Global regularity for the 2D Boussinesq equations with partial viscosity terms

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Abstract

In this paper we prove the global in time regularity for the 2D Boussinesq system with either the zero diffusivity or the zero viscosity. We also prove that as diffusivity(viscosity) goes to zero the solutions of the fully viscous equations converges stongly to those of zero diffusion(viscosity) equations. Our result for the zero diffusion system, in particular, solves the Problem no. 3 posed by M. K. Moffatt in [8].

1 Introduction

The Boussineq system for the homogeneous incompressible fluid flows in $\mathbb{R}^2$ is

$$(B)\begin{cases} \dfrac{\partial v}{\partial t} + (v\cdot\nabla)v = -\nabla p + \nu\Delta v + \theta e_2, \\ \dfrac{\partial \theta}{\partial t} + (v\cdot\nabla)\theta = \kappa\Delta\theta \\ \mathrm{div}\ v = 0, \\ v(x,0) = v_0(x), \qquad \theta(x,0)=\theta_0(x), \end{cases}$$

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where $v = (v_1, v_2)$, $v_j = v_j(x,t), j = 1, 2$, $(x,t) \in \mathbb{R}^2 \times (0, \infty)$, is the velocity vector field, $p = p(x,t)$ is the scalar pressure, $\theta(x,t)$ is the scalar temperature, $\nu \geq 0$ is the viscosity, and $\kappa \geq 0$ is the molecular diffusivity, and $e_2 = (0,1)$. The Boussinesq system has important roles in the atmospheric sciences(See e.g. [7]).

The global in time regularity of (B) with $\nu > 0$ and $\kappa > 0$ is well-known(See e.g. [2]). On the other hand, the regularity/singularity questions of the case of (B) with $\kappa = \nu = 0$ is an outstanding open problem in the mathematical fluid mechanics(See e.g. [3, 5, 9] for studies in this direction.). Even the regularity problem for 'partial viscosity cases'(i.e. either the zero diffusivity case, $\kappa = 0$ and $\nu > 0$, or the zero viscosity case, $\kappa > 0$ and $\nu = 0$), has been open to author's knowledge. Actually, the author has been recently informed of the article by M. K. Moffatt, where the question of singularity in the case $\kappa = 0, \nu > 0$ and its possible development in the limit $\kappa \to 0$ is listed as one of the 21th century problems(See the Problem no. 3 in [8]). For this problem very recent progress is obtained by Cordoba, Fefferman and De La LLave([4]), where the authors proved that special type of singularities, called 'squirt singularities', is absent. In this paper we consider the both of two partial viscosity cases, and prove the global in time regularity for both of the cases. We also prove that as diffusivity(viscosity) goes to zero the solutions of (B) converge strongly to those of zero diffusivity(viscosity) equations. In particular the Problem no. 3 in [8] is solved. More precise statements of our results are stated in Theorem 1.1 and Theorem 1.2 below. For later references we write down the zero diffusivity Boussinesq equations:

$$(B_1) \begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla) v = -\nabla p + \nu \Delta v + \theta e_2, \\ \dfrac{\partial \theta}{\partial t} + (v \cdot \nabla)\theta = 0 \\ \mathrm{div}\ v = 0, \\ v(x,0) = v_0(x), \qquad \theta(x,0) = \theta_0(x), \end{cases}$$

where $\nu > 0$ is fixed. For this system the following is our main result.

Theorem 1.1 *Let $\nu > 0$ be fixed, and div $v_0 = 0$. Let $m > 2$ be an integer, and $(v_0, \theta_0) \in H^m(\mathbb{R}^2)$. Then, there exists unique solution (v, θ) with $\theta \in C([0,\infty); H^m(\mathbb{R}^2))$ and $v \in C([0,\infty); H^m(\mathbb{R}^2)) \cap L^2(0,T; H^{m+1}(\mathbb{R}^2))$ of the system (B_1). Moreover, for each $s < m$, solutions (v, θ) of (B) converges to the corresponding solutions of (B_1) in $C([0,T]; H^s(\mathbb{R}^2))$ as $\kappa \to 0$.*

We also write down the zero viscosity Boussinesq equations.

$$
(B_2)\begin{cases} \dfrac{\partial v}{\partial t}+(v\cdot\nabla)v=-\nabla p+\theta e_2, \\ \dfrac{\partial \theta}{\partial t}+(v\cdot\nabla)\theta=\kappa\Delta\theta \\ \mathrm{div}\ v=0, \\ v(x,0)=v_0(x), \qquad \theta(x,0)=\theta_0(x), \end{cases}
$$

where $\kappa>0$ is given. The following is our result on (B_2).

Theorem 1.2 *Let $\kappa>0$ be fixed, and div $v_0=0$. Let $m>2$ be an integer. Let $m>2$ be an integer, and $(v_0,\theta_0)\in H^m(\mathbb{R}^2)$. Then, there exists unique solutions (v,θ) with $v\in C([0,\infty);H^m(\mathbb{R}^2))$ and $\theta\in C([0,\infty);H^m(\mathbb{R}^2))\cap L^2(0,T;H^{m+1}(\mathbb{R}^2))$ of the system (B_2). Moreover, for each $s<m$, solutions (v,θ) of (B) converges to the corresponding solutions of (B_2) in $C([0,T];H^s(\mathbb{R}^2))$ as $\nu\to 0$.*

The author would like to thank deeply to Professor Diego Cordoba for introducing his preprint([4]) to him, which motivated this research.

2 The Proof of Theorem 1.1

We first recall the following result on the system (B) with $\kappa=\nu=0$, proved in [3],[5]:

Theorem 2.1 *Suppose $(v_0,\theta_0)\in H^m(\mathbb{R}^2)$ with $m>2$ is an integer. Then, there exists a unique local classical solution $(v,\theta)\in C([0,T_1);H^m(\mathbb{R}^2))$ for some $T_1=T_1(\|v_0\|_{H^m},\|\theta_0\|_{H^m})$. Moreover, the solution remains in $H^m(\mathbb{R}^2)$ up to time $T>T_1$, namely $(v,\theta)\in C([0,T];H^m(\mathbb{R}^2))$ if and only if*

$$
\int_0^T \|\nabla\theta(t)\|_{L^\infty}dt<\infty. \tag{2.1}
$$

By obvious changes of the proof in [3] we can easily infer that the similar conclusion holds for the system (B_1) and (B_2) respectively. Hence, for the proof of the global regularity part of Theorem 1.1 and Theorem 1.2 it suffices to prove the estimate (2.1) for all $T\in(0,\infty)$ for the classical solutions (v,θ) of (B_1) and (B_2).

(i) Preliminary estimates:

Let $T>0$ be a given fixed time. From the second equation of (B_1) we immediately have

$$
\|\theta(t)\|_{L^p}\le\|\theta_0\|_{L^p} \qquad \forall t\in[0,T], p\in[1,\infty]. \tag{2.2}
$$

Taking L^2 inner product the first equation of (B_1) with v, we have, after integration by part,

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 + \nu\|\nabla v\|_{L^2}^2 \le \|\theta\|_{L^2}\|v\|_{L^2}.$$

We have

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 \le \|\theta\|_{L^2}\|v\|_{L^2} \le \|\theta_0\|_{L^2}\|v\|_{L^2},$$

where we used (2.2) for $p = 2$. Hence, $\frac{d}{dt}\|v\|_{L^2} \le \|\theta_0\|$, and we obtain

$$\|v(t)\|_{L^2} \le \|v_0\|_{L^2} + \|\theta_0\|_{L^2}T \qquad \forall t \in [0,T]. \tag{2.3}$$

Taking the operation curl on the both sides of the first equation of (B), we obtain

$$\omega_t + (v\cdot\nabla)\omega = -\theta_{x_1} + \nu\Delta\omega, \tag{2.4}$$

where $\omega = \partial_{x_1}v_2 - \partial_{x_2}v_1$. Let $p \ge 2$. Multiplying (2.4) by $\omega|\omega|^{p-2}$ and integrating it over $\mathbb{R}^2$, we find, after integration by part,

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|\omega|^p dx + (p-1)\nu\int_{\mathbb{R}^2}|\nabla\omega|^2|\omega|^{p-2}dx\\
&\qquad = \frac{1}{p}\int_{\mathbb{R}^2}(v\cdot\nabla)|\omega|^p dx - \int_{\mathbb{R}^2}\theta_{x_1}\omega|\omega|^{p-2}dx\\
&\qquad = -\frac{1}{p}\int_{\mathbb{R}^2}\operatorname{div} v\,|\omega|^p dx + (p-1)\int_{\mathbb{R}^2}\theta\omega_{x_1}|\omega|^{p-2}dx\\
&\qquad \le \frac{(p-1)\nu}{2}\int_{\mathbb{R}^2}|\nabla\omega|^2|\omega|^{p-2}dx + \frac{(p-1)}{2\nu}\int_{\mathbb{R}^2}\theta^2|\omega|^{p-2}dx\\
&\qquad \le \frac{(p-1)\nu}{2}\int_{\mathbb{R}^2}|\nabla\omega|^2|\omega|^{p-2}dx + \frac{(p-1)}{2\nu}\|\theta\|_{L^p}^2\|\omega\|_{L^p}^{p-2},
\end{aligned}$$

Absorbing the term, $\frac{(p-1)\nu}{2}\int_{\mathbb{R}^2}|\nabla\omega|^2|\omega|^{p-2}dx$ to the left hand, we find

$$\frac{1}{p}\frac{d}{dt}\|\omega\|_{L^p}^p + \frac{(p-1)\nu}{2}\int_{\mathbb{R}^2}|\nabla\omega|^2|\omega|^{p-2}dx \le \frac{(p-1)}{2\nu}\|\theta\|_{L^p}^2\|\omega\|_{L^p}^{p-2}. \tag{2.5}$$

For $p = 2$ in particular, we obtain after integration over $[0,T]$,

$$\|\omega(t)\|_{L^2}^2 + \nu\int_0^T\|\nabla\omega(s)\|_{L^2}^2 ds \le 2\|\omega_0\|_{L^2}^2 + \frac{2}{\nu}\|\theta_0\|_{L^2}^2 T \quad \forall t \in [0,T]. \tag{2.6}$$

Hence, we find that, by Hölder's inequality,

$$\begin{aligned}
\int_0^T\|\nabla\omega(s)\|_{L^2}ds &\le C\sqrt{T}\left(\int_0^T\|\nabla\omega(s)\|_{L^2}^2 ds\right)^{\frac{1}{2}}\\
&\le C\|\omega_0\|_{L^2}\sqrt{T} + C\|\theta_0\|_{L^2}T \qquad \forall t \in [0,T].
\end{aligned} \tag{2.7}$$

On the other hand, from (2.5), we have for $p \in [2, \infty)$

$$\|\omega(t)\|_{L^p}^2 \le \|\omega_0\|_{L^p}^2 + \frac{(p-1)}{\nu}\|\theta_0\|_{L^p}^2 T \le \left(\|\omega_0\|_{L^p} + \frac{\sqrt{p-1}}{\sqrt{\nu}}\|\theta_0\|_{L^p}\sqrt{T}\right)^2,$$

and

$$\|\omega(t)\|_{L^p} \le \|\omega_0\|_{L^p} + \frac{\sqrt{p-1}}{\sqrt{\nu}}\|\theta_0\|_{L^p}\sqrt{T} \qquad \forall t \in [0,T], p \in [2,\infty). \quad (2.8)$$

We now recall the Gagliardo-Nirenberg interpolation inequality in $\mathbb{R}^2$.

$$\|f\|_{L^\infty} \le C\|f\|_{L^p}^{\frac{p-2}{2p-2}}\|Df\|_{L^p}^{\frac{p}{2p-2}}, \qquad f \in W^{1,p}(\mathbb{R}^2), p > 2, \quad (2.9)$$

By this and the Calderon-Zygmund inequality combined with the estimates (2.3) and (2.8) we find for $p \in (2, \infty)$

$$\begin{aligned} \|v(t)\|_{L^\infty} &\le C\|v(t)\|_{L^2}^{\frac{p-2}{2p-2}}\|\nabla v(t)\|_{L^p}^{\frac{p}{2p-2}} \le C\|v(t)\|_{L^2}^{\frac{p-2}{2p-2}}\|\omega(t)\|_{L^p}^{\frac{p}{2p-2}} \\ &\le C(v_0, \theta_0, T, \nu, p) \qquad \forall t \in [0,T]. \end{aligned} \quad (2.10)$$

(ii) $W^{2,p}$ estimate for v:

We take derivative operation $D = (\partial_{x_1}, \partial_{x_2})$ on (2.4), and then take L^2 inner product it with $D\omega|D\omega|^{p-2}$, $p > 2$. After integration by part we obtain

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|D\omega\|_{L^p}^p + (p-1)\nu\int_{\mathbb{R}^2}|D^2\omega|^2|D\omega|^{p-2}dx \\ &= -\int_{\mathbb{R}^2}[D(v\cdot\nabla)\omega]\,D\omega|D\omega|^{p-2}dx - \int_{\mathbb{R}^2}D\theta_{x_1}\,D\omega|D\omega|^{p-2}dx \\ &= (p-1)\int_{\mathbb{R}^2}[(v\cdot\nabla)\omega]\,D^2\omega|D\omega|^{p-2}dx + (p-1)\int_{\mathbb{R}^2}\theta_{x_1}D^2\omega|D\omega|^{p-2}dx \\ &\le \frac{(p-1)\nu}{4}\int_{\mathbb{R}^2}|D^2\omega|^2|D\omega|^{p-2}dx + \frac{(p-1)}{\nu}\int_{\mathbb{R}^2}|v(x)|^2|D\omega|^p dx \\ &\quad + \frac{(p-1)\nu}{4}\int_{\mathbb{R}^2}|D^2\omega|^2|D\omega|^{p-2}dx + \frac{(p-1)}{\nu}\int_{\mathbb{R}^2}|\nabla\theta|^2|D\omega|^{p-2}dx, \end{aligned}$$

where we used the inequality, $ab \le \frac{\nu}{4}a^2 + \frac{b^2}{\nu}$. Absorbing the first and the third terms to the left hand side, we have

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|D\omega\|_{L^p}^p + \frac{(p-1)\nu}{2}\int_{\mathbb{R}^2}|D^2\omega|^2|D\omega|^{p-2}dx \\ &\le \frac{(p-1)}{\nu}\int_{\mathbb{R}^2}|v(x)|^2|D\omega|^p dx + \frac{(p-1)}{\nu}\int_{\mathbb{R}^2}|\nabla\theta|^2|D\omega|^{p-2}dx \\ &\le \frac{(p-1)}{\nu}\|v\|_{L^\infty}^2\|D\omega\|_{L^p}^p + \frac{2(p-1)}{p\,\nu}\|\nabla\theta\|_{L^p}^p + \frac{(p-1)(p-2)}{p\,\nu}\|D\omega\|_{L^p}^p, \end{aligned}$$

where we used Young's inequality, $a^2b^{p-2} \le \frac{2}{p}a^p + \frac{p-2}{p}b^p$ for $p \ge 2$. Recalling the estimate of $\|v(t)\|_{L^\infty}$ in (2.10), we find that

$$\frac{d}{dt}\|D\omega\|_{L^p}^p \le C\|D\omega\|_{L^p}^p + C\|\nabla\theta\|_{L^p}^p, \qquad \forall t \in [0,T], \tag{2.11}$$

where $C = C(v_0, \theta_0, T, \nu, p)$.
Now taking $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$ to the second equation of (B_1), we obtain

$$\nabla^\perp\theta_t + (v\cdot\nabla)\nabla^\perp\theta = \nabla^\perp\theta\cdot\nabla v. \tag{2.12}$$

Taking L^2 inner product (2.12) with $\nabla^\perp\theta|\nabla^\perp\theta|^{p-2}$, we deduce, after integration by part, that

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla\theta|^p dx &= \frac{1}{p}\int_{\mathbb{R}^2}(v\cdot\nabla)|\nabla\theta|^p dx + \int_{\mathbb{R}^2}(\nabla^\perp\theta\cdot\nabla)v\cdot\nabla^\perp\theta|\nabla\theta|^{p-2}dx \\ &\le \int_{\mathbb{R}^2}|\nabla v||\nabla\theta|^p dx.\end{aligned}$$

Hence, for $p > 2$ we have

$$\begin{aligned}&\frac{d}{dt}\|\nabla\theta\|_{L^p}^p \le p\|\nabla v\|_{L^\infty}\|\nabla\theta\|_{L^p}^p \\ &\quad\le C(1+\|\nabla v\|_{L^2}+\|D^2v\|_{L^2})[1+\log^+(\|D^2v\|_{L^p})]\|\nabla\theta\|_{L^p}^p \\ &\quad\le C(1+\|\omega\|_{L^2}+\|D\omega\|_{L^2})[1+\log^+(\|D\omega\|_{L^p}^p+\|\nabla\theta\|_{L^p}^p)]\|\nabla\theta\|_{L^p}^p, \\ &\quad\le C(1+\|D\omega\|_{L^2})[1+\log^+(\|D\omega\|_{L^p}^p+\|\nabla\theta\|_{L^p}^p)]\|\nabla\theta\|_{L^p}^p, \end{aligned} \tag{2.13}$$

where $C = (v_0, \theta_0, T, \nu, , p)$, and we used the following form of the Brezis-Wainger inequality[1](See also [6]),

$$\|f\|_{L^\infty} \le C(1+\|\nabla f\|_{L^2})\left[1+\log^+(\|\nabla f\|_{L^p})\right]^{\frac{1}{2}} + C\|f\|_{L^2} \tag{2.14}$$

for $f \in L^2(\mathbb{R}^2)\cap W^{1,p}(\mathbb{R}^2)$, which holds for $p > 2$, and the Calderon-Zygmund inequality as well as the estimate (2.6). Adding (2.11) and (2.13) together, and setting $X(t) = \|\nabla\theta\|_{L^p}^p + \|D\omega\|_{L^p}^p$, we find that

$$\frac{dX}{dt} \le C(1+\|D\omega(t)\|_{L^2})(1+\log^+ X)X \le C(1+\|D\omega(t)\|_{L^2})(1+\log^+ X)X$$

for all $t \in [0,T]$, where $C = C(v_0, \theta_0, T, \nu, p)$. By Gronwall's lemma we have

$$X(t) \le X(0)\exp\left[\exp\left\{CT + C\int_0^T \|D\omega(s)\|_{L^2}ds\right\}\right] \qquad \forall t \in [0,T],$$

which, combined with the estimate (2.7), implies that for $p > 2$

$$\|D\omega(t)\|_{L^p} \le C(v_0, \theta_0, T, \nu, p) \qquad \forall t \in [0,T]. \tag{2.15}$$

By the Gagliardo-Nirenberg (2.9) and the Calderon-Zygmund inequalities we have

$$\begin{aligned}\|\nabla v(t)\|_{L^\infty} &\le C\|\nabla v(t)\|_{L^2}^{\frac{p-2}{2p-2}}\|D^2 v(t)\|_{L^p}^{\frac{p}{2p-2}} \le C\|\omega(t)\|_{L^2}^{\frac{p-2}{2p-2}}\|D\omega(t)\|_{L^p}^{\frac{p}{2p-2}}\\ &\le C(v_0,\theta_0,T,\nu,p) \quad \forall t\in[0,T], p\in(2,\infty], \end{aligned} \tag{2.16}$$

where we used the Gagliardo-Nirenberg inequality(2.9), estimates (2.8), and (2.15). From the first part of the inequalities of (2.13), we find that

$$\frac{d}{dt}\|\nabla\theta\|_{L^p} \le \|\nabla v\|_{L^\infty}\|\nabla\theta\|_{L^p},$$

and by Gronwall's lemma

$$\|\nabla\theta(t)\|_{L^p} \le \|\nabla\theta_0\|_{L^p}\exp\left(\int_0^t \|\nabla v(s)\|_{L^\infty}ds\right),$$

where taking $p\to\infty$, we obtain finally

$$\begin{aligned}\|\nabla\theta(t)\|_{L^\infty} &\le \|\nabla\theta_0\|_{L^\infty}\exp\left(\int_0^T \|\nabla v(s)\|_{L^\infty}ds\right)\\ &\le C \qquad \forall t\in[0,T],\end{aligned} \tag{2.17}$$

where $C=C(\|v_0\|_{W^{2,p}},\|\theta_0\|_{W^{2,p}},T,p,\nu)$, and we used the estimate (2.16). Since we have the embedding, $H^m(\mathbb{R}^2)\hookrightarrow W^{2,p}(\mathbb{R}^2)$, for all $m>2$ and $p>2$ we have just reached to the estimate (2.1). for any given $T\in(0,\infty)$ and for all $v_0,\theta_0\in H^m(\mathbb{R}^2)$ with $m>2$.

(iii) Vanishing diffusivity limit:

Let (v,p,θ) and $(\tilde v,\tilde p,\tilde\theta)$ be solutions of (B_1) and (B) respectively with the same initial conditions (v_0,θ_0). We first observe that all the estimates derived in (i) and (ii) above are also valid for solutions of (B). Moreover these estimates are independent of κ. Summarizing these estimates, we have the key κ-independent estimates for the solutions $(\tilde v,\tilde\theta)$.

$$\|\nabla\tilde v\|_{L^\infty}+\|\nabla\tilde\theta\|_{L^\infty}+\|\tilde v\|_{W^{2,p}}+\|\tilde\theta\|_{W^{2,p}} \le C(\|v_0\|_{W^{2,p}},\|\theta_0\|_{W^{2,p}},\nu,T,p). \tag{2.18}$$

From $(B)-(B)_1$ we obtain for $\Theta=\theta-\tilde\theta$, $P=p-\tilde p$, $V=v-\tilde v$

$$\Theta_t+(v\cdot\nabla)\Theta+(V\cdot\nabla)\tilde\theta=\kappa\Delta\Theta+\kappa\Delta\tilde\theta, \tag{2.19}$$

and

$$V_t+(v\cdot\nabla)V+(V\cdot\nabla)\tilde v=-\nabla P+\Theta e_2+\nu\Delta V \tag{2.20}$$

together with div $V=0$. Taking L^2 inner product (2.19) with Θ, we have after integration by part

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Theta\|_{L^2}^2+\kappa\|\nabla\Theta\|_{L^2}^2&=-\int_{\mathbb{R}^2}(V\cdot\nabla)\tilde{\theta}\,\Theta dx-\kappa\int_{\mathbb{R}^2}\nabla\tilde{\theta}\cdot\nabla\Theta dx\\&\le\|\nabla\tilde{\theta}\|_{L^\infty}\|V\|_{L^2}\|\Theta\|_{L^2}+\kappa\|\nabla\tilde{\theta}\|_{L^2}\|\nabla\Theta\|_{L^2}\\&\le C\|V\|_{L^2}^2+C\|\Theta\|_{L^2}^2+\frac{\kappa}{2}\|\nabla\tilde{\theta}\|_{L^2}^2+\frac{\kappa}{2}\|\nabla\Theta\|_{L^2}^2,\end{aligned}$$

where $C=C(v_0,\theta_0,T,\nu)$, and we have used the estimate (2.18). Absorbing the term, $\frac{\kappa}{2}\|\nabla\Theta\|_{L^2}^2$ to the left hand side, and using the estimate (2.18), we obtain that

$$\frac{d}{dt}\|\Theta\|_{L^2}^2+\kappa\|\nabla\Theta\|_{L^2}^2\le C\|\Theta\|_{L^2}^2+C\|V\|_{L^2}^2+C\kappa\|\nabla\tilde{\theta}\|_{L^2}^2. \tag{2.21}$$

On the other hand, we take L^2 inner product (2.20) with V, and integrate by part to obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|V\|_{L^2}^2+\nu\|\nabla V\|_{L^2}^2&=-\int_{\mathbb{R}^2}(V\cdot\nabla)\tilde{v}\cdot V dx+\int_{\mathbb{R}^2}\Theta e_2\cdot V dx\\&\le\|\nabla\tilde{v}\|_{L^\infty}\|V\|_{L^2}^2+\|\Theta\|_{L^2}\|V\|_{L^2}\\&\le C(\|V\|_{L^2}^2+\|\Theta\|_{L^2}^2),\end{aligned} \tag{2.22}$$

where $C=C(v_0,\theta_0,T,\nu)$, where we used (2.18) again. Adding (2.22) to (2.21), and setting $X(t)=\|\Theta(t)\|_{L^2}^2+\|V(t)\|_{L^2}^2$, we obtain that

$$\frac{d}{dt}X(t)\le CX(t)+C\kappa\|\nabla\tilde{\theta}\|_{L^2}^2.$$

By Gronwall's lemma we find that

$$\begin{aligned}X(t)&\le X(0)e^{Ct}+C\kappa\int_0^t\|\nabla\tilde{\theta}(s)\|_{L^2}^2e^{C(t-s)}ds\\&\le Ce^{CT}\kappa\int_0^T\|\nabla\tilde{\theta}(t)\|_{L^2}^2dt\le C\kappa,\end{aligned}$$

where we used the fact $X(0)=0$ and the estimate (2.18). Hence, we obtain

$$\sup_{0\le t\le T}(\|v(t)-\tilde{v}(t)\|_{L^2}+\|\theta(t)-\tilde{\theta}(t)\|_{L^2})\le C\sqrt{\kappa}, \tag{2.23}$$

where $C=C(v_0,\theta_0,T,\nu)$. From the Gagliardo-Nirenberg interpolation inequality, and the estimate (2.18) together with the embedding, $H^m(\mathbb{R}^2)\hookrightarrow W^{2,p}(\mathbb{R}^2)$ for $m>2$, we deduce that for $0\le s<m$,

$$\begin{aligned}\sup_{0\le t\le T}\|v(t)-\tilde{v}(t)\|_{H^s}&\le C\sup_{0\le t\le T}\|v(t)-\tilde{v}(t)\|_{L^2}^\sigma\|v(t)-\tilde{v}(t)\|_{H^m}^{1-\sigma}, \quad ,\\&\le C(\|v_0\|_{H^m}+\|\tilde{v}_0\|_{H^m})^{1-\sigma}\sup_{0\le t\le T}\|v(t)-\tilde{v}(t)\|_{L^2}^\sigma\\&\le C\kappa^{\frac{m-s}{2m}}, \quad \text{where } \sigma=1-\frac{s}{m} \text{ and } C=C(v_0,\theta_0,T,\nu,s,m),\end{aligned}$$

and similarly for $\|\theta-\tilde{\theta}\|_{H^s}$, and we obtain the desired convergence $(\tilde{v},\tilde{\theta}) \to (v,\theta)$ in $C([0,T];H^s((\mathbb{R}^2))$ as $\kappa \to 0$. $\square$

Remark after the proof: The local existence and finite time blow-up criterion for solutions in the functional setting, $H^m(\mathbb{R}^2)$, $m>2$, which is proved in ([3],[5]) can be easily modified using function spaces $W^{2,p}(\mathbb{R}^2)$, $p>2$. Combining this with the above proof, we can actually prove the following:

Corollary 2.1 *Let $2<p<\infty$, and $(v_0,\theta_0)\in W^{2,p}(\mathbb{R}^2)$. Then, there exists unique solution $(v,\theta)\in C([0,\infty);W^{2,p}(\mathbb{R}^2))$ of the system (B_1). Moreover, for each $q\in[1,p)$ and $T\in(0,\infty)$, solutions (v,θ) of (B) converges to the corresponding solutions of (B_1) in $C([0,T];W^{1,q}(\mathbb{R}^2))$ as $\kappa\to 0$.*

3 The Proof of Theorem 1.2

Similarly to the preliminary remark in beginning of the previous section, in order to prove the global regularity part of Theorem 1.2 we have only to prove the estimate (2.1) for the classical solution of (B_2) for all $T\in(0,\infty)$.

<u>(i) Preliminary estimates:</u>

Taking L^2 inner product the second equation of (B_2) with θ, we have immediately,

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2+\kappa\|\nabla\theta\|_{L^2}^2=0.$$

Integrating this over $[0,T]$ we have

$$\frac{1}{2}\|\theta(t)\|_{L^2}^2+\int_0^T\|\nabla\theta\|_{L^2}^2dt\le\frac{1}{2}\|\theta_0\|_{L^2}^2\qquad\forall t\in[0,T]. \tag{3.1}$$

Next, taking L^2 inner product the first equation of (B_2) with v, we have after integration by part

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2=\int_{\mathbb{R}^2}\theta e_2\cdot v dx\le\|\theta\|_{L^2}\|v\|_{L^2}.$$

Combining this with (3.1), we easily obtain

$$\|v(t)\|_{L^2}\le\|v_0\|_{L^2}+\int_0^T\|\theta(s)\|_{L^2}ds=\|v_0\|_{L^2}+T\|\theta_0\|_{L^2} \tag{3.2}$$

for all $t\in[0,T]$. Taking curl of the first equation of (B_2), we have

$$\omega_t+(v\cdot\nabla)\omega=-\theta_{x_1}. \tag{3.3}$$

Taking L^2 inner product (3.3) with ω, and integrating by part, we deduce

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 \le \int_{\mathbb{R}^2} |\nabla\theta||\omega| dx \le \|\nabla\theta\|_{L^2}\|\omega\|_{L^2},$$

and

$$\frac{d}{dt}\|\omega\|_{L^2} \le \|\nabla\theta\|_{L^2}.$$

Hence, using the estimate (3.1), we derive

$$\begin{aligned} \|\omega(t)\|_{L^2} &\le \int_0^T \|\nabla\theta\|_{L^2} dt + \|\omega_0\|_{L^2} \\ &\le T^{\frac{1}{2}} \left(\int_0^T \|\nabla\theta\|_{L^2}^2 dt\right)^{\frac{1}{2}} + \|\omega_0\|_{L^2} \\ &\le \frac{T^{\frac{1}{2}}}{\sqrt{2}}\|\theta_0\|_{L^2} + \|\omega_0\|_{L^2} \qquad \forall t \in [0,T]. \end{aligned} \tag{3.4}$$

(ii) $W^{1,p}$ estimate for (θ, v):

Taking operation $\nabla^\perp$ on the second equation of (B_2), we have

$$\nabla^\perp\theta + (v\cdot\nabla)\nabla^\perp\theta = (\nabla^\perp\theta\cdot\nabla)v + \kappa\Delta\nabla^\perp\theta. \tag{3.5}$$

We now take scalar product (3.5) in L^2 by $\nabla^\perp\theta|\nabla^\perp\theta|^{p-2}$, $p > 2$, we obtain after integration by part

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\|\nabla^\perp\theta\|_{L^p}^p + (p-1)\kappa\int_{\mathbb{R}^2} |D^2\theta|^2|\nabla^\perp\theta|^{p-2}dx &= \int_{\mathbb{R}^2} (\nabla^\perp\theta\cdot\nabla)v \cdot \nabla^\perp\theta|\nabla^\perp\theta|^{p-2}dx \\ &= -(p-1)\int_{\mathbb{R}^2} v\cdot(\nabla^\perp\theta\cdot\nabla)\nabla^\perp\theta|\nabla^\perp\theta|^{p-2}dx \\ &\le (p-1)\int_{\mathbb{R}^2} |v||\nabla^\perp\theta||D^2\theta||\nabla^\perp\theta|^{p-2}dx \\ &\le \frac{(p-1)}{2\kappa}\int_{\mathbb{R}^2} |v|^2|\nabla^\perp\theta|^p dx + \frac{(p-1)\kappa}{2}\int_{\mathbb{R}^2} |D^2\theta|^2|\nabla^\perp\theta|^{p-2}dx, \end{aligned}$$

where we used the inequality, $ab \le \frac{a^2}{2\kappa} + \frac{\kappa b^2}{2}$. We absorb the second term to the left hand side to have

$$\begin{aligned} \frac{d}{dt}\|\nabla\theta\|_{L^p}^p + \frac{p(p-1)\kappa}{2}\int_{\mathbb{R}^2} |D^2\theta|^2|\nabla\theta|^{p-2}dx &\le \frac{(p-1)p}{2\kappa}\|v\|_{L^\infty}^2\|\nabla\theta\|_{L^p}^p \\ &\le C(1+\|v\|_{L^2}+\|\nabla v\|_{L^2})^2\left(1+\log^+(\|\nabla v\|_{L^p}^p)\right)\|\nabla\theta\|_{L^p}^p \\ &\le C(1+\|v\|_{L^2}+\|\omega\|_{L^2}^2)\left[1+\log^+(\|\omega\|_{L^p}^p+\|\nabla\theta\|_{L^p}^p)\right]\|\nabla\theta\|_{L^p}^p \\ &\le C\left[1+\log^+(\|\omega\|_{L^p}^p+\|\nabla\theta\|_{L^p}^p)\right]\|\nabla\theta\|_{L^p}^p, \end{aligned} \tag{3.6}$$

where we applied the Brezis-Wainger inequality (2.14) keeping the power, $\frac{1}{2}$, of the log $-$term preserved, the Calderon-Zygmund inequality, and the estimates (3.2), (3.4). On the other hand, taking L^2 inner product (3.3) with $\omega|\omega|^{p-2}$, we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|\omega\|_{L^p}^p + \frac{1}{p}\int_{\mathbb{R}^2}(v\cdot\nabla)|\omega|^p dx &= -\int_{\mathbb{R}^2}\theta_{x_2}\omega|\omega|^{p-2}dx\\ &\le \int_{\mathbb{R}^2}|\nabla\theta||\omega|^{p-1}dx\\ &\le \frac{1}{p}\|\nabla\theta\|_{L^p}^p + \frac{(p-1)}{p}\|\omega\|_{L^p}^p, \end{aligned} \tag{3.7}$$

where we used Young's inequality, $ab^{p-1} \le \frac{1}{p}a^p + \frac{p-1}{p}b^p$, $1 < p < \infty$. Adding (3.7) to (3.6), and setting $X(t) = \|\nabla\theta(t)\|_{L^p}^p + \|\omega\|_{L^p}^p$, we have

$$\frac{d}{dt}X(t) \le C(1+\log X(t))X(t) \qquad \forall t\in[0,T].$$

The Gronwall lemma provides us with

$$X(t) \le X(0)e^{e^{CT}} \qquad \forall t\in[0,T].$$

Hence,

$$\|\nabla\theta(t)\|_{L^p}^p + \|\omega\|_{L^p}^p \le C(v_0,\theta_0,T,p,\kappa). \tag{3.8}$$

We also note that similarly to (2.10), the estimate (3.8), combined with (3.2) and (2.9) implies that

$$\|v(t)\|_{L^\infty} \le C(v_0,\theta_0,T,p) \qquad \forall t\in[0,T]. \tag{3.9}$$

(iii) $W^{2,p}$ estimate for θ:

Taking operation D^2 on the second equation of (B_2), and then taking L^2 inner product this with $D^2\theta|D^2\theta|^{p-2}$, $p > 2$, we have after integration by part

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|D^2\theta\|_{L^p}^p + (p-1)\kappa\int_{\mathbb{R}^2}|D^3\theta|^2|D^2\theta|^{p-2}dx\\ &\quad= -\int_{\mathbb{R}^2}D^2(v\cdot\nabla)\theta\, D^2\theta|D^2\theta|^{p-2} = (p-1)\int_{\mathbb{R}^2}D[(v\cdot\nabla)\theta]\, D^3\theta|D^2\theta|^{p-2}dx\\ &\quad= (p-1)\int_{\mathbb{R}^2}Dv\cdot D\theta D^3\theta|D^2\theta|^{p-2}dx + (p-1)\int_{\mathbb{R}^2}[(v\cdot\nabla)D\theta]\, D^3\theta|D^2\theta|^{p-2}dx\\ &\quad\le \frac{(p-1)}{\kappa}\|\nabla\theta\|_{L^\infty}^2\int_{\mathbb{R}^2}|\nabla v|^2|D^2\theta|^{p-2}dx + \frac{(p-1)\kappa}{4}\int_{\mathbb{R}^2}|D^3\theta|^2|D^2\theta|^{p-2}dx\\ &\quad+\frac{(p-1)}{\kappa}\|v\|_{L^\infty}^2\int_{\mathbb{R}^2}|D^2\theta|^p dx + \frac{(p-1)\kappa}{4}\int_{\mathbb{R}^2}|D^3\theta|^2|D^2\theta|^{p-2}dx,\end{aligned}$$

where we used the inequality, $ab \le \frac{a^2}{\kappa} + \frac{\kappa b^2}{4}$, again. Absorbing the terms, $\frac{(p-1)\kappa}{4}\int_{\mathbb{R}^2} |D^3\theta|^2 |D^2\theta|^{p-2}dx$ to the left hand side, we derive

$$\begin{aligned}
\frac{d}{dt}\|D^2\theta\|_{L^p}^p &\le C\|\nabla\theta\|_{L^\infty}^2\|\nabla v\|_{L^p}^2\|D^2\theta\|_{L^p}^{p-2} + C\|v\|_{L^\infty}^2\|D^2\theta\|_{L^p}^p \\
&\le C\|\nabla\theta\|_{L^p}^{\frac{2p-4}{2p-2}}\|\omega\|_{L^p}^2\|D^2\theta\|_{L^p}^{p-\frac{2p-4}{2p-2}} + C\|v\|_{L^\infty}^2\|D^2\theta\|_{L^p}^p \\
&\le C + C\|D^2\theta\|_{L^p}^p,
\end{aligned}$$

where we used the Gagliardo-Nirenberg interpolation inequality (2.9), the estimates, (3.8), (3.9), and Young's inequality(Note $p - \frac{2p-4}{2p-2} < p$ when $p > 2$). Thanks to Gronwall's lemma, we have the estimate,

$$\|D^2\theta(t)\|_{L^p} \le C(v_0, \theta_0, T, p, \kappa) \qquad \forall t \in [0,T],\ \forall p > 2.$$

Using the interpolation inequality (2.9) as previously, we obtain that

$$\|\nabla\theta(t)\|_{L^\infty} \le C \qquad \forall t \in [0,T], \tag{3.10}$$

where $C = C(\|v_0\|_{W^{2,p}}, \|\theta_0\|_{W^{2,p}}, p, \kappa)$. Similarly to the proof of Theorem 1.1, we have the embedding, $H^m(\mathbb{R}^2) \hookrightarrow W^{2,p}(\mathbb{R}^2)$, for all $m > 2$ and $p > 2$, and thus we have reached to the estimate (2.1) for all $T \in (0,\infty)$ and for all $v_0, \theta_0 \in H^m(\mathbb{R}^2)$ with $m > 2$.

(vi) Vanishing viscosity limit:

Let (v, p, θ) and $(\tilde{v}, \tilde{p}, \tilde{\theta})$ be solutions of (B_2) and (B) respectively with the same initial conditions (v_0, θ_0). Similarly to the case of zero diffusivity problem in Section 2, we first note that all the estimates in (i), (ii) and (iii) above are valid for solutions of (B) also, and these estimates are independent of ν. The key ν-independent estimate for the solutions $(\tilde{v}, \tilde{\theta})$ is

$$\|\nabla\tilde{v}\|_{L^\infty} + \|\nabla\tilde{\theta}\|_{L^\infty} + \|\tilde{v}\|_{W^{2,p}} + \|\tilde{\theta}\|_{W^{2,p}} \le C(\|v_0\|_{W^{2,p}}, \|\theta_0\|_{W^{2,p}}, \kappa, T, p). \tag{3.11}$$

From $(B) - (B_2)$ we obtain for $\Theta = \theta - \tilde{\theta}$, $P = p - \tilde{p}$, $V = v - \tilde{v}$

$$\Theta_t + (v\cdot\nabla)\Theta + (V\cdot\nabla)\tilde{\theta} = \kappa\Delta\Theta \tag{3.12}$$

with div $V = 0$, and

$$V_t + (v\cdot\nabla)V + (V\cdot\nabla)\tilde{v} = -\nabla P + \Theta e_2 + \nu\Delta V + \nu\Delta\tilde{v}. \tag{3.13}$$

Taking L^2 inner product (3.12) with Θ, we have after integration by part

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Theta\|_{L^2}^2 + \kappa\|\nabla\Theta\|_{L^2}^2 &= -\int_{\mathbb{R}^2} (V\cdot\nabla)\tilde{\theta}\,\Theta dx \\
&\le \|\nabla\tilde{\theta}\|_{L^\infty}\|V\|_{L^2}\|\Theta\|_{L^2} \le C\|V\|_{L^2}^2 + C\|\Theta\|_{L^2}^2,
\end{aligned}$$

where $C = C(v_0, \theta_0, T, \kappa)$, and we have used the estimate (3.11). Hence,

$$\frac{d}{dt}\|\Theta\|_{L^2}^2 \le C\|\Theta\|_{L^2}^2 + C\|V\|_{L^2}^2. \tag{3.14}$$

On the other hand, we take L^2 inner product (3.13) with V, and integrate by part to obtain:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|V\|_{L^2}^2 + \nu\|\nabla V\|_{L^2}^2 &= -\int_{\mathbb{R}^2}(V\cdot\nabla)\tilde{v}\,\cdot V dx \\
&+ \int_{\mathbb{R}^2}\Theta e_2\cdot V dx - \nu\int_{\mathbb{R}^2}\nabla\tilde{v}\cdot\nabla V dx \\
&\le \|\nabla\tilde{v}\|_{L^\infty}\|V\|_{L^2}^2 + \|\Theta\|_{L^2}\|V\|_{L^2} + \nu\|\nabla\tilde{v}\|_{L^2}\|\nabla V\|_{L^2} \\
&\le C(\|V\|_{L^2}^2 + \|\Theta\|_{L^2}^2) + \frac{\nu}{2}\|\nabla V\|_{L^2}^2 + \frac{\nu}{2}\|\nabla\tilde{v}\|_{L^2}^2,
\end{aligned}$$

where $C = C(v_0, \theta_0, T, \kappa)$, and we used the estimate (3.11) again. Absorbing the term, $\frac{\nu}{2}\|\nabla V\|_{L^2}^2$ to the left hand side, we obtain that

$$\frac{d}{dt}\|V\|_{L^2}^2 \le C(\|V\|_{L^2}^2 + \|\Theta\|_{L^2}^2) + \frac{\nu}{2}\|\nabla\tilde{v}\|_{L^2}^2. \tag{3.15}$$

Adding (3.15) to (3.14), and setting $X(t) = \|\Theta(t)\|_{L^2}^2 + \|V(t)\|_{L^2}^2$, we obtain that

$$\frac{d}{dt}X(t) \le CX(t) + C\nu\|\nabla\tilde{v}\|_{L^2}^2.$$

By Gronwall's lemma we find that

$$\begin{aligned}
X(t) &\le X(0)e^{Ct} + C\nu\int_0^t \|\nabla\tilde{v}(s)\|_{L^2}^2 e^{C(t-s)}ds \\
&\le Ce^{CT}\nu\int_0^T \|\nabla\tilde{v}(t)\|_{L^2}^2 dt \le C\nu,
\end{aligned}$$

where we used (3.11) and the fact that $X(0) = 0$. Hence, we obtain

$$\sup_{0\le t\le T}(\|v(t) - \tilde{v}(t)\|_{L^2} + \|\theta(t) - \tilde{\theta}(t)\|_{L^2}) \le C\sqrt{\nu}, \tag{3.16}$$

where $C = C(v_0, \theta_0, T, \kappa)$. Similarly to the case of vanishing diffusivity limit, the interpolation inequality, and the uniform in ν estimate (3.11) lead us to the estimate, for $0 \le s < m$,

$$\begin{aligned}
\sup_{0\le t\le T}\|v(t) - \tilde{v}(t)\|_{H^s} &\le C\sup_{0\le t\le T}\|v(t) - \tilde{v}(t)\|_{L^2}^{\sigma}\|v(t) - \tilde{v}(t)\|_{H^m}^{1-\sigma}, \quad , \\
&\le C(\|v_0\|_{H^m} + \|\tilde{v}_0\|_{H^m})^{1-\sigma}\sup_{0\le t\le T}\|v(t) - \tilde{v}(t)\|_{L^2}^{\sigma} \\
&\le C\nu^{\frac{m-s}{2m}}, \quad \text{where } \sigma = 1 - \tfrac{s}{m} \text{ and } C = C(v_0, \theta_0, T, \kappa, s, m),
\end{aligned}$$

and similarly for $\|\theta - \tilde{\theta}\|_{H^s}$, we obtain the desired convergence $(\tilde{v}, \tilde{\theta}) \to (v, \theta)$ in $C([0, T]; H^s((\mathbb{R}^2))$ as $\nu \to 0$. □

Remark after the proof: Similarly to the remark at the end of Section 2 we can actually prove the following:

Corollary 3.1 *Let $2 < p < \infty$, and $(v_0, \theta_0) \in W^{2,p}(\mathbb{R}^2)$. Then, there exists unique solutions $(v, \theta) \in C([0, \infty); W^{2,p}(\mathbb{R}^2))$ of the system (B_2). Moreover, for each $q \in [1, p)$, $T \in (0, \infty)$, solutions (v, θ) of (B) converges to the corresponding solutions of (B_1) in $C([0, T]; W^{1,q}(\mathbb{R}^2))$ as $\nu \to 0$.*

References

[1] H. Brezis and S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities,* Comm. P.D.E., **5** , no. 7, (1980), pp.773-789.

[2] J.R. Cannon and E. DiBenedetto, *The initial problem for the Boussinesq equations with data in L^p,* Lect. Note Math. no. 771, Springer, Berlin, (1980), pp. 129-144.

[3] D. Chae, H-S. Nam, *Local existence and blow-up criterion for the Boussinesq equations,* Proc. Roy. Soc. Edinburgh, Sect. A **127**, no. 5, (1997), pp. 935-946.

[4] D. Córdoba, C. Fefferman and R. De La LLave, *On squirt singularities in hydrodynamics,* SIAM J. Math. Anal., to appear.

[5] W. E. and C. Shu, *Small scale structures un Boussinesq convection,* Phys. Fluids, **6**, (1994), pp. 48-54.

[6] H. Engler, *An alternative proof of the Brezis-Wainger inequality,* Commun. P.D.E., **14** , no. 4, (1989), pp.541-544.

[7] A. Majda, *Introduction to PDEs and Waves for the Atmosphere and Ocean,* Courant Lecture Notes in Mathematics, no. 9, AMS/CIMS, (2003).

[8] H. K. Moffatt, *Some remarks on topological fluid mechanics,* in An Introduction to the Geometry and Topology of Fluid Flows, R. L. Ricca, ed., Kluwer Academic Publishers, Dordrecht, The Netherlands, (2001), pp. 3-10.

[9] Y. Taniuchi, *A note on the blow-up criterion for the inviscid 2-D Boussinesq equations,* Lecture Notes in Pure and Applied Math., **223**, "the Navier-Stokes equations:theory and numerical methods", edited by R. Salvi. (2002), pp. 131-140.