION OPERATOR

In this section, we present a new model collision operator for like-particle collisions that satisfies the criteria stated above. The interspecies collisions will be considered in Sec. IV.

Let us start by introducing some standard notation. In discussing collision operators on phase space, we shall denote  $\mathbf{r}$  the position variable in the physical space and use the  $(v, \xi, \vartheta)$  coordinates in velocity space, where  $v = |\mathbf{v}|$  is the energy variable,  $\xi = v_{\parallel}/v$  is the pitch-angle variable, and  $\vartheta$  the gyroangle about the equilibrium magnetic field. One can easily adapt the operators presented here to unmagnetized plasmas, but as we are interested in gyrokinetic plasmas, we shall concentrate on the strongly magnetized case. Taking the notation of Ref. 19 as the standard, we introduce the normalized velocity variable  $x = v/v_{th}$  and a set of velocity-dependent collision fre-

quencies for like-particle collisions:

$$\nu_D(v) = \nu \frac{\Phi(x) - G(x)}{x^3}, \quad (3)$$

$$\nu_s(v) = \nu \frac{4G(x)}{x}, \quad (4)$$

$$\nu_{\parallel}(v) = \nu \frac{2G(x)}{x^3} = \frac{1}{2x^2} \nu_s, \quad (5)$$

$$\Delta\nu(v) = \nu_D - \nu_s = \frac{1}{2v^3 F_0} \frac{\partial}{\partial v} v^4 \nu_{\parallel} F_0, \quad (6)$$

$$\nu_E(v) = -2\Delta\nu - \nu_{\parallel} = -\frac{1}{v^4 F_0} \frac{\partial}{\partial v} v^5 \nu_{\parallel} F_0, \quad (7)$$

where  $\Phi(x) = (2/\sqrt{\pi}) \int_0^x e^{-y^2} dy$  is the error function,  $G(x) = [\Phi(x) - x\Phi'(x)]/2x^2$  is the Chandrasekhar function, and  $\nu = \sqrt{2\pi} n_0 q^4 \ln \Lambda T_0^{-3/2} m^{-1/2}$  is the dimensional like-particle collision frequency (here  $\ln \Lambda$  is the Coulomb logarithm and  $q$  is the particle charge). Note that the two differential identities given in Eqs. (6) and (7) will prove very useful in what follows.

If one wishes to construct a model linearized collision operator, the following general form constitutes a natural starting point

$$C[\delta f] = \frac{\partial}{\partial \mathbf{v}} \cdot \left[ \hat{D}(\mathbf{v}) \cdot \frac{\partial \delta f}{\partial \mathbf{v}} \frac{1}{F_0} \right] + P[\delta f](\mathbf{v}) F_0, \quad (8)$$

where the first term is the ‘‘test-particle’’ collision operator and the second term the ‘‘field-particle’’ operator. Most model operators can be obtained by picking a suitably simple form for the velocity-space diffusion tensor  $\hat{D}$  and the functional  $P$ , subject to the constraints that one chooses to impose on the model operator.

In constructing our model operator, we retain the exact form of  $\hat{D}$  for the linearized Landau collision operator:<sup>19</sup>

$$C[\delta f] = \nu_D L[\delta f] + \frac{1}{v^2} \frac{\partial}{\partial v} \left( \frac{1}{2} v^4 \nu_{\parallel} F_0 \frac{\partial \delta f}{\partial v} \frac{1}{F_0} \right) + P[\delta f](\mathbf{v}) F_0, \quad (9)$$

where we have explicitly separated the energy-diffusion part (the second term) and the angular part (the first term), which includes pitch-angle scattering and is described by the Lorentz operator:

$$L[\delta f] = \frac{1}{2} \left[ \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial \delta f}{\partial \xi} + \frac{1}{1 - \xi^2} \frac{\partial^2 \delta f}{\partial \theta^2} \right]. \quad (10)$$

Our modelling choice is to pick  $P$  to be of the form

$$P[\delta f](\mathbf{v}) = \nu_s \frac{2\mathbf{v} \cdot \mathbf{U}[\delta f]}{v_{\text{th}}^2} + \nu_E \frac{v^2}{v_{\text{th}}^2} Q[\delta f]. \quad (11)$$

One can view this prescription as first expanding  $P$  in spherical harmonics (one can easily show that they are eigenfunctions of the full field-particle operator), retaining only the first two terms, and then arbitrarily factorizing the explicit  $v$  and  $\delta f$  dependence of each harmonic. The functionals  $\mathbf{U}[\delta f]$  and  $Q[\delta f]$  are mandated to have no explicit velocity dependence. In this ansatz the  $v$  dependence is chosen so that the final operator is self adjoint and also to ensure automatic particle conservation by the field-particle operator:  $\int P[\delta f](\mathbf{v}) F_0 d\mathbf{v} = 0$ . Indeed the first term in Eq. (11) gives a vanishing contribution to this integral because it is proportional to  $\mathbf{v}$ , and so does the second term because of the differential identity given in Eq. (7). The functionals  $\mathbf{U}[\delta f]$  and  $Q[\delta f]$  are now uniquely chosen so as to ensure that the model operator conserves momentum and energy: a straightforward calculation gives

$$\mathbf{U}[\delta f] = \frac{3}{2} \int \nu_s \mathbf{v} \delta f d\mathbf{v} \bigg/ \int (v/v_{\text{th}})^2 \nu_s F_0 d\mathbf{v}, \quad (12)$$

$$Q[\delta f] = \int v^2 \nu_E \delta f d\mathbf{v} \bigg/ \int v^2 (v/v_{\text{th}})^2 \nu_E F_0 d\mathbf{v}. \quad (13)$$

These are in fact just the standard correction expressions used for the model pitch-angle-scattering operator<sup>19,20</sup> and for more complex operators including energy diffusion.<sup>21</sup>

Note that the numerical implementation of our collision operator documented in Paper II<sup>9</sup> achieves exact satisfaction of the conservation laws by choosing the discretization scheme that exactly captures the differential identities Eqs. (6) and (7) and the double integration by parts needed in deriving Eqs. (12) and (13).

To summarize, we now have the following model operator for like particle collisions:

$$C[\delta f] = \frac{\nu_D}{2} \left[ \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial \delta f}{\partial \xi} + \frac{1}{1 - \xi^2} \frac{\partial^2 \delta f}{\partial \theta^2} \right] + \frac{1}{v^2} \frac{\partial}{\partial v} \left( \frac{1}{2} v^4 \nu_{\parallel} F_0 \frac{\partial \delta f}{\partial v} \frac{1}{F_0} \right) + \nu_s \frac{2\mathbf{v} \cdot \mathbf{U}[\delta f]}{v_{\text{th}}^2} F_0 + \nu_E \frac{v^2}{v_{\text{th}}^2} Q[\delta f] F_0, \quad (14)$$

where the functionals  $\mathbf{U}[\delta f]$  and  $Q[\delta f]$  are given by Eqs. (12) and (13). The modelling choice of the field-particle operator that we have made [Eq. (11)] means that, in

order to compute our collision operator, we have only to calculate definite integrals over the entirety of the velocity space — a significant simplification in terms of

computational complexity and ease of use in numerical simulations (see Paper II<sup>9</sup>).

As we have shown above, our operator conserves particles, momentum and energy by construction. It is also not hard to see that it vanishes precisely when  $\delta f/F_0 = 1, \mathbf{v}, v^2$  and linear combinations thereof, i.e. if  $\delta f$  is a perturbed Maxwellian. From this and the fact that the operator is self adjoint, it can be easily shown that the operator *only* conserves particles, momentum and energy and that no spurious conservation laws have been introduced by our model. Because the operator contains the exact test-particle part, it provides velocity-space diffusion both in energy and in angle and thus will efficiently dissipate small-scale structure. Finally, it satisfies the  $H$ -theorem — this is proved in Appendix A.

This operator thus fulfills the criteria set forth in Sec. I to be satisfied by any physically reasonable model operator. We now proceed to convert this operator into a form suitable for use in gyrokinetics.

### III. COLLISIONS IN GYROKINETICS

The gyrokinetic theory is traditionally derived for a collisionless plasma.<sup>23,24</sup> However, as we have argued in Sec. I, even when the collision frequency is small, collisions must be included in order to regularize the phase space and to ensure convergence of fluxes to statistically stationary values. Mathematically, collisions can be included in gyrokinetics if the collision frequency is formally ordered to be comparable to the fluctuation frequency,<sup>12</sup>  $\nu \sim \omega \sim k_{\parallel} v_{\text{th}}$  — the weakly collisional limit (collisionality larger than this leads simply to fluid equations). In practice, collision frequency tends to be smaller than the fluctuation frequency, but this need not upset the formal ordering as long as it is not too small: the cases  $\nu \gg \omega$  and  $\nu \ll \omega$  can be treated as subsidiary limits.<sup>7</sup>

Under the formal ordering  $\nu \sim \omega$  (and, in fact, also under an even less restrictive ordering allowing for even smaller collisions<sup>31</sup>), it is possible to show that the equilibrium distribution function (lowest order in the gyrokinetic expansion) is a Maxwellian<sup>12</sup> and the full distribution function can be represented as

$$f = \left(1 - \frac{q\varphi}{T_0}\right) F_0 + h(t, \mathbf{R}, \mu, \varepsilon), \quad (15)$$

where  $F_0$  is a Maxwellian,  $\varphi$  the electrostatic potential (a

fluctuating quantity) and  $h$  the (perturbed) distribution function of the particle guiding centers. Here  $\varepsilon = mv^2/2$  is the particle energy,  $\mu = mv_{\perp}^2/2B_0$  the first adiabatic invariant,  $B_0$  the strength of the equilibrium magnetic field,  $\mathbf{R} = \mathbf{r} - \boldsymbol{\rho} = \mathbf{r} - \hat{\mathbf{b}} \times \mathbf{v}/\Omega$  the guiding center position,  $\Omega$  the cyclotron frequency, and  $\hat{\mathbf{b}} = \mathbf{B}_0/B_0$ . The gyrokinetic equation, written in general geometry and including the collision operator is then

$$\begin{aligned} \frac{\partial h}{\partial t} + (v_{\parallel} \hat{\mathbf{b}} + \mathbf{v}_D) \cdot \frac{\partial h}{\partial \mathbf{R}} + \frac{c}{B_0} \{ \langle \chi \rangle_{\mathbf{R}}, h \} \\ = -q \frac{\partial F_0}{\partial \varepsilon} \frac{\partial \langle \chi \rangle_{\mathbf{R}}}{\partial t} + \frac{c}{B_0} \{ F_0, \langle \chi \rangle_{\mathbf{R}} \} + C_{\text{GK}}[h], \end{aligned} \quad (16)$$

where  $\chi = \varphi - \mathbf{v} \cdot \mathbf{A}/c$  the gyrokinetic potential,  $\langle \chi \rangle_{\mathbf{R}} = (1/2\pi) \int \chi(\mathbf{R} + \boldsymbol{\rho}) d\vartheta$  is an average over gyroangles holding  $\mathbf{R}$  fixed (the ‘‘gyroaverage’’),  $\mathbf{v}_D$  is the guiding center drift velocity.

The gyrokinetic collision operator  $C_{\text{GK}}[h]$  is the gyroaverage of the linearized collision operator. The latter acts on the perturbed distribution  $h$  holding the particle position  $\mathbf{r}$  (not the guiding center  $\mathbf{R}$ !) fixed. The latter nuance must be kept in mind working out the explicit form of  $C_{\text{GK}}[h(\mathbf{R})]$  from the unaveraged linearized operator  $C[h(\mathbf{r} - \boldsymbol{\rho})]$ .

Let us restrict our consideration to local simulations, which are carried out in a flux tube of long parallel extent, but short perpendicular extent. In such simulations, one assumes that the equilibrium profiles are constant across the tube, but have non-zero gradients across the tube so as to keep all the appropriate drifts and instabilities. This permits one to use periodic boundary conditions and perform the simulations spectrally perpendicular to field lines.<sup>25</sup> Thus

$$h = \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{R}} h_{\mathbf{k}}(l, v, \mu), \quad (17)$$

where  $l$  is a coordinate along the field line and the Fourier transform is understood to be only with respect to the perpendicular components of  $\mathbf{R}$ , i.e.,  $\mathbf{k} \equiv \mathbf{k}_{\perp}$ . Treating the perpendicular coordinates spectrally confines all dependence on the gyroangle  $\vartheta$  to the exponent, thus we can compute the gyroangle dependence explicitly and carry out the gyroaveraging of the collision operator in a particularly transparent analytical way:<sup>7,22</sup>

$$C_{\text{GK}}[h] = \left\langle C \left[ \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{R}} h_{\mathbf{k}} \right] \right\rangle_{\mathbf{R}} = \sum_{\mathbf{k}} \langle e^{i\mathbf{k} \cdot \mathbf{r}} C[e^{-i\mathbf{k} \cdot \boldsymbol{\rho}} h_{\mathbf{k}}] \rangle_{\mathbf{R}} = \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{R}} \langle e^{i\mathbf{k} \cdot \boldsymbol{\rho}} C[e^{-i\mathbf{k} \cdot \boldsymbol{\rho}} h_{\mathbf{k}}] \rangle_{\mathbf{R}}, \quad (18)$$

where  $\boldsymbol{\rho} = \hat{\mathbf{b}} \times \mathbf{v}_{\perp}/\Omega$ . Thus, in Fourier space

$$C_{\text{GK}}[h_{\mathbf{k}}] = \langle e^{i\mathbf{k} \cdot \boldsymbol{\rho}} C[e^{-i\mathbf{k} \cdot \boldsymbol{\rho}} h_{\mathbf{k}}] \rangle, \quad (19)$$

where  $\langle \dots \rangle$  refers to the explicit averaging over the  $\vartheta$

dependence. Some general properties of this operator are discussed in Appendix B of Ref. 7.

We now apply the general gyroaveraging formula Eq. (19) to our model operator given by Eq. (14). The gyrokinetic transformation of variables  $(\mathbf{r}, v, \xi, \vartheta) \rightarrow (\mathbf{R}, \mu, \varepsilon, \vartheta)$  mixes position and velocity space. However, in the collision operator, to the lowest order in the gyrokinetic expansion, we can neglect spatial dependence of  $\mu$  that

comes via the equilibrium magnetic field  $B_0(\mathbf{r})$  and thus use the  $(v, \xi)$  velocity variables. After some straightforward algebra, which involves converting velocity derivatives at constant  $\mathbf{r}$  to those at constant  $\mathbf{R}$  and evaluating the arising gyroaverages as detailed in Appendix B, we arrive at the following model gyrokinetic collision operator

$$C_{\text{GK}}[h_{\mathbf{k}}] = \frac{\nu_D}{2} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial h_{\mathbf{k}}}{\partial \xi} + \frac{1}{v^2} \frac{\partial}{\partial v} \left( \frac{1}{2} v^4 \nu_{\parallel} F_0 \frac{\partial h_{\mathbf{k}}}{\partial v} \right) - \frac{1}{4} [\nu_D (1 + \xi^2) + \nu_{\parallel} (1 - \xi^2)] \frac{v^2}{v_{\text{th}}^2} k_{\perp}^2 \rho^2 h_{\mathbf{k}} + 2\nu_s \frac{v_{\perp} J_1(a) U_{\perp}[h_{\mathbf{k}}] + v_{\parallel} J_0(a) U_{\parallel}[h_{\mathbf{k}}]}{v_{\text{th}}^2} F_0 + \nu_E \frac{v^2}{v_{\text{th}}^2} J_0(a) Q[h_{\mathbf{k}}] F_0, \quad (20)$$

where  $\rho = v_{\text{th}}/\Omega$  is the thermal Larmor radius (not to be confused with the velocity-dependent  $\rho$ ),  $a = k_{\perp} v_{\perp}/\Omega$ ,  $J_0$  and  $J_1$  are Bessel functions and

$$U_{\perp}[h_{\mathbf{k}}] = \frac{3}{2} \int \nu_s v_{\perp} J_1(a) h_{\mathbf{k}} d\mathbf{v} \Big/ \int (v/v_{\text{th}})^2 \nu_s F_0 d\mathbf{v}, \quad (21)$$

$$U_{\parallel}[h_{\mathbf{k}}] = \frac{3}{2} \int \nu_s v_{\parallel} J_0(a) h_{\mathbf{k}} d\mathbf{v} \Big/ \int (v/v_{\text{th}})^2 \nu_s F_0 d\mathbf{v}, \quad (22)$$

$$Q[h_{\mathbf{k}}] = \int v^2 \nu_E J_0(a) h_{\mathbf{k}} d\mathbf{v} \Big/ \int v^2 (v/v_{\text{th}})^2 \nu_E F_0 d\mathbf{v}. \quad (23)$$

Note that since the position and velocity space are mixed by the gyrokinetic transformation of variables,  $\mathbf{R} = \mathbf{r} - \rho$ , the collision operator now contains not just pitch-angle and  $v$  derivatives but also a spatial perpendicular ‘‘gyrodiffusion’’ term.

It is important to make sure that the operator we have derived behaves in a physically sensible way in the long- and short-wavelength limits. When  $k_{\perp} \rho \ll 1$ , the all finite-Larmor-radius effects, including the gyrodiffusion, disappear and we end up with pitch-angle scattering and energy diffusion corrected for energy and parallel momentum conservation — the drift-kinetic limit. In the opposite limit,  $k_{\perp} \rho \gg 1$ , we can estimate the behavior of our operator by adopting the scaling of the velocity derivatives based on the nonlinear perpendicular phase mixing mechanism for gyrokinetic turbulence proposed in Ref. 1: this produces velocity-space structure with characteristic gradients  $v_{\text{th}} \partial/\partial v_{\perp} \sim k_{\perp} \rho$  (see also Refs. 3,7). With this estimate, we see that all the field-particle terms in the operator are subdominant by a factor of  $(k_{\perp} \rho)^{-3}$ . Thus the operator reduces to the gyrokinetic form of the test-particle Landau operator in this limit. All diffusive terms are also equally large in this scaling, supporting our supposition that energy diffusion needs to be included. These considerations give us some confidence that we correctly model the diffusive aspects of the collisional physics in a short-wavelength turbulent regime. Indeed, if one applies the same estimates to the

full linearized Landau operator, the Rosenbluth potentials of the perturbation are small when  $k_{\perp} \rho \gg 1$  because they are integrals of a rapidly oscillating function, so the dominant effect does, indeed, come entirely from the test-particle part of the operator.

The gyrokinetic  $H$ -theorem, which has to be satisfied in order for heating and transport to be correctly calculated, is given by<sup>7,12</sup>

$$\iint \frac{h}{F_0} C_{\text{GK}}[h] d\mathbf{R} d\mathbf{v} \leq 0. \quad (24)$$

The gyrokinetic collision operator given by Eq. (20) respects this inequality, as can either be shown directly from Eq. (20) (analogously to the proof in Appendix A) or inferred from Eq. (2) by transforming to gyrokinetic variables. The operator also manifestly diffuses small-scale structure both in velocity and in (perpendicular) position space.

How to express the conservation-law tests upon this operator is a somewhat subtler question. This is because after the gyroaveraging has been done, one cannot explicitly separate the position- and velocity-space dynamics in the gyrokinetic phase space. However, it is still possible to express the evolution of particles, momentum and energy as local conservation laws. Let us take the velocity moments of the gyrokinetic equation Eq. (16) corresponding to these conserved quantities and taken at constant position  $\mathbf{r} = \mathbf{R} + \rho$ . Defining  $\langle \dots \rangle_{\mathbf{r}}$  as the aver-

age over gyroangles while holding  $\mathbf{r}$  fixed and using Eq. (18), we arrive at the following evolution equation for the conserved moments:

$$\begin{aligned}
& \frac{\partial}{\partial t} \int \left( \frac{1}{v^2} \right) \delta f(t, \mathbf{r}, \mathbf{v}) d\mathbf{v} + \nabla \cdot \hat{\Gamma}^{(0)} \\
&= \int \left\langle \left( \frac{1}{v^2} \right) C_{\text{GK}}[h] \right\rangle_{\mathbf{r}} d\mathbf{v} \\
&= \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \int \left\langle \left( \frac{1}{v^2} \right) e^{-i\mathbf{k} \cdot \boldsymbol{\rho}} \right\rangle \langle e^{i\mathbf{k} \cdot \boldsymbol{\rho}} C[e^{-i\mathbf{k} \cdot \boldsymbol{\rho}} h_{\mathbf{k}}] \rangle d\mathbf{v} \\
&= \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \left[ \int \left( \frac{1}{v_{\parallel} \hat{\mathbf{b}}} \right) \langle C[h_{\mathbf{k}}] \rangle d\mathbf{v} - i\mathbf{k} \cdot \hat{\Gamma}_{\mathbf{k}}^{(\text{coll})} \right] \\
&= -\nabla \cdot \hat{\Gamma}^{(\text{coll})}, \tag{25}
\end{aligned}$$

where  $\hat{\Gamma}^{(0)}$  denotes the fluxes arising from the terms in Eq. (16) other than the collision operator. The moments of the gyroaveraged collision operator have been expressed as the moments of the same operator formally taken at  $\mathbf{k} \cdot \boldsymbol{\rho} = 0$  (i.e., dropping all finite-Larmor-radius contributions:  $\langle C[h_{\mathbf{k}}] \rangle$  instead of  $\langle e^{i\mathbf{k} \cdot \boldsymbol{\rho}} C[e^{-i\mathbf{k} \cdot \boldsymbol{\rho}} h_{\mathbf{k}}] \rangle$ ) plus the divergence of the collisional flux arising from the finite-Larmor-radius part of  $C_{\text{GK}}[h]$ . This representation was achieved by representing all gyrophase factors in Eq. (25) as infinite Taylor series and noticing that they take the form  $e^{\pm i\mathbf{k} \cdot \boldsymbol{\rho}} = 1 \pm i\mathbf{k} \cdot \boldsymbol{\rho} - (1/2)(\mathbf{k} \cdot \boldsymbol{\rho})^2 + \dots = 1 - i\mathbf{k} \cdot [\dots]$ , where the square brackets contain the rest of the series. Without the finite-Larmor-radius terms, the particle, momentum and energy moments of the gyrokinetic collision operator vanish,

$$\int \left( \frac{1}{v_{\parallel}} \right) \langle C[h_{\mathbf{k}}] \rangle d\mathbf{v} = 0, \tag{26}$$

and Eq. (25) represents the local conservation law for these quantities, with the fluxes containing both collisionless and collisional contributions:  $\hat{\Gamma} = \hat{\Gamma}^{(0)} + \hat{\Gamma}^{(\text{coll})}$ . Thus, a conservative numerical implementation of the gyrokinetic collision operator can be achieved if Eq. (26) is hard-wired into the numerical scheme. How to do this is explained in Paper II,<sup>9</sup> where we also demonstrate the correct performance of our model operator on a number of test problems.

#### IV. ELECTRON-ION COLLISIONS

Let us now turn to the collisions between different species and focus on a plasma containing only electrons

and one species of ions with a mass ratio  $m_e/m_i \ll 1$ . The smallness of the mass ratio allows for a significant simplification of the interspecies collision terms. Since, ion-electron collisions are subdominant to the ion-ion ones,<sup>19</sup>  $\nu_{ie}/\nu_{ii} \sim (m_e/m_i)^{1/2}$ , it is safe for most physical purposes to neglect the ion-electron collisions and effects associated with them (such as the small slow collisional change in the mean ion momentum). Thus, the ion collisions can be modelled using the like-particle operator proposed above [Eq. (20)].

The situation is different for the electron-ion collisions, which are same order in mass ratio as the electron-electron collisions,<sup>19</sup>  $\nu_{ei} \sim \nu_{ee}$ . Thus, the full electron collision operator has two parts:

$$C[\delta f_e] = C_{ee}[\delta f_e] + C_{ei}[\delta f_e]. \tag{27}$$

The electron-electron operator  $C_{ee}[\delta f_e]$  can be modelled by the like-particle operator proposed above [Eq. (14)], the electron-ion collision operator can be expanded in the mass ratio and to lowest order reads<sup>19</sup>

$$C_{ei}[\delta f_e] = \nu_D^{ei} \left( L[\delta f_e] + \frac{2\mathbf{v} \cdot \mathbf{u}_i}{v_{\text{the}}^2} F_{0e} \right), \tag{28}$$

$$\nu_D^{ei}(v) = \nu_{ei} \left( \frac{v_{\text{the}}}{v} \right)^3 \tag{29}$$

where  $\nu_{ei} = \sqrt{2}\pi n_{0i} Z^2 e^4 \ln \Lambda T_{0e}^{-3/2} m_e^{-1/2}$  is the dimensional electron-ion collision frequency,  $Z = q_i/e$ ,  $e$  is the electron charge,  $L$  is the Lorentz operator given by Eq. (10), and

$$\mathbf{u}_i = \frac{1}{n_{0i}} \int \mathbf{v} \delta f_i d\mathbf{v} \tag{30}$$

is the ion flow velocity. Thus, the electron-ion collisions are correctly modelled to lowest order in the mass ratio by electron pitch-angle scattering off static ions plus electron drag against the bulk ion flow. Note that the ion drag term is necessary to correctly capture electron-ion friction and hence resistivity; failure to include it leads to incorrect results, with mean electron momentum relaxed towards zero rather than towards equality with the mean ion momentum.

Performing the conversion of  $C_{ei}$  to the gyroaveraged form in a way entirely analogous to what was done in Sec. III and Appendix B, we get

$$C_{\text{GK}}^{ei}[h_{e\mathbf{k}}] = \nu_D^{ei} \left[ \frac{1}{2} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial h_{e\mathbf{k}}}{\partial \xi} - \frac{1}{4} (1 + \xi^2) \frac{v^2}{v_{\text{the}}^2} k_{\perp}^2 \rho_e^2 h_{e\mathbf{k}} + \frac{2v_{\parallel} J_0(a_e) u_{\parallel i\mathbf{k}}}{v_{\text{the}}^2} F_{0e} - \frac{Zm_e}{m_i} \frac{v_{\perp}^2}{v_{\text{the}}^2} \frac{J_1(a_e)}{a_e} F_{0e} k_{\perp}^2 \rho_i^2 \frac{1}{n_{0i}} \int \frac{2v_{\perp}'^2}{v_{\text{thi}}^2} \frac{J_1(a_i')}{a_i'} h_{i\mathbf{k}}(\mathbf{v}') d\mathbf{v}' \right], \quad (31)$$

where

$$u_{\parallel i\mathbf{k}} = \frac{1}{n_{0i}} \int v_{\parallel} J_0(a_i) h_{i\mathbf{k}} d\mathbf{v} \quad (32)$$

and  $a_s = k_{\perp} v_{\perp} / \Omega_s$  for species  $s$  and the rest of the notation as the same as in previous sections, with species indices this time.

Let us estimate the size of the four terms in Eq. (31) at the ion (long) and electron (short) scales. The first term (pitch-angle scattering) is always important. At the ion scales,  $k_{\perp} \rho_i \sim 1$ , the third term (parallel ion drag) is equally important, while the second term (electron gyrodiffusion) and the fourth term are subdominant by a factor of  $m_e/m_i$ . At the electron scales,  $k_{\perp} \rho_e \sim 1$ , the pitch-angle scattering and the electron gyrodiffusion (the first two terms) are both important.

Since at these scales  $k_{\perp} \rho_i \sim (m_i/m_e)^{1/2} \gg 1$ , the third and fourth terms are subdominant by a factor (resulting from the Bessel functions under the velocity integrals) of  $1/\sqrt{k_{\perp} \rho_i} \sim (m_e/m_i)^{1/4}$ . In fact, they are smaller than this estimate because at these short wavelengths, the ion distribution function has small-scale structure in velocity space with characteristics scales  $\delta v_{\perp} / v_{\text{thi}} \sim 1/k_{\perp} \rho_i$ , which leads to the reduction of the velocity integrals by another factor of  $1/\sqrt{k_{\perp} \rho_i}$ . Thus, at the electron scales, the third and fourth terms in Eq. (31) are subdominant by a factor of  $(m_e/m_i)^{1/2}$ .

These considerations mean that the fourth term in Eq. (31) is always negligible and can safely be dropped. The full model gyrokinetic electron collision operator is, therefore,

$$C_{\text{GK}}[h_{e\mathbf{k}}] = C_{\text{GK}}^{ee}[h_{e\mathbf{k}}] + \nu_D^{ei} \left[ \frac{1}{2} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial h_{e\mathbf{k}}}{\partial \xi} - \frac{1}{4} (1 + \xi^2) \frac{v^2}{v_{\text{the}}^2} k_{\perp}^2 \rho_e^2 h_{e\mathbf{k}} + \frac{2v_{\parallel} J_0(a_e) u_{\parallel i\mathbf{k}}}{v_{\text{the}}^2} F_{0e} \right], \quad (33)$$

where the electron-electron model operator  $C_{\text{GK}}^{ee}[h_{e\mathbf{k}}]$  is given by Eq. (20) and  $u_{\parallel i\mathbf{k}}$  by Eq. (32).

Finally, we note that since  $en_{0e}(u_{\parallel i} - u_{\parallel e}) = j_{\parallel}$  is the parallel current, the parallel Ampère's law can be used to express  $u_{\parallel i}$  in Eq. (33) in a form that does not contain an explicit dependence on the ion distribution function:

$$u_{\parallel i\mathbf{k}} = \frac{1}{n_{0e}} \int v_{\parallel} J_0(a_e) h_{e\mathbf{k}} d\mathbf{v} + \frac{c}{4\pi en_{0e}} k_{\perp}^2 A_{\parallel \mathbf{k}}. \quad (34)$$

This turns out to be useful in the numerical implementation of the electron operator, detailed in Paper II.<sup>9</sup>

## V. SUMMARY

Thus in Sec. I we have argued the necessity of dissipation in turbulence simulations, justified the direct modelling of collisions in order to provide such dissipation and postulated a set of constraints for a physically reasonable model collision operator. Previously used model operators mostly do not contain energy diffusion and were thus deemed unsatisfactory for these purposes. Of the existing model operators that contain energy diffusion, two

are detailed in Ref. 22 and Ref. 21. The former however does not satisfy the  $H$ -theorem [Eq. (2)] and the latter incorrectly captures the smallest scales. These problems are demonstrated and discussed in detail in Appendix C.

In Sec. II we presented a new operator [Eq. (14)] that successfully introduces energy diffusion whilst maintaining the  $H$ -Theorem and conservation laws, thus satisfying the conditions set forth in the introduction. This operator is then transformed into gyrokinetic form in Sec. III correctly accounting for the gyrodiffusive terms and FLR effects [Eq. (20)]. In order to provide a complete recipe for modelling the collisional effects in simulations, the same gyroaveraging procedure is applied in Sec. IV to electron-ion collisions, somewhat simplified by the mass-ratio expansion [Eq. (33)]. This leaves us with a complete picture of collisions in gyrokinetic simulations, capturing gyrodiffusion, resistivity and small-scale energy diffusion.

When we discussed the gyroaveraging procedure in Sec. III we presented the specific case of the application to Eulerian flux-tube  $\delta f$  gyrokinetic simulations.<sup>26,27</sup> However, the form presented in Eq. (14) is suitable for inclusion in most  $\delta f$  kinetic systems and even amenable to use in Lagrangian codes by applying the methods of

Refs. 28 or 29 to the gyroaveraged operator given by Eq. (20). Indeed, by suitable discretization of the gyroaveraging procedure<sup>27</sup> it would also be usable in a global Eulerian code.

We conclude by noting that the final arbiter of the practicality and effectiveness of this collision model is the numerical implementation and testing performed in Paper II,<sup>9</sup> where our operator is integrated into the GS2 code. The battery of tests shows that our operator not only reproduces the correct physics in the weakly collisional regime but even allows a gyrokinetic code to capture correctly the collisional (reduced-MHD) limit.

### Acknowledgments

We thank J. Hastie, D. Ernst, P. Ricci, C. Roach, B. Rogers and T. Tatsuno for useful discussions. I.G.A. was supported by a CASE EPSRC studentship in association with UKAEA Fusion (Culham). M.B. was supported by the US DOE Center for Multiscale Plasma Dynamics. A.A.S. was supported by an STFC (UK) Advanced Fellowship and STFC Grant ST/F002505/1. M.B. and W.D. would also like to thank the Leverhulme Trust (UK) International Network for Magnetized Plasma Turbulence for travel support.

### Appendix A: Proof of The $H$ -Theorem for Eq. (14)

In the case of the expansion  $f = F_0 + \delta f$  about a Maxwellian the entropy generation by like particle collisions takes the form

$$\begin{aligned} \frac{dS}{dt} &= -\frac{d}{dt} \iint f \ln f \, dv \, dr \\ &= -\iint \hat{f} C[\hat{f} F_0] \, dv \, dr \geq 0, \end{aligned} \quad (\text{A1})$$

where we use the compact notation  $\hat{f} = \delta f / F_0$ . The statement of the  $H$ -theorem is that the right-hand side of Eq. (A1) is nonnegative and that it is exactly zero when  $\delta f$  is a perturbed Maxwellian.

We represent  $\hat{f}$  as a Cartesian tensor expansion (or equivalently spherical harmonic expansion) in velocity space:

$$\hat{f}(\mathbf{r}, \mathbf{v}) = \hat{f}_0(\mathbf{r}, v) + \mathbf{v} \cdot \hat{\mathbf{f}}_1(\mathbf{r}, v) + R[\hat{f}](\mathbf{r}, \mathbf{v}), \quad (\text{A2})$$

where  $R[\hat{f}]$  comprises the higher order terms. It is then possible to recast the statement of the  $H$ -Theorem in terms of this expansion using linearity of the model collision operator  $C$  [Eq. (14)], orthogonality of the expansion and the fact that spherical harmonics are eigenfunctions of the Lorentz operator  $L$ . By construction,  $R[\hat{f}]$  satisfies  $\int R[\hat{f}] F_0 \, dv = 0$  and  $\int \mathbf{v} R[\hat{f}] F_0 \, dv = \mathbf{0}$ , from which it follows that  $R[\hat{f}]$  does not contribute to the field-particle parts of the model operator:  $\mathbf{U}[R[\hat{f}] F_0] = \mathbf{0}$  and

$Q[R[\hat{f}] F_0] = 0$ . Substituting Eq. (A2) into the right-hand side of Eq. (A1), where the operator  $C$  is given by Eq. (14), and integrating by parts those terms involving derivatives of  $R[\hat{f}]$ , we find that they all give nonnegative contributions, so we have

$$-\int \hat{f} C[\hat{f} F_0] \, dv \geq \sigma_0 + \sigma_1, \quad (\text{A3})$$

where

$$\sigma_0 = -\int \hat{f}_0 C[\hat{f}_0 F_0] \, dv, \quad (\text{A4})$$

$$\sigma_1 = -\int \mathbf{v} \cdot \hat{\mathbf{f}}_1 C[\mathbf{v} \cdot \hat{\mathbf{f}}_1 F_0] \, dv. \quad (\text{A5})$$

In order to prove the  $H$ -theorem, it is now sufficient to show that  $\sigma_0 \geq 0$  and  $\sigma_1 \geq 0$ .

Starting with  $\sigma_0$  and using Eq. (14), we integrate over angles and use the differential identity given in Eq. (7) to express the term containing  $Q$ :

$$\sigma_0 = -2\pi \int \hat{f}_0 \frac{\partial}{\partial v} \left[ v^4 \nu_{\parallel} F_0 \frac{\partial}{\partial v} \left( \hat{f}_0 - \frac{v^2}{v_{\text{th}}^2} Q[\hat{f}_0 F_0] \right) \right] \, dv, \quad (\text{A6})$$

Using the aforementioned identity again in the expression for  $Q$  [Eq. (13)] and integrating by parts where opportune, we get

$$\begin{aligned} \sigma_0 &= 4\pi \left[ \frac{1}{2} \int \left( \frac{\partial \hat{f}_0}{\partial v} \right)^2 v^4 \nu_{\parallel} F_0 \, dv \right. \\ &\quad \left. - \left( \int \frac{\partial \hat{f}_0}{\partial v} v^5 \nu_{\parallel} F_0 \, dv \right)^2 / \int v^6 \nu_E F_0 \, dv \right] \end{aligned} \quad (\text{A7})$$

It is easy to see from the Cauchy-Schwarz inequality that

$$\left( \int \frac{\partial \hat{f}_0}{\partial v} v^5 \nu_{\parallel} F_0 \, dv \right)^2 \leq \int \left( \frac{\partial \hat{f}_0}{\partial v} \right)^2 v^4 \nu_{\parallel} F_0 \, dv \int v^6 \nu_{\parallel} F_0 \, dv. \quad (\text{A8})$$

Using this in the second term of Eq. (A7), we infer

$$\begin{aligned} \sigma_0 &\geq 4\pi \int \left( \frac{\partial \hat{f}_0}{\partial v} \right)^2 v^4 \nu_{\parallel} F_0 \, dv \\ &\quad \times \left( \frac{1}{2} - \int v^6 \nu_{\parallel} F_0 \, dv / \int v^6 \nu_E F_0 \, dv \right) = 0, \end{aligned} \quad (\text{A9})$$

where to prove that the right-hand side vanishes, we again used the differential identity given in Eq. (7) and integrated by parts. Thus, we have proved that  $\sigma_0 \geq 0$ .

Turning now to  $\sigma_1$  [Eq. (A5)], using Eq. (14), and integrating by parts where opportune, we get

$$\begin{aligned} \sigma_1 &= \int (\mathbf{v} \cdot \hat{\mathbf{f}}_1)^2 \nu_D F_0 \, dv - \frac{1}{2} \int \left( \frac{\partial}{\partial v} \mathbf{v} \cdot \hat{\mathbf{f}}_1 \right)^2 v^2 \nu_{\parallel} F_0 \, dv \\ &\quad - 3v_{\text{th}}^2 \left| \int \mathbf{x} \mathbf{x} \cdot \hat{\mathbf{f}}_1 \nu_s F_0 \, dv \right|^2 / \int x^2 \nu_s F_0 \, dv, \end{aligned} \quad (\text{A10})$$

where we have used the standard notation that  $\mathbf{x} = \mathbf{v}/v_{\text{th}}$  and  $x = v/v_{\text{th}}$ . Integrating over angles and using the simple identity  $\mathbf{a} \cdot \int \hat{\mathbf{v}} \hat{\mathbf{v}} d\Omega = (4\pi/3)\mathbf{a}$ , where  $\hat{\mathbf{v}} = \mathbf{v}/v$  and  $\mathbf{a}$  is an arbitrary vector, we have

$$\sigma_1 = \frac{4\pi v_{\text{th}}^5}{3} \left( \int |\hat{\mathbf{f}}_1|^2 x^4 \nu_D F_0 dx + \frac{1}{2} \int \left| \frac{\partial}{\partial x} x \hat{\mathbf{f}}_1 \right|^2 x^4 \nu_{\parallel} F_0 dx - \left| \int \hat{\mathbf{f}}_1 x^4 \nu_s F_0 dx \right|^2 / \int x^4 \nu_s F_0 dx \right). \quad (\text{A11})$$

Once again applying the Cauchy-Schwarz inequality, we find that

$$\left| \int \hat{\mathbf{f}}_1 x^4 \nu_s F_0 dx \right|^2 \leq \int |\hat{\mathbf{f}}_1|^2 x^4 \nu_s F_0 dx \int x^4 \nu_s F_0 dx. \quad (\text{A12})$$

Using this in the last term in Eq. (A11), we get

$$\sigma_1 \geq \frac{4\pi}{3} v_{\text{th}}^5 \left( \int |\hat{\mathbf{f}}_1|^2 x^4 \Delta \nu F_0 dx + \frac{1}{2} \int \left| \frac{\partial}{\partial x} x \hat{\mathbf{f}}_1 \right|^2 x^4 \nu_{\parallel} F_0 dx \right), \quad (\text{A13})$$

where  $\Delta \nu$  is defined in Eq. (6). Upon using the differential identity given in Eq. (6) to express  $\Delta \nu$  in the first term of the above expression and integrating the resulting expression by parts, we finally obtain

$$\sigma_1 \geq \frac{4\pi}{3} v_{\text{th}}^5 \int \left| \frac{\partial \hat{\mathbf{f}}_1}{\partial x} \right|^2 x^6 \nu_{\parallel} F_0 dx \geq 0. \quad (\text{A14})$$

We now consider when these inequalities becomes equalities, i.e., when the right-hand side of Eq. (A1) is zero. Firstly, this requires  $\partial R[\hat{f}]/\partial \xi = 0$  and thus  $R[\hat{f}] = 0$ , so  $\hat{f}$  contains no 2nd or higher-order spherical harmonics. Secondly,  $\sigma_0 = 0$  if either  $\hat{f}_0$  is independent of  $v$  or we have equality in the invocation of the Cauchy-Schwarz inequality [Eq. (A8)], which occurs if  $\hat{f}_0 \propto v^2$ . Similarly  $\sigma_1 = 0$  iff  $\hat{\mathbf{f}}_1$  is independent of  $v$ . Thus, the right-hand side of Eq. (A1) vanishes iff  $\hat{f} \propto 1, \mathbf{v}, v^2$ , i.e.,  $\delta f = \hat{f} F_0$  is a perturbed Maxwellian.

This completes the proof of the  $H$ -theorem for our model operator.

## Appendix B: Gyroaveraging

To transform the derivatives in Eq. (14) from the original phase-space coordinates  $(\mathbf{r}, v, \xi, \vartheta)$  to the new coor-

dinates  $(\mathbf{R}, v, \xi, \vartheta)$ , we require the following formulae:

$$\left( \frac{\partial}{\partial v} \right)_{\mathbf{r}} = \left( \frac{\partial}{\partial v} \right)_{\mathbf{R}} - \frac{1}{v} \boldsymbol{\rho} \cdot \left( \frac{\partial}{\partial \mathbf{R}} \right)_{\mathbf{v}}, \quad (\text{B1})$$

$$\left( \frac{\partial}{\partial \xi} \right)_{\mathbf{r}} = \left( \frac{\partial}{\partial \xi} \right)_{\mathbf{R}} + \frac{\xi}{1 - \xi^2} \boldsymbol{\rho} \cdot \left( \frac{\partial}{\partial \mathbf{R}} \right)_{\mathbf{v}}, \quad (\text{B2})$$

$$\left( \frac{\partial}{\partial \vartheta} \right)_{\mathbf{r}} = \left( \frac{\partial}{\partial \vartheta} \right)_{\mathbf{R}} + \frac{\mathbf{v}_{\perp}}{\Omega} \cdot \left( \frac{\partial}{\partial \mathbf{R}} \right)_{\mathbf{v}}, \quad (\text{B3})$$

where  $\boldsymbol{\rho} = \hat{\mathbf{b}} \times \mathbf{v}_{\perp}/\Omega$ . In Fourier-transformed perpendicular guiding center variables, we can replace in the above formulae  $(\partial/\partial \mathbf{R})_{\mathbf{v}} \rightarrow i\mathbf{k}$ , where  $\mathbf{k} \equiv \mathbf{k}_{\perp}$ . It is convenient to align (without loss of generality) the  $\vartheta = 0$  axis with  $\mathbf{k}$ , so we have  $\mathbf{v}_{\perp} \cdot \mathbf{k} = k_{\perp} v \sqrt{1 - \xi^2} \cos \vartheta$  and  $\boldsymbol{\rho} \cdot \mathbf{k} = -k_{\perp} v \sqrt{1 - \xi^2} \sin \vartheta$ . Using the above formulae, we gyroaverage the Lorentz operator in Eq. (14):

$$\langle L[h_{\mathbf{k}}] \rangle = \frac{1}{2} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial h_{\mathbf{k}}}{\partial \xi} - \frac{v^2 (1 + \xi^2)}{4\Omega^2} k_{\perp}^2 h_{\mathbf{k}}, \quad (\text{B4})$$

where we have used  $\langle \boldsymbol{\rho} \boldsymbol{\rho} \rangle_{\mathbf{R}} = \langle \mathbf{v}_{\perp} \mathbf{v}_{\perp} \rangle_{\mathbf{R}} = (1/2)\mathbf{I}$ . Note that both the terms containing  $\xi$  and  $\vartheta$  derivatives in the original operator Eq. (10) produce non-zero gyrodiffusive contributions [the second term in Eq. (B4)]. Another such gyrodiffusive term, equal to  $-\nu_{\parallel} [v^2(1 - \xi^2)/4\Omega] k_{\perp}^2 h_{\mathbf{k}}$ , arises from the energy-diffusion part of the test-particle operator in Eq. (14). Collecting these terms together and defining the thermal Larmor radius  $\rho = v_{\text{th}}/\Omega$ , we arrive at the gyrodiffusion term in Eq. (20).

It remains to gyroaverage the field-particle terms. For the energy-conservation term [Eq. (13)] we have

$$\langle e^{i\mathbf{k} \cdot \boldsymbol{\rho}} Q[e^{-i\mathbf{k} \cdot \boldsymbol{\rho}} h_{\mathbf{k}}] \rangle = \langle e^{i\mathbf{k} \cdot \boldsymbol{\rho}} \rangle Q[e^{-i\mathbf{k} \cdot \boldsymbol{\rho}} h_{\mathbf{k}}], \quad (\text{B5})$$

where

$$Q[e^{-i\mathbf{k} \cdot \boldsymbol{\rho}} h_{\mathbf{k}}] = \int v^2 \nu_E \langle e^{-i\mathbf{k} \cdot \boldsymbol{\rho}} \rangle h_{\mathbf{k}} d\mathbf{v} / \int v^2 (v/v_{\text{th}})^2 \nu_E F_0 d\mathbf{v}. \quad (\text{B6})$$

Note that the  $\vartheta$  integration in  $Q$  only affected  $e^{-i\mathbf{k} \cdot \boldsymbol{\rho}}$ , hence the above expression. Using the standard Bessel function identity<sup>30</sup>  $\int_0^{2\pi} e^{ia \sin \vartheta} d\vartheta = 2\pi J_0(a)$ , we find  $\langle e^{-i\mathbf{k} \cdot \boldsymbol{\rho}} \rangle = J_0(a)$ , where  $a = k_{\perp} v_{\perp}/\Omega$ . Substituting this into Eqs. (B5) and (B6), we arrive at the energy-conservation term in Eq. (20), where expression in the right-hand side of Eq. (B6) is denoted  $Q[h_{\mathbf{k}}]$  [Eq. (23)].

The momentum-conserving terms are handled in an analogous way: details can be found in Appendix B of Ref. 7, where a simpler model operator was gyroaveraged.

## Appendix C: Comparison with previous model operators

In order to compare and contrast with previously suggested operators that do include energy diffusion, we first

rewrite in our notation the operator derived by Catto and

Tsang<sup>22</sup> [their Eqs. (14) and (16)],

$$C_{CT}[\delta f] = \nu_D L[\delta f] + \frac{1}{v^2} \frac{\partial}{\partial v} \left( \frac{1}{2} v^4 \nu_{\parallel} F_0 \frac{\partial \delta f}{\partial v} \right) + \frac{2F_0}{n_0 v_{th}^2} \mathbf{v} \cdot \int \mathbf{v} \nu_s \delta f d\mathbf{v} + \frac{2F_0}{3n_0} \left( \frac{v^2}{v_{th}^2} - \frac{3}{2} \right) \int \frac{v^2}{v_{th}^2} \nu_E \delta f d\mathbf{v}. \quad (C1)$$

This operator, whilst it conserves particle number, momentum and energy, neither obeys the  $H$ -Theorem nor vanishes on a perturbed Maxwellian.

The latter point can be demonstrated most easily by letting  $\delta f = x^2 F_0$ , where  $x = v/v_{th}$ . This  $\delta f$  is proportional to a perturbed Maxwellian with non-zero  $\delta n$  and  $\delta T$ . We can then evaluate the test-particle and field-particle parts of the operator to find

$$\frac{1}{v^2} \frac{\partial}{\partial v} \left( \frac{1}{2} v^4 \nu_{\parallel} F_0 \frac{\partial \delta f}{\partial v} \right) = \frac{1}{x^2} \frac{\partial}{\partial x} (x^5 \nu_{\parallel} F_0) = -x^2 \nu_E F_0 \quad (C2)$$

and

$$\int x^2 \nu_E \delta f d\mathbf{v} = \frac{4n_0}{\sqrt{\pi}} \int_0^{\infty} x^6 \nu_E e^{-x^2} dx = \sqrt{\frac{2}{\pi}} n_0 \nu. \quad (C3)$$

Substituting into Eq. (C1), we get

$$C_{CT}[\delta f] = -x^2 \nu_E F_0 + \sqrt{\frac{2}{\pi}} \left( x^2 - \frac{3}{2} \right) \nu F_0, \quad (C4)$$

which is non-zero despite  $\delta f$  being a perturbed Maxwellian.

In order to show that the  $H$ -theorem can be violated by the operator Eq. (C1), let us consider a perturbed distribution function of the form  $\delta f = x^3 F_0$ . Then

$$C_{CT}[\delta f] = \left[ \frac{3}{2} (\nu_{\parallel} - \nu_E) + \left( x^2 - \frac{3}{2} \right) \nu \right] F_0, \quad (C5)$$

so the entropy generation is,

$$\frac{dS}{dt} = - \iint \frac{\delta f}{F_0} C_{CT}[\delta f] d\mathbf{v} d\mathbf{r} = -\frac{3}{64} (32 - 21\sqrt{2}) \nu V < 0, \quad (C6)$$

where  $V$  is the volume of the system. The above expression is negative, which breaks the  $H$ -theorem and produces unphysical plasma cooling for the particular form of the perturbed distribution function that we have examined.

The second case we examine here is the sequence of operators derived by Hirshman and Sigmar.<sup>21</sup> The general operator proposed by these authors is given in their Eq. (25). In our notation, we rewrite here the  $N = 0$ ,  $l = 0, 1$  restriction of the like-particle form of their operator with  $\Delta\nu$  set to 0 for simplicity (this does not affect the discussion that follows):

$$C_{HS}[\delta f] = \nu_D L[\delta f] + \frac{1}{v^2} \frac{\partial}{\partial v} \left[ \frac{1}{2} v^4 \nu_{\parallel} F_0 \frac{\partial}{\partial v} \left( \frac{1}{4\pi} \int_{-1}^1 d\xi \int_0^{2\pi} d\vartheta \frac{\delta f}{F_0} - \frac{v^2}{v_{th}^2} Q[\delta f] \right) \right] + \nu_D \frac{2\mathbf{v} \cdot \mathbf{U}}{v_{th}^2} F_0, \quad (C7)$$

where  $\mathbf{U}$  and  $Q$  are defined by Eqs. (12) and (13). The primary concern here comes from the angle averaging operation in the energy-diffusion part of the operator. Firstly, the energy diffusion only acts on the spherically symmetric (in velocity space) part of the perturbed distribution function. However, there is no reason why there cannot arise perturbations that have very large energy derivatives but angle-average to zero (for example,  $\delta f \propto \xi$ ). Clearly, such perturbations will not damped correctly. Secondly, upon conversion to gyrokinetic coordinates and gyroaveraging (see Sec. III and Appendix B), the operator becomes

$$C_{HS,GK}[h_{\mathbf{k}}] = \nu_D(v) \left[ \frac{1}{2} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial h_{\mathbf{k}}}{\partial \xi} - \frac{1}{4} (1 + \xi^2) \frac{v^2}{v_{th}^2} k_{\perp}^2 \rho^2 h_{\mathbf{k}} \right] + \frac{J_0(a)}{v^2} \frac{\partial}{\partial v} \left[ \frac{1}{2} v^4 \nu_{\parallel} F_0 \frac{\partial}{\partial v} \int_{-1}^1 d\xi \frac{J_0(a) h_{\mathbf{k}}}{2\pi F_0} \right] + 2\nu_D \frac{v_{\perp} J_1(a) U_{\perp}[h_{\mathbf{k}}] + v_{\parallel} J_0(a) U_{\parallel}[h_{\mathbf{k}}]}{v_{th}^2} F_0 + \nu_E \frac{v^2}{v_{th}^2} J_0(a) Q[h_{\mathbf{k}}] F_0, \quad (C8)$$

where the conservation functionals  $U_{\perp}$ ,  $U_{\parallel}$  and  $Q$  are the same as defined in Eqs. (21)–(23). The immediately obvi-

ous problem is that the angle averaging has introduced

two new Bessel functions into the energy diffusion term. The energy diffusion is therefore suppressed by one power of  $k_{\perp}\rho$  in the limit  $k_{\perp}\rho \gg 1$ , while it is precisely in this limit that we expect the small-scale structure in the velocity space to be particularly important.<sup>1,7</sup> This means that the energy cutoff in phase space is artificially pushed to smaller scales and one might encounter all the problems associated with insufficient energy diffusion.<sup>9</sup>

While, for the reasons outlined above, we expect the Hirshman–Sigmar operator not to be a suitable model for

collisions, we would like to note that for many purposes the Hirshman–Sigmar operators are superior to the model operator we presented in Sec. II. Taken as a sequence, they provide a rigorous way of obtaining classical and neoclassical transport coefficients to any desired degree of accuracy, and it is relatively easy to solve the Spitzer problem for them, while the Spitzer functions for our operator are hard to find analytically.

---

\* Electronic address: [i.abel107@imperial.ac.uk](mailto:i.abel107@imperial.ac.uk)

† Electronic address: [mabarnes@umd.edu](mailto:mabarnes@umd.edu)

- <sup>1</sup> A. A. Schekochihin, S. C. Cowley, W. Dorland, G. W. Hammett, G. G. Howes, G. G. Plunk, E. Quataert, and T. Tatsuno, *Plasma Phys. Controlled Fusion* (2008), in press (e-print arXiv:0806.1069).
- <sup>2</sup> G. W. Hammett, W. Dorland, and F. W. Perkins, *Phys. Fluids B* **4**, 2052 (1991).
- <sup>3</sup> W. Dorland and G. W. Hammett, *Phys. Fluids B* **5**, 812 (1993).
- <sup>4</sup> J. A. Krommes and G. Hu, *Phys. Plasmas* **1**, 3211 (1994).
- <sup>5</sup> J. A. Krommes, *Phys. Plasmas* **6**, 1477 (1999).
- <sup>6</sup> T. H. Watanabe and H. Sugama, *Phys. Plasmas* **11**, 1476 (2004).
- <sup>7</sup> A. A. Schekochihin, S. C. Cowley, W. Dorland, G. W. Hammett, G. G. Howes, E. Quataert, and T. Tatsuno, *Astrophys. J. Suppl.* (2007), submitted (e-print arXiv:0704.0044).
- <sup>8</sup> G. G. Howes, *Phys. Plasmas* **15**, 055904 (2008).
- <sup>9</sup> M. Barnes, I. G. Abel, T. Tatsuno, A. A. Schekochihin, S. C. Cowley, and W. Dorland, *Phys. Plasmas* (2008).
- <sup>10</sup> T. Tatsuno, W. Dorland, A. A. Schekochihin, S. C. Cowley, M. Barnes, G. G. Plunk, and G. G. Howes, *Phys. Rev. Lett.* (2008), submitted.
- <sup>11</sup> J. Candy and R. E. Waltz, *Phys. Plasmas* **13**, 032310 (2006).
- <sup>12</sup> G. G. Howes, S. C. Cowley, W. Dorland, G. W. Hammett, E. Quataert, and A. A. Schekochihin, *Astrophys. J.* **651**, 590 (2006).
- <sup>13</sup> H. Sugama, M. Okamoto, W. Horton, and M. Wakatani, *Phys. Plasmas* **3**, 2379 (1996).
- <sup>14</sup> K. Hallatschek, *Phys. Rev. Lett.* **93**, 125001 (2004).

<sup>15</sup> B. Scott, *Phys. Plasmas* (2007), submitted (e-print arXiv:0710.4899).

- <sup>16</sup> T. H. Watanabe and H. Sugama, *Phys. Plasmas* **9**, 3659 (2002).
- <sup>17</sup> L. Boltzmann, *Sitzungsber. Akad. Wiss. Wien* **66**, 275 (1872).
- <sup>18</sup> L. D. Landau, *Phys. Z. Sowjet.* **10**, 154 (1936).
- <sup>19</sup> P. Helander and D. J. Sigmar, *Collisional Transport in Magnetized Plasmas* (Cambridge University Press, 2002).
- <sup>20</sup> P. H. Rutherford, L. M. Kovrizhnikh, M. N. Rosenbluth, and F. L. Hinton, *Phys. Rev. Lett.* **25**, 1090 (1970).
- <sup>21</sup> S. P. Hirshman and D. J. Sigmar, *Phys. Fluids* **19**, 1532 (1976).
- <sup>22</sup> P. J. Catto and K. T. Tsang, *Phys. Fluids* **20**, 396 (1977).
- <sup>23</sup> E. A. Frieman and L. Chen, *Phys. Fluids* **25**, 502 (1982).
- <sup>24</sup> A. Brizard and T. S. Hahm, *Rev. Mod. Phys.* **79**, 421 (2007).
- <sup>25</sup> M. A. Beer, S. C. Cowley, and G. W. Hammett, *Phys. Plasmas* **2**, 2687 (1995).
- <sup>26</sup> F. Jenko, W. Dorland, M. Kotschenreuther, and B. N. Rogers, *Phys. Plasmas* **7**, 1904 (2000), URL <http://link.aip.org/link/?PHP/7/1904/1>.
- <sup>27</sup> J. Candy and R. E. Waltz, *J. Comp. Phys.* **186**, 545 (2003).
- <sup>28</sup> X. Q. Xu and M. N. Rosenbluth, *Phys. Fluids B* **3**, 627 (1991).
- <sup>29</sup> A. M. Dimits and B. I. Cohen, *Phys. Rev. E* **49**, 709 (1994).
- <sup>30</sup> G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, 1966).
- <sup>31</sup> S. C. Cowley, unpublished