Selfgravitating Electroweak strings

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Abstract

We obtain selfgravitating multi-string configurations for the Einstein-Weinberg-Salam model, in terms of solutions for a nonlinear elliptic system of Liouville type whose solvability was posed as an open problem in [15].

1 Introduction

Aim of this paper is to establish the existence of gravitating strings for the Einstein-Weinberg-Salam theory, where the non-abelian $SU(2) \times U(1)$-Electroweak theory is coupled with Einstein's equation to take into account the effect of gravity. We shall be interested to obtain static strings, parallel along a given direction. Thus, in the Minkowski space $\mathbb{R}^{1+3}$ with time variable $t = x^0$ and space variables (x^1, x^2, x^3), we consider the x_3-direction as a fixed(vertical) direction. Accordingly, we restrict the choice of gravitational metrics to take the form:

$$ds^2 = (dx^0)^2 - (dx^3)^2 - e^{\eta}((dx^1)^2 + (dx^2)^2), \tag{1.1}$$

so that the conformal factor η will define one of our unknown. Furthermore, by formulating the Electroweak theory in terms of the unitary gauge variables, we may introduce a setting (suggested by the Ambjorn-Olesen's vortex ansatz [1, 2, 3]) so that, with the physical parameters specified according to a "critical" condition, the second order Euler-Lagrange equations reduces to selfdual first order equations of Bogomolnyi type when restricted to time independent solutions. The resulting selfdual equations are expressed in terms of a complex valued massive field W, a scalar field φ and real valued 2-vector fields $P = (P_\mu)_{\mu=1,2}$ and $Z = (Z_\mu)_{\mu=1,2}$, which together with the conformal factor η are assumed to depend only on the (x^1, x^2)-variables. The massive field W is (weakly) coupled with the fields P and Z through the covariant derivative in the form:

$$D_j W = \partial_j W - ig_1(P_j \sin\theta + Z_j \cos\theta)W, \quad j = 1, 2 \tag{1.2}$$

where g_1 is the $SU(2)$-coupling constant, $\theta \in (0, \pi/2)$ is the Weinberg's mixing angle, that relates to the $U(1)$-coupling constant g_2 via the identity:

$$\cos\theta = \frac{g_1}{(g_1^2 + g_2^2)^{1/2}}.$$

Let $P_{12} = \partial_1 P_2 - \partial_2 P_1$ and $Z_{12} = \partial_1 Z_2 - \partial_2 Z_1$ be the curls of the vector fields P and Z respectively, we may formulate the selfdual equations as follows:

$$D_1 W + iD_2 W = 0 \tag{1.3}$$

$$P_{12} = \frac{g_1}{2\sin\theta}\phi_0^2 e^{\eta} + 2g_1 \sin\theta |W|^2 \tag{1.4}$$

$$Z_{12} = \frac{g_1}{2\cos\theta}(\varphi^2 - \phi_0^2) + 2g_1 \cos\theta |W|^2 \tag{1.5}$$

$$Z_j = -\frac{2\cos\theta}{g_1}\varepsilon^{kj}\partial_k \log\varphi \tag{1.6}$$

where ϕ_0 is the symmetry breaking constant and ε^{kj} denotes the totally antisymmetric symbol fixed with $\varepsilon^{12} = 1$. In this setting the reduced 2-dimensional energy density $\mathcal{H}$ takes the form:

$$\mathcal{H} = \frac{1}{8}\frac{g_1^2\phi_0^4}{\sin^2\theta} + \frac{g_1^2}{4\cos^2\theta}(\varphi^2 - \phi_0^2)^2 + g_1^2\varphi^2|W|^2 e^{-\eta} + 2e^{-\eta}|\nabla\varphi|^2, \tag{1.7}$$

and we also obtain the Gauss curvature $K_\eta = -\frac{1}{2}e^{-\eta}\Delta\eta$ relative to the Riemann surface $(\mathbb{R}^2, e^{\eta}\delta_{jk})$ by means of the relation:

$$K_\eta = 8\pi G\mathcal{H} + \Lambda, \tag{1.8}$$

where G is Newton's gravitational constant, and Λ is the cosmological constant that, by Einstein's equation, must be fixed as follows:

$$\Lambda = \frac{\pi G g_1^2 \phi_0^2}{\sin^2 \theta}. \tag{1.9}$$

We refer to Chapter 10 of Yang's monograph[15] for a detailed discussion about the derivation of those relations. We only observe that in view of (1.3), W is required to satisfy a sort of gauge invariant version of the Cauchy-Riemann equation. In particular this implies ([12]) that W can vanish at isolated zeros, say $\{z_1, \cdots, z_N\}$ (repeated according to multiplicity), which determine the string's location.
Therefore, following [12], we may introduce new variables (u, v) such that,

$$e^u = |W|^2, \qquad e^v = \varphi^2 \tag{1.10}$$

and see that the selfgravitating Electroweak string solution to (1.3)-(1.6) may be expressed in terms of a triplet (u, v, η) solution in $\mathbb{R}^2$ for the following elliptic system:

$$\begin{cases} -\Delta u = g_1^2 e^{v+\eta} + 4g_1^2 e^u - 4\pi \displaystyle\sum_{k=1}^{N} \delta(z - z_k) \\ \Delta v = \dfrac{g_1^2}{2\cos^2\theta}[e^v - \phi_0^2]e^\eta + 2g_1^2 e^u \\ -\Delta \eta = 4\pi G g_1^2 e^\eta \left[\dfrac{(e^v - \phi_0^2)^2}{\cos^2\theta} + \dfrac{\phi_0^4}{\sin^2\theta} \right] \\ \qquad + 16\pi G g_1^2 e^{u+v} + 8\pi G |\nabla v|^2 e^v, \end{cases} \tag{1.11}$$

where $\{z_1, \cdots, z_N\}$ are given points (repeated with multiplicity) in $\mathbb{R}^2$and correspond to the zeros of the massive field:

$$W(z) = \exp\left(\frac{u}{2} + i \sum_{k=1}^{N} arg \frac{z - z_k}{|z - z_k|} \right). \tag{1.12}$$

Indeed, by virtue of (1.2), (1.10) and (1.12) we can easily recover the full string (W, φ, P, Z, η) solution of (1.3)-(1.6) out of the triplet (u, v, η) satisfying (1.11). Again, we refer to [15] for details, where in fact the solvability of (1.11) is listed as a challenging open problem, in contrast, for instance, to the analogous Einstein-Abelian-Higgs system whose string solutions have been classified rather accurately in [13, 14]. See [15] also for more references. Satisfactory results are available also in case we neglect the effect of gravity, and take $\eta = G = 0$ in (1.11). In this case the resulting (2×2) system has been treated in [11] and [7] to yield various classes of planar Electroweak

vortex-like configurations, while Electroweak periodic vortices have been established in [10] and [5].
It is the main goal of this paper to show that, if

$$\frac{\sin^2\theta}{4\pi G\phi_0^2} > N+1 \tag{1.13}$$

then, for any assigned set of points $\{z_1, \cdots, z_N\} \subset \mathbb{R}^2$ (repeated according to their multiplicity) the system (1.11) admits (a one-parameter family of) solutions satisfying the boundary conditions:

$$\int_{\mathbb{R}^2} e^u < +\infty, \qquad \int_{\mathbb{R}^2} e^\eta < +\infty, \qquad |\nabla e^v| \in L^2(\mathbb{R}^2). \tag{1.14}$$

Notice that the boundary conditions (1.14) appear as "natural" in this context, as they imply a finite energy property for the corresponding selfdual string, in the sense that,

$$\int_{\mathbb{R}^2} \mathcal{H} e^\eta < +\infty \qquad \text{and} \qquad \int_{\mathbb{R}^2} K_\eta e^\eta < +\infty \tag{1.15}$$

(see (1.7) and (1.8)). Moreover they ensure finite flux for the vector fields P and Z. More precisely, concerning (1.3)-(1.6) we obtain the following result:

Theorem 1.1 *Let $N \in \mathbb{N}$ be an integer such that (1.13) holds. For a given set of points $\{z_1, \cdots, z_N\} \subset \mathbb{R}^2$ (repeated according to their multiplicity) there exists $\varepsilon_1 > 0$ such that for every $\varepsilon \in (0, \varepsilon_1)$ there exists $(W^\varepsilon, \varphi^\varepsilon, P^\varepsilon, \eta^\varepsilon)$, a selfgravitating Electroweak string solution of (1.3)-(1.6) satisfying the finite energy condition (1.15) and with W^ε vanishing exactly at the points $\{z_1, \cdots, z_N\}$ according to their multiplicity.*

On the basis of the above discussion, to establish Theorem 1.1 we only need to focus about system (1.11). We are going to attack (1.11) by perturbation techniques in a spirit similar to the work of Chae-Imanuvilov in [6] for the study of non-topological Chern-Simons vortices. In fact, the perturbative approach introduced in [6] has proven particularly useful to handle elliptic systems of Liouville type in the plane. In this respect it is important to notice that the conformal invariance of the Liouville operator: $\Delta u + e^u$ in $\mathbb{R}^2$, is the origin of some degeneracies that are manifested by an extreme sensitivity of the operator under perturbations. Therefore, it is never a standard task to make perturbation technique work successfully in this context. Concerning our system (1.11), we show how to take advantage of the specific structure of the perturbation terms in order to limit the degeneracy effect on the corresponding operator, so to restore a crucial invertibility property. In this way we are able to identify a certain neighborhood in a suitable function space where to locate our solutions. This allows us to provide a rather accurate control on the behavior of the solution at infinity, and therefore verify (1.14). The details of our perturbative method are carried out in the following section.

2 Preliminaries and Statement of the Main Result

We start by transforming (1.11) to an equivalent system. To this purpose multiply the second equation of (1.11) by e^v, and use the identity $\Delta e^v = e^v \Delta v + |\nabla v|^2 e^v$ to obtain

$$\Delta e^v = \frac{g_1^2}{2\cos^2\theta}[e^v - \phi_0^2]e^{\eta+v} + 2g_1^2 e^{u+v} + |\nabla v|^2 e^v. \tag{2.1}$$

The third equation in (1.11) added to (2.1) $\times\, 8\pi G$ gives;

$$\Delta(\eta + 8\pi G e^v) = -4\pi G g_1^2 \phi_0^4 \left(\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}\right) e^\eta + \frac{4\pi G g_1^2 \phi_0^2}{\cos^2\theta} e^{\eta+v}.$$

Thus, if we introduce the notations:

$$\lambda_1 = 4g_1^2, \lambda_2 = 4\pi G g_1^2 \phi_0^4 \left(\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}\right), \lambda_3 = \frac{g_1^2\phi_0^2}{2\cos^2\theta}, \lambda_4 = 8\pi G, \tag{2.2}$$

we arrive to the following equivalent formulation of (1.11)

$$\Delta u = -\frac{\lambda_1}{4} e^{v+\eta} - \lambda_1 e^u + 4\pi \sum_{k=1}^{N} \delta(z - z_k) \tag{2.3}$$

$$\Delta(\eta + \lambda_4 e^v) = -\lambda_2 e^\eta + \lambda_3\lambda_4 e^{\eta+v} \tag{2.4}$$

$$\Delta v = \frac{\lambda_3}{\phi_0^2} e^{v+\eta} - \lambda_3 e^\eta + \frac{\lambda_1}{2} e^u, \qquad \text{in} \quad \mathbb{R}^2. \tag{2.5}$$

To construct solutions for (2.3)-(2.5) notice that the first equation (2.3) admits a "singular" Liouville-type structure, which motivates to take

$$\int_{\mathbb{R}^2} e^u < +\infty \tag{2.6}$$

as a "natural" boundary condition. Since (2.6) is scale invariant under the transform:

$$u(x) \longrightarrow u_\varepsilon(x) = u(\frac{x}{\varepsilon}) + 2\log(\frac{1}{\varepsilon}),$$

$\forall \varepsilon > 0$, we can consider the $\varepsilon-$scaled version of (2.3)-(2.5) obtained by also transforming:

$$v(x) \longrightarrow v_\varepsilon(x) = v(\frac{x}{\varepsilon}) + 2\log(\frac{1}{\varepsilon})$$

$$\eta(x) \longrightarrow \eta_\varepsilon(x) = \eta(\frac{x}{\varepsilon}) + 2\log(\frac{1}{\varepsilon}).$$

In fact, in terms of the unknowns $(u_\varepsilon, v_\varepsilon, \eta_\varepsilon)$ system (2.3)-(2.5) takes the form:

$$\Delta u = -\varepsilon^2 \frac{\lambda_1}{4} e^{v+\eta} - \lambda_1 e^u + 4\pi \sum_{k=1}^{N} \delta(z - \varepsilon z_k) \tag{2.7}$$

$$\Delta(\eta + \varepsilon^2 \lambda_4 e^v) = -\lambda_2 e^\eta + \varepsilon^2 \lambda_3 \lambda_4 e^{\eta+v} \tag{2.8}$$

$$\Delta v = \frac{\varepsilon^2 \lambda_3}{\phi_0^2} e^{v+\eta} - \lambda_3 e^\eta + \frac{\lambda_1}{2} e^u, \qquad \text{in} \quad \mathbb{R}^2. \tag{2.9}$$

This suggests to look for solution of (2.7)-(2.9) "close" in a suitable sense to those of the system

$$\Delta u^0 = -\lambda_1 e^{u^0} + 4\pi \sum_{k=1}^{N} \delta(z - \varepsilon z_k) \tag{2.10}$$

$$\Delta \eta^0 = -\lambda_2 e^{\eta^0} \tag{2.11}$$

$$\Delta v^0 = -\lambda_3 e^{\eta^0} + \frac{\lambda_1}{2} e^{u^0}, \tag{2.12}$$

for which we can exhibit an explicit solution. To this purpose, we introduce complex notation, by setting $z = x_1 + ix_2$ for every $(x_1, x_2) \in \mathbb{R}^2$, and define:

$$f(z) = (N+1) \prod_{k=1}^{N} (z - z_k), \quad F(z) = \int_0^z f(\xi) d\xi.$$

Set

$$f_\varepsilon(z) = (N+1) \prod_{k=1}^{N} (z - \varepsilon z_k), \quad \text{and} \quad F_\varepsilon(z) = \int_0^z f_\varepsilon(\xi) d\xi,$$

then, by Liouville formula [8], we know that for every $\varepsilon > 0$ and $a, b \in \mathbb{C}$, the functions

$$u^0_{\varepsilon,a}(z) = \log \left[\frac{8|f_\varepsilon(z)|^2}{\lambda_1 \left(1 + |F_\varepsilon(z) + a|^2\right)^2} \right], \quad \eta^0_b(z) = \log \left[\frac{8}{\lambda_2 (1 + |z + b|^2)^2} \right]$$

satisfy (2.10) and (2.11) respectively. Furthermore, if we set,

$$\kappa = \frac{2\lambda_3}{\lambda_2} \tag{2.13}$$

then, we also solve (2.12) by taking,

$$v^0_{\varepsilon,a,b} = \log \left[\frac{1 + |F_\varepsilon(z) + a|^2}{(1 + |z + b|^2)^\kappa} \right].$$

Reasonably we may look for solution of (2.3)-(2.5) in the form:

$$\begin{aligned} u(z) &= u^0_{\varepsilon,a}(\varepsilon z) + 2\log\varepsilon + \varepsilon^2\sigma_1(\varepsilon z) && (2.14)\\ \eta(z) &= \eta^0_b(\varepsilon z) + 2\log\varepsilon + \varepsilon^2\sigma_2(\varepsilon z) && (2.15)\\ v(z) &= v^0_{\varepsilon,a,b}(\varepsilon z) + 2\log\varepsilon + \varepsilon^2\sigma_3(\varepsilon z) && (2.16)\end{aligned}$$

with $\sigma_1, \sigma_2, \sigma_3$ suitable functions which identify the error terms in the expansion (2.14)-(2.16) as $\varepsilon \to 0$. Introducing the notation:

$$\begin{aligned} u^0_{\varepsilon,a}(\varepsilon z) + 2\log\varepsilon &:= \log\rho^I_{\varepsilon,a}(z)\\ \eta^0_b(\varepsilon z) + 2\log\varepsilon &:= \log\rho^{II}_{\varepsilon,b}(z)\\ v^0_{\varepsilon,a,b}(\varepsilon z) + 2\log\varepsilon &:= \log\rho^{III}_{\varepsilon,a,b}(z)\end{aligned}$$

we see that,

$$\begin{aligned} \rho^I_{\varepsilon,a}(z) &= \frac{8\varepsilon^{2N+2}|f(z)|^2}{\lambda_1\left(1+\varepsilon^{2N+2}\left|F(z)+\frac{a}{\varepsilon^{N+1}}\right|^2\right)^2}\\ \rho^{II}_{\varepsilon,b}(z) &= \frac{8\varepsilon^2}{\lambda_2(1+|\varepsilon z+b|^2)^2}\\ \rho^{III}_{\varepsilon,a,b}(z) &= \frac{\varepsilon^2\left(1+\varepsilon^{2N+2}\left|F(z)+\frac{a}{\varepsilon^{N+1}}\right|^2\right)}{(1+|\varepsilon z+b|^2)^\kappa}\end{aligned}$$

are well defined also for negative ε. We prove:

Theorem 2.1 *Let $N \in \mathbb{N}$ be such that*

$$\kappa = \frac{2\lambda_3}{\lambda_2} > N+1. \qquad (2.17)$$

For given points $\{z_j\}_{j=1}^N \in \mathbb{R}^2$ (repeated according to their multiplicity), there exists $\varepsilon_1 > 0$, such that for every $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, $\varepsilon \neq 0$, problem (2.3)-(2.5) admits a solution $(u^\varepsilon, \eta^\varepsilon, v^\varepsilon)$ of the following form:

$$\begin{aligned} u^\varepsilon(z) &= \log\rho^I_{\varepsilon,a^*_\varepsilon}(z) + \varepsilon^2 w_1(\varepsilon|z|) + \varepsilon^2 u^*_{1,\varepsilon}(\varepsilon z), && (2.18)\\ \eta^\varepsilon(z) &= \log\rho^{II}_{\varepsilon,b^*_\varepsilon}(z) + \varepsilon^2 w_2(\varepsilon|z|) + \varepsilon^2 u^*_{2,\varepsilon}(\varepsilon z) && (2.19)\\ v^\varepsilon(z) &= \log\rho^{III}_{\varepsilon,a^*_\varepsilon,b^*_\varepsilon}(z) + \varepsilon^2 w_3(\varepsilon|z|) + \varepsilon^2 u^*_{3,\varepsilon}(\varepsilon z), && (2.20)\end{aligned}$$

with $\rho^I_{\varepsilon,a^*_\varepsilon}(z), \rho^{II}_{\varepsilon,b^*_\varepsilon}(z), \rho^{III}_{\varepsilon,a^*_\varepsilon,b^*_\varepsilon}(z)$ defined above and $|a^*_\varepsilon| + |b^*_\varepsilon| \to 0$, as $\varepsilon \to 0$. Furthermore, the functions w_1, w_2, w_3 are radial, and satisfy:

$$\begin{aligned} w_1(|z|) &= C_1\log|z| + O(1), && (2.21)\\ w_2(|z|) &= -C_2\log|z| + O(1), && (2.22)\\ w_3(|z|) &= C_3\log|z| + O(1) && (2.23)\end{aligned}$$

as $|z| \to \infty$, with explicit constants C_1, C_2, C_3(determined in Lemma 3.1 below); while $u^*_{1,\varepsilon}, u^*_{2,\varepsilon}, u^*_{3,\varepsilon}$ satisfy:

$$\sup_{z\in\mathbb{R}^2} \frac{\sum_{j=1}^3 |u^*_{j,\varepsilon}(\varepsilon z)|}{1+\log^+|z|} = o(1), \qquad \text{as } \varepsilon \to 0. \tag{2.24}$$

In particular, $(u^\varepsilon, \eta^\varepsilon, v^\varepsilon)$ verifies the boundary condition (1.14).

Remark: By our construction the sufficient condition (2.17) is clearly necessary to ensure the validity of the last of the boundary conditions in (1.14). Notice that in case the parameters λ_j, $j = 1, \cdots, 4$ are chosen according to (2.2), then (2.17) reads as follws,

$$\frac{\sin^2\theta}{4\pi G\phi_0^2} > N + 1,$$

and provides a sufficient condition for the existence of Electroweak selfgravitating strings as stated in Theorem 1.1, which becomes an easy consequence of Theorem 2.1. This condition is analogous to the necessary and sufficient condition obtained in [14] for the existence of Abelian Higgs strings in the Einstein-Maxwell-Higgs system. In a sense it imposes a restriction between the total string number N and the gravitational constant G which should be considered small. Here ϕ_0 plays a role of symmetry breaking parameter analogous to that in the Abelian Higgs strings model.

3 The Proof of Theorem 2.1

Following [6], we derive our result by making an appropriate use of the Implicit Function Theorem([9],[16]) over the spaces:

$$X_\alpha = \{u \in L^2_{loc}(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} (1+|x|^{2+\alpha})|u(x)|^2 dx < \infty\}$$

equipped with the norm $\|u\|^2_{X_\alpha} = \int_{\mathbb{R}^2}(1+|x|^{2+\alpha})|u(x)|^2 dx$, and

$$Y_\alpha = \{u \in W^{2,2}_{loc}(\mathbb{R}^2) \mid \|\Delta u\|^2_{X_\alpha} + \left\| \frac{u(x)}{1+|x|^{1+\frac{\alpha}{2}}} \right\|^2_{L^2(\mathbb{R}^2)} < \infty\}$$

equipped with the norm $\|u\|^2_{Y_\alpha} = \|\Delta u\|^2_{X_\alpha} + \left\| \frac{u(x)}{1+|x|^{1+\frac{\alpha}{2}}} \right\|^2_{L^2(\mathbb{R}^2)}$, where $\alpha \in (0, \frac{1}{2})$ is fixed throughout this paper. For this purpose we recall the following useful facts proved in [6].

Proposition 3.1 *For $\alpha \in (0, \frac{1}{2})$ we have:*

(i) $v \in Y_\alpha$ is harmonic if and only if $v \equiv$ constant.

(ii) There exists a constant $C_0 > 0$ such that for all $v \in Y_\alpha$ we have:

$$|v(x)| \leq C_0 \|v\|_{Y_\alpha} (\log^+ |x| + 1), \qquad \forall x \in \mathbb{R}^2,$$

where $\log^+ |x| = \max\{ \log |x|, 0\}$.

Since we are going to search for solutions (u, η, v) in the form (2.14)-(2.16), by direct inspection we see that the functions σ_j , $j = 1, 2, 3$ must satisfy:

$$\Delta\sigma_1 = -\frac{\lambda_1}{4} g_b^{II}(z) g_{\varepsilon,a,b}^{III}(z) e^{\varepsilon^2(\sigma_2+\sigma_3)} - \frac{\lambda_1}{\varepsilon^2} g_{\varepsilon,a}^{I}(z)(e^{\varepsilon^2\sigma_1} - 1) \tag{3.1}$$

$$\Delta\sigma_2 = -\lambda_4 \Delta[g_{\varepsilon,a,b}^{III}(z) e^{\varepsilon^2\sigma_3}] - \frac{\lambda_2}{\varepsilon^2} g_b^{II}(z)(e^{\varepsilon^2\sigma_2} - 1) + \lambda_3\lambda_4 g_b^{II}(z) g_{\varepsilon,a,b}^{III}(z) e^{\varepsilon^2(\sigma_2+\sigma_3)} \tag{3.2}$$

$$\Delta\sigma_3 = \frac{\lambda_3}{\phi_0^2} g_b^{II}(z) g_{\varepsilon,a,b}^{III}(z) e^{\varepsilon^2(\sigma_2+\sigma_3)} - \frac{\lambda_3}{\varepsilon^2} g_b^{II}(z)(e^{\varepsilon^2\sigma_2} - 1) + \frac{\lambda_1}{2\varepsilon^2} g_{\varepsilon,a}^{I}(z)(e^{\varepsilon^2\sigma_1} - 1), \tag{3.3}$$

where we have set

$$g_{\varepsilon,a}^{I}(z) = e^{u_{\varepsilon,a}}, \quad g_b^{II}(z) = e^{\eta_b^0}, \quad g_{\varepsilon,a,b}^{III}(z) = e^{v_{\varepsilon,a,b}^0}.$$

In order to determine the triplet $(\sigma_1, \sigma_2, \sigma_3)$ we are going to consider the free parameters $a, b \in \mathbb{C}$ above as part of our unknowns. More precisely, we concentrate around the values $a = 0$, $b = 0$, and consider the radial functions:

$$\rho_1 = \lim_{\varepsilon \to 0} g_{\varepsilon,0}^{I} = \frac{8(N+1)^2 r^{2N}}{\lambda_1 (1 + r^{2N+2})^2}, \quad \rho_2 = g_0^{II} = \frac{8}{\lambda_2 (1 + r^2)^2},$$

and

$$\rho_3 = \lim_{\varepsilon \to 0} g_{\varepsilon,0}^{III} = \frac{1 + r^{2N+2}}{(1 + r^2)^\kappa}.$$

Thus, by taking $a = b = 0$ in (3.1), (3.2) and (3.3) and letting $\varepsilon \to 0$, (formally) we obtain the linear system:

$$\Delta w_1 + \lambda_1 \rho_1 w_1 = -\frac{\lambda_1}{4} \rho_2 \rho_3 \tag{3.4}$$

$$\Delta w_2 + \lambda_2 \rho_2 w_2 = -\lambda_4 \Delta\rho_3 + \lambda_3 \lambda_4 \rho_2 \rho_3 \tag{3.5}$$

$$\Delta w_3 = \frac{1}{2} \lambda_1 \rho_1 w_1 - \lambda_3 \rho_2 w_2 + \frac{\lambda_3}{\phi_0^2} \rho_2 \rho_3. \tag{3.6}$$

Consequently, if we let (w_1, w_2, w_3) be a solution of (3.4), (3.5), (3.6) then, under the decomposition

$$\sigma_j(z) = w_j(z) + u_j(z), \qquad j = 1, 2, 3, \tag{3.7}$$

we reduce to solve for $(u_1,u_2,u_3)$the following implicit problem:

$$\begin{aligned} P_1(u_1,u_2,u_3,a,b,\varepsilon) &= \Delta u_1 + \frac{\lambda_1}{4} g_b^{II}(z) g_{\varepsilon,a,b}^{III}(z) e^{\varepsilon^2(u_2+u_3+w_2+w_3)} \\ &\quad + \frac{\lambda_1}{\varepsilon^2} g_{\varepsilon,a}^{I}(z)(e^{\varepsilon^2(u_1+w_1)}-1) + \Delta w_1 = 0, \end{aligned}$$

$$P_2(u_1,u_2,u_3,a,b,\varepsilon) = \Delta\left(u_2 + \lambda_4 g_{\varepsilon,a,b}^{III}(z) e^{\varepsilon^2(u_3+w_3)}\right)$$
$$+\frac{\lambda_2}{\varepsilon^2} g_b^{II}(z)(e^{\varepsilon^2(u_2+w_2)}-1) - \lambda_3\lambda_4 g_b^{II}(z) g_{\varepsilon,a,b}^{III}(z) e^{\varepsilon^2(u_2+u_3+w_2+w_3)} + \Delta w_2 = 0,$$

and

$$P_3(u_1,u_2,u_3,a,b,\varepsilon) = \Delta u_3 - \frac{\lambda_3}{\phi_0^2} g_b^{II}(z) g_{\varepsilon,a,b}^{III}(z) e^{\varepsilon^2(u_2+u_3+w_2+w_3)}$$
$$+\frac{\lambda_3}{\varepsilon^2} g_b^{II}(z)(e^{\varepsilon^2(u_2+w_2)}-1) - \frac{\lambda_1}{2\varepsilon^2} g_{\varepsilon,a}^{I}(z)(e^{\varepsilon^2(u_1+w_1)}-1) + \Delta w_3 = 0.$$

We aim to apply the Implicit Function Theorem to the operator $P=(P_1,P_2,P_3)$ around the origin. For this purpose we start by constructing a suitable solution set for the above linear system (3.4)-(3.6).

Lemma 3.1 *For $\kappa > N$ there exists a radial solution (w_1,w_2,w_3) of (3.4)-(3.6) in Y_α^3 satisfying:*

$$\begin{aligned} w_1(r) &= C_1 \log r + O(1), \quad \text{and} \quad w_1'(r) = \frac{C_1}{r} + O(1) && (3.8)\\ w_2(r) &= -C_2 \log r + O(1), \quad \text{and} \quad w_2'(r) = -\frac{C_2}{r} + O(1) && (3.9)\\ w_3(r) &= C_3 \log r + O(1), \quad \text{and} \quad w_3'(r) = \frac{C_3}{r} + O(1) && (3.10) \end{aligned}$$

as $r \to \infty$, with

$$\begin{aligned} C_1 &= \frac{\lambda_1}{\lambda_2}\left[\frac{\kappa(\kappa-1)\cdots(\kappa-N)-(N+1)!}{(1+\kappa)\kappa\cdots(\kappa-N)}\right], \quad \text{and } C_1>0 \text{ for } \kappa > N+1;\\ C_2 &= \frac{4(\lambda_2+\lambda_3)\lambda_4[\kappa^2(\kappa-1)\cdots(\kappa-N)+(\kappa-2N-2)(N+1)!]}{\lambda_2(2+\kappa)(1+\kappa)\cdots(\kappa-N)},\\ &\qquad \text{and } C_2>0 \text{ for } \kappa > N+1;\\ C_3 &= -\frac{C_1}{2} - C_2\frac{\lambda_3}{\lambda_2} + \frac{4\mu}{(\kappa+1)\lambda_2}; \end{aligned}$$

respectively, with $\mu = \frac{\lambda_3}{\phi_0^2} - \frac{\lambda_3^2\lambda_4}{\lambda_2} - \frac{\lambda_1}{8}$ and κ defined in (2.13).

Before going into the proof of Lemma 3.1, we recall the following properties relative to the operators defined by the right hand side of (3.4) and (3.5), useful also in the sequel. We refer to [6] and [4] for the proof.

Proposition 3.2 *For $\alpha \in (0, \frac{1}{2})$ and $j = 1, 2$, set*

$$L_j = \Delta + \lambda_j \rho_j : Y_\alpha \to X_\alpha.$$

We have

$$KerL_j = Span\{\varphi_{j,+}, \varphi_{j,-}, \varphi_{j,0}\}, \tag{3.11}$$

where,

$$\varphi_{1,+} = \frac{r^{N+1}\cos(N+1)\theta}{1+r^{2N+2}}, \qquad \varphi_{1,-} = \frac{r^{N+1}\sin(N+1)\theta}{1+r^{2N+2}},$$

$$\varphi_{2,+} = \frac{r\cos\theta}{1+r^2}, \qquad \varphi_{2,-} = \frac{r\sin\theta}{1+r^2},$$

$$\varphi_{1,0} = \frac{1-r^{2(N+1)}}{1+r^{2(N+1)}}, \qquad \varphi_{2,0} = \frac{1-r^2}{1+r^2}.$$

Moreover,

$$ImL_j = \{f \in X_\alpha | \int_{\mathbb{R}^2} f\varphi_{j,\pm} = 0\}. \tag{3.12}$$

Proof of Lemma 3.1: Taking into account Proposition 3.2, it is possible to use a variation of parameters formula, in order to see that a radial solution of

$$\Delta w(r) + \lambda_1 \rho_1 w(r) = f(r), \tag{3.13}$$

may be obtained by means of the formula:

$$w(r) = \varphi_{1,0}(r)\left\{\int_0^r \frac{\phi_f(s)-\phi_f(1)}{(1-s)^2}ds + \frac{\phi_f(1)r}{1-r}\right\} \tag{3.14}$$

with

$$\phi_f(r) := \left(\frac{1+r^{2N+2}}{1-r^{2N+2}}\right)^2 \frac{(1-r)^2}{r}\int_0^r \varphi_{1,0}(t)tf(t)dt,$$

and

$$\varphi_{1,0}(r) := \frac{1-r^{2N+2}}{1+r^{2N+2}},$$

where $\phi_f(1)$ and $w_1(1)$ are the well-defined limits of $\phi_f(r)$ and $w_1(r)$, as $r \to 1$. See [6] and [4]. To obtain w_1 we use formula (3.14) with $f(r) = -\frac{\lambda_1}{4}\rho_2(r)\rho_3(r)$. We find,

$$w_1(r) = -\frac{\lambda_1}{4}\varphi_{1,0}(r)\int_2^r \left(\frac{1+s^{2N+2}}{1-s^{2N+2}}\right)^2 \frac{A_1(s)}{s}ds + O(1) \tag{3.15}$$

as $r \to \infty$, where

$$A_1(s) = \int_0^s \varphi_{1,0}(t)t\rho_2(t)\rho_3(t)dt.$$

Since $\varphi_{1,0}(r) \to -1$ and $\varphi'_{1,0}(r) \to 0$ as $r \to \infty$, to obtain (3.8) we only need to evaluate,

$$
\begin{aligned}
A_1 = A_1(\infty) &= \int_0^\infty \varphi_{1,0}(r) r \rho_2(r)\rho_3(r)dr \\
&= \frac{8}{\lambda_2}\int_0^\infty \frac{(1-r^{2N+2})r}{(1+r^2)^{2+\kappa}}dr \\
&= \frac{4}{\lambda_2}\int_0^\infty \frac{1-t^{N+1}}{(1+t)^{2+\kappa}}dt \\
&= \frac{4}{\lambda_2}\left[\frac{1}{1+\kappa} - \frac{(N+1)!}{(1+\kappa)\kappa\cdots(\kappa-N)}\right] \\
&= \frac{4}{\lambda_2}\left[\frac{\kappa(\kappa-1)\cdots(\kappa-N)-(N+1)!}{(1+\kappa)\kappa\cdots(\kappa-N)}\right].
\end{aligned}
$$

So, $A_1 > 0$ for $\kappa > N+1$, and (3.8) is proved. To obtain w_2 we use the analogous of formula (3.14) for the operator L_2 which now holds with $N = 0$ and $\varphi_{2,0}$ to replace $\varphi_{1,0}$. Exactly as above we reduce to evaluate,

$$
A_2 = A_2(\infty) = \int_0^\infty \varphi_{2,0}(r) f(r) r dr, \tag{3.16}
$$

with $f(r) = \lambda_3\lambda_4\rho_2\rho_3 - \lambda_4\Delta\rho_3$. Since $\varphi_{2,0} \in KerL_2$, integration by part, yields to the identity,

$$
\int_0^\infty \varphi_{2,0}\Delta\rho_3 r dr = \int_0^\infty \Delta\varphi_{2,0}\rho_3 r dr = -\lambda_2\int_0^\infty \varphi_{2,0}\rho_2\rho_3 r dr. \tag{3.17}
$$

Consequently,

$$
\begin{aligned}
A_2 &= (\lambda_2+\lambda_3)\lambda_4\int_0^\infty \varphi_{2,0}\rho_2\rho_3 r dr \\
&= \frac{8(\lambda_2+\lambda_3)\lambda_4}{\lambda_2}\int_0^\infty \frac{(1-r^2)(1+r^{2N+2})}{(1+r^2)^{3+\kappa}} r dr \\
&= \frac{4(\lambda_2+\lambda_3)\lambda_4}{\lambda_2}\int_0^\infty \frac{(1-t)(1+t^{N+1})}{(1+t)^{3+\kappa}}dt \\
&= \frac{4(\lambda_2+\lambda_3)\lambda_4}{\lambda_2}\int_0^\infty \left[\frac{1}{(1+t)^{3+\kappa}} - \frac{t}{(1+t)^{3+\kappa}} + \frac{t^{N+1}}{(1+t)^{3+\kappa}} - \frac{t^{N+2}}{(1+t)^{3+\kappa}}\right]dt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{4(\lambda_2+\lambda_3)\lambda_4}{\lambda_2}\left[\frac{1}{2+\kappa}-\frac{1}{(2+\kappa)(1+\kappa)}+\frac{(N+1)!}{(2+\kappa)(1+\kappa)\cdots(1+\kappa-N)}\right.\\
&\qquad\left.-\frac{(N+2)!}{(2+\kappa)(1+\kappa)\cdots(\kappa-N)}\right]\\
&= \frac{4(\lambda_2+\lambda_3)\lambda_4}{\lambda_2(2+\kappa)(1+\kappa)\cdots(\kappa-N)}[(\kappa+1)\kappa\cdots(\kappa-N)-\kappa(\kappa-1)\cdots(\kappa-N)\\
&\qquad+(\kappa-N)(N+1)!-(N+2)!]\\
&= \frac{4(\lambda_2+\lambda_3)\lambda_4[\kappa^2(\kappa-1)\cdots(\kappa-N)+(\kappa-2N-2)(N+1)!]}{\lambda_2(2+\kappa)(1+\kappa)\cdots(\kappa-N)}, \qquad (3.18)
\end{aligned}
$$

and, (3.9) is also proved. In order to obtain w_3 with the given asymptotic expansion, we use the following decomposition:

$$
w_3(r) = -\frac{w_1(r)}{2}+\frac{\lambda_3}{\lambda_2}w_2(r)+\frac{\lambda_3\lambda_4}{\lambda_2}\rho_3(r)+\varphi(r), \qquad (3.19)
$$

where φ is a regular radial function satisfying:

$$
\Delta\varphi = \left(\frac{\lambda_3}{\phi_0^2}-\frac{\lambda_3^2\lambda_4}{\lambda_2}-\frac{\lambda_1}{8}\right)\rho_2\rho_3.
$$

Set

$$
\mu = \frac{\lambda_3}{\phi_0^2}-\frac{\lambda_3^2\lambda_4}{\lambda_2}-\frac{\lambda_1}{8}, \qquad (3.20)
$$

Incidentally notice that by the choice of λ_j, $j=1,\cdots,4$, as in (2.2) we have $\mu = \frac{g_1^2}{2}\sin^4\theta(1+\cos^2\theta)$. Hence,

$$
\begin{aligned}
r\varphi'(r) &= \frac{8\mu}{\lambda_2}\int_0^r \frac{(1+r^{2N+2})r}{(1+r^2)^{\kappa+2}}dr = \frac{4\mu}{\lambda_2}\int_0^{r^2}\frac{1+t^{N+1}}{(1+t)^{\kappa+2}}dt\\
&= \frac{4\mu}{\lambda_2(\kappa+1)}\left(1-\frac{1}{(1+r^2)^{\kappa+1}}\right)+\frac{4\mu}{\lambda_2}\int_0^{r^2}\frac{t^{N+1}}{(1+t)^{\kappa+2}}dt.
\end{aligned}
$$

Consequently, using the fact that $\kappa > N$, as $r\to+\infty$ we find $r\varphi'(r)\to\frac{4\mu}{\lambda_2(\kappa+1)}$ and,

$$
\varphi(r) = \frac{4\mu}{(\kappa+1)\lambda_2}\log r + O(1).
$$

In view of (3.19) we derive the desired conclusion for w_3, and complete the proof.□

Remark: Observe that with the choice of (w_1,w_2,w_3) as in Lemma 3.1 and the condition $\kappa > N+1$, for $0<\alpha<\min\{\frac{1}{2},\kappa-N-1\}$ there exists $\varepsilon_0>0$ such that the operator $P=(P_1,P_2,P_3)$ defined above is a continuous mapping from $\Omega_{\varepsilon_0} = \{(u,a,b,\varepsilon)\in {Y_\alpha}^3\times\mathbb{C}^2\times\mathbb{R} : \|u\|_{{Y_\alpha}^3}+|a|+|b|+|\varepsilon|<\varepsilon_0\}$

into ${X_\alpha}^3$ and $P(0,0,0,0,0,0)=0$.

Next we proceed to compute the linearized operator of P around zero. From tedious but not difficult computations we see that, for $a=a_1+ia_2$ and $b=b_1+ib_2$, we have

$$\frac{\partial g^I_{\varepsilon,a}(z)}{\partial a_1}\bigg|_{(a,\varepsilon)=(0,0)} = -4\rho_1\varphi_{1,+}, \qquad \frac{\partial g^I_{\varepsilon,a}(z)}{\partial a_2}\bigg|_{(a,\varepsilon)=(0,0)} = -4\rho_1\varphi_{1,-},$$

$$\frac{\partial g^{II}_b(z)}{\partial b_1}\bigg|_{b=0} = -4\rho_2\varphi_{2,+}, \qquad \frac{\partial g^{II}_b(z)}{\partial b_2}\bigg|_{b=0} = -4\rho_2\varphi_{2,-},$$

$$\frac{\partial g^{III}_{\varepsilon,a,b}(z)}{\partial a_1}\bigg|_{(a,b,\varepsilon)=(0,0,0)} = 2\rho_3\varphi_{1,+}, \qquad \frac{\partial g^{III}_{\varepsilon,a,b}(z)}{\partial a_2}\bigg|_{(a,b,\varepsilon)=(0,0,0)} = 2\rho_3\varphi_{1,-},$$

$$\frac{\partial g^{III}_{\varepsilon,a,b}(z)}{\partial b_1}\bigg|_{(a,b,\varepsilon)=(0,0,0)} = -\frac{4\lambda_3}{\lambda_2}\rho_3\varphi_{2,+}, \qquad \frac{\partial g^{III}_{\varepsilon,a,b}(z)}{\partial b_2}\bigg|_{(a,b,\varepsilon)=(0,0,0)} = -\frac{4\lambda_3}{\lambda_2}\rho_3\varphi_{2,-},$$

$$\frac{\partial g^{II}_b(z)g^{III}_{\varepsilon,a,b}(z)}{\partial a_1}\bigg|_{(a,b,\varepsilon)=(0,0,0)} = 2\rho_2\rho_3\varphi_{1,+}, \qquad \frac{\partial g^{II}_b(z)g^{III}_{\varepsilon,a,b}(z)}{\partial a_2}\bigg|_{(a,b,\varepsilon)=(0,0,0)} = 2\rho_2\rho_3\varphi_{1,-},$$

$$\frac{\partial g^{II}_b(z)g^{III}_{\varepsilon,a,b}(z)}{\partial b_1}\bigg|_{(a,b,\varepsilon)=(0,0,0)} = -4(1+\frac{\lambda_3}{\lambda_2})\rho_2\rho_3\varphi_{2,+},$$

$$\frac{\partial g^{II}_b(z)g^{III}_{\varepsilon,a,b}(z)}{\partial b_2}\bigg|_{(a,b,\varepsilon)=(0,0,0)} = -4(1+\frac{\lambda_3}{\lambda_2})\rho_2\rho_3\varphi_{2,-}.$$

Therefore, setting

$$P'_{(u_1,u_2,u_3,a,b)}(0,0,0,0,0,0)[v_1,v_2,v_3,\alpha,\beta] = \mathcal{A}[v_1,v_2,v_3,\alpha,\beta],$$

we can check that for $\mathcal{A}=(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)$, $\alpha=\alpha_1+i\alpha_2$ and $\beta=\beta_1+i\beta_2$ we have:

$$\begin{aligned}\mathcal{A}_1[v_1,v_2,v_3,\alpha,\beta] &= \Delta v_1+\lambda_1\rho_1 v_1\\ &\quad +\lambda_1\left[-4\rho_1 w_1+\frac{1}{2}\rho_2\rho_3\right](\varphi_{1,+}\alpha_1+\varphi_{1,-}\alpha_2)\\ &\quad -\lambda_1(\frac{\lambda_3}{\lambda_2}+1)\rho_2\rho_3(\varphi_{2,+}\beta_1+\varphi_{2,-}\beta_2),\end{aligned} \tag{3.21}$$

$$\begin{aligned}&\mathcal{A}_2[v_1,v_2,v_3,\alpha,\beta] = \Delta v_2+\lambda_2\rho_2 v_2\\ &\quad -2\lambda_3\lambda_4\rho_2\rho_3(\varphi_{1,+}\alpha_1+\varphi_{1,-}\alpha_2)-2\lambda_4\Delta[\rho_3(\varphi_{1,+}\alpha_1+\varphi_{1,-}\alpha_2)]\\ &\quad -4\left[\lambda_2\rho_2 w_2-\lambda_3\lambda_4(1+\frac{\lambda_3}{\lambda_2})\rho_2\rho_3\right](\varphi_{2,+}\beta_1+\varphi_{2,-}\beta_2)\\ &\quad -4\frac{\lambda_4\lambda_3}{\lambda_2}\Delta[\rho_3(\varphi_{2,+}\beta_1+\varphi_{2,-}\beta_2)],\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
\mathcal{A}_3[v_1,v_2,v_3,\alpha,\beta] &= \Delta v_3+\lambda_3\rho_2 v_2-\frac{\lambda_1}{2}\rho_1 v_1 \\
&\quad+\left[2\lambda_1\rho_1 w_1-\frac{2\lambda_3}{\phi_0^2}\rho_2\rho_3\right](\varphi_{1,+}\alpha_1+\varphi_{1,-}\alpha_2) \\
&\quad-\left[4\lambda_3\rho_2 w_1-\frac{4\lambda_3}{\phi_0^2}(\frac{\lambda_3}{\lambda_2}+1)\rho_2\rho_3\right](\varphi_{2,+}\beta_1+\varphi_{2,-}\beta_2).
\end{aligned} \tag{3.23}$$

It is interesting to note that although we need the condition $\kappa > N+1$ in order to have that the operator P is well defined from ${Y_\alpha}^3\times\mathbb{C}^2\times(-\varepsilon_0,\varepsilon_0)$ into ${X_\alpha}^3$, its linearized operator at the origin, $\mathcal{A}=(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)$, given in (3.21)-(3.23), appears to be well defined from ${Y_\alpha}^3\times\mathbb{C}^2$ into ${X_\alpha}^3$ only under the weaker assumption $\kappa > N$, which also suffices to ensure the following crucial properties:

Proposition 3.3 *If $\kappa > N$, then the operator $\mathcal{A}:(Y_\alpha)^3\times(\mathbb{C})^2\to(X_\alpha)^3$ given by (3.21)-(3.23) is onto. Moreover,*

$$\begin{aligned}
Ker\mathcal{A} &= Span\Big\{(0,0,1);(\varphi_{1,\pm},\varphi_{2,\pm},-\frac{1}{2}\varphi_{1,\pm}+\frac{\lambda_3}{\lambda_2}\varphi_{2,\pm}); \\
&\quad(\varphi_{1,0},\varphi_{2,0},-\frac{1}{2}\varphi_{1,0}+\frac{\lambda_3}{\lambda_2}\varphi_{2,0});(\varphi_{1,\pm},\varphi_{2,0},-\frac{1}{2}\varphi_{1,\pm}+\frac{\lambda_3}{\lambda_2}\varphi_{2,0}); \\
&\quad(\varphi_{1,0},\varphi_{2,\pm},-\frac{1}{2}\varphi_{1,0}+\frac{\lambda_3}{\lambda_2}\varphi_{2,\pm})\Big\}\times\{(0,0)\}^2.
\end{aligned} \tag{3.24}$$

In order to prove the proposition above we establish the following,

Lemma 3.2 *Let $\kappa > N$, then*

$$I_1^\pm := \int_{\mathbb{R}^2}\left[-4\rho_1 w_1+\frac{1}{2}\rho_2\rho_3\right]\varphi_{1,\pm}^2 dx=\frac{2\pi}{\lambda_2(\kappa+1)}, \tag{3.25}$$

and

$$\begin{aligned}
I_2^\pm &:= \int_{\mathbb{R}^2}\left[-\lambda_2\rho_2 w_2+\lambda_3\lambda_4(\frac{\lambda_3}{\lambda_2}+1)\rho_2\rho_3\right]\varphi_{2,\pm}^2 dx \\
&\qquad -\frac{\lambda_3\lambda_4}{\lambda_2}\int_{\mathbb{R}^2}\Delta(\rho_3\varphi_{2,\pm})\varphi_{2,\pm}dx \\
&= \frac{\pi\lambda_4(N+1)!(N+1)}{(1+\kappa)\kappa\cdots(1+\kappa-N)}
\end{aligned} \tag{3.26}$$

with w_1 and w_2 as given by Lemma 3.1.

Proof: We prove (3.25) by recalling the formula

$$L_1\left[\frac{1}{(1+r^{2N+2})^2}\right]=\frac{16(N+1)^2r^{4N+2}}{(1+r^{2N+2})^4},$$

and computing,

$$\begin{aligned}
I_1^{\pm} &= \int_0^{2\pi}\int_0^{\infty}\left[-4\rho_1 w_1+\frac{1}{2}\rho_2\rho_3\right]\frac{r^{2N+2}}{(1+r^{2N+2})^2}\left\{\begin{array}{c}\cos^2(N+1)\theta\\ \sin^2(N+1)\theta\end{array}\right\}rdrd\theta\\
&= \pi\int_0^{\infty}\left[-\frac{32(N+1)^2r^{2N}}{\lambda_1(1+r^{2N+2})^2}w_1+\frac{1}{2}\rho_2\rho_3\right]\frac{r^{2N+2}}{(1+r^{2N+2})^2}rdr\\
&= \pi\int_0^{\infty}\left\{-\frac{2}{\lambda_1}L_1\left[\frac{1}{(1+r^{2N+2})^2}\right]w_1+\frac{\rho_2\rho_3r^{2N+2}}{2(1+r^{2N+2})^2}\right\}rdr\\
&= \pi\int_0^{\infty}\left\{-\frac{2}{\lambda_1}\frac{L_1w_1}{(1+r^{2N+2})^2}+\frac{\rho_2\rho_3r^{2N+2}}{2(1+r^{2N+2})^2}\right\}rdr\\
&= \pi\int_0^{\infty}\left\{\frac{\rho_2\rho_3}{2(1+r^{2N+2})^2}+\frac{\rho_2\rho_3r^{2N+2}}{2(1+r^{2N+2})^2}\right\}rdr\\
&= \frac{\pi}{2}\int_0^{\infty}\frac{\rho_2\rho_3}{(1+r^{2N+2})}rdr=\frac{4\pi}{\lambda_2}\int_0^{\infty}\frac{rdr}{(1+r^2)^{\kappa+2}}=\frac{2\pi}{\lambda_2(\kappa+1)},
\end{aligned}$$

where, the integration by parts performed above is justified by the asymptotic behavior (in Lemma 3.1) of w_1 and its derivative, as $r\to+\infty$. In order to prove (3.26) we use integration by part to obtain:

$$\begin{aligned}
I_2^{\pm} &= \int_{\mathbb{R}^2}\left[-\lambda_2\rho_2w_2+\lambda_3\lambda_4(1+\frac{\lambda_3}{\lambda_2})\rho_2\rho_3\right]\varphi_{2,\pm}^2dx\\
&\qquad -\frac{\lambda_3\lambda_4}{\lambda_2}\int_{\mathbb{R}^2}\rho_3\varphi_{2,\pm}\Delta\varphi_{2,\pm}dx\\
&= \int_{\mathbb{R}^2}\left[-\lambda_2\rho_2w_2+\lambda_3\lambda_4(2+\frac{\lambda_3}{\lambda_2})\rho_2\rho_3\right]\varphi_{2,\pm}^2dx, \qquad (3.27)
\end{aligned}$$

where again by (3.11) we used that $-\Delta\varphi_{2,\pm}=\lambda_2\rho_2\varphi_{2,\pm}$. In view of the identity:

$$L_2\left[\frac{1}{(1+r^2)^2}\right]=\frac{16r^2}{(1+r^2)^4},$$

we may transform the first term of $I_2^\pm$ as follows

$$\begin{aligned}
&-\int_{\mathbb{R}^2} \lambda_2\rho_2 w_2 \varphi_{2,\pm}^2 dx \\
&= -\int_0^\infty\int_0^{2\pi} \lambda_2\rho_2 w_2 \frac{r^2}{(1+r^2)^2}\left\{\begin{array}{c}\cos^2\theta \\ \sin^2\theta\end{array}\right\} r dr d\theta \\
&= -8\pi\int_0^\infty \frac{r^2}{(1+r^2)^4} w_2 r dr = -\frac{\pi}{2}\int_0^\infty L_2\left[\frac{1}{(1+r^2)^2}\right] w_2 r dr \\
&= -\frac{\pi}{2}\int_0^\infty \frac{L_2 w_2}{(1+r^2)^2} r dr \\
&= -\frac{\pi}{2}\int_0^\infty \frac{1}{(1+r^2)^2}\left[\lambda_3\lambda_4\rho_2\rho_3 - \lambda_4\Delta\rho_3\right] r dr,
\end{aligned}$$

where we used (3.5) to derive the last identity. Substituting this result into (3.27), we find,

$$\begin{aligned}
I_2^\pm &= -\frac{\pi}{2}\lambda_3\lambda_4\int_0^\infty \frac{\rho_2\rho_3}{(1+r^2)^2} r dr + \frac{\pi}{2}\lambda_4\int_0^\infty \frac{\Delta\rho_3}{(1+r^2)^2} r dr \\
&\quad +\pi\lambda_3\lambda_4(2+\frac{\lambda_3}{\lambda_2})\int_0^\infty \frac{\rho_2\rho_3 r^3}{(1+r^2)^2} dr \\
&= J_1 + J_2 + J_3.
\end{aligned}$$

We can rewrite J_1, J_3 as follows

$$J_1 = -\frac{\pi}{16}\lambda_2\lambda_3\lambda_4\int_0^\infty \rho_2^2\rho_3 r dr, \tag{3.28}$$

$$J_3 = \frac{\pi}{8}\lambda_2\lambda_3\lambda_4(2+\frac{\lambda_3}{\lambda_2})\int_0^\infty \rho_2^2\rho_3 r^3 dr. \tag{3.29}$$

Also observe that,

$$\Delta\rho_2 = \lambda_2(2r^2-1)\rho_2^2,$$

as it can be easily checked. Therefore, for $\kappa > N$ we can perform integration by parts and obtain,

$$\begin{aligned}
J_2 &= \frac{\pi}{16}\lambda_2\lambda_4\int_0^\infty \Delta\rho_3\rho_2 r dr = \frac{\pi}{16}\lambda_2\lambda_4\int_0^\infty \rho_3\Delta\rho_2 r dr \\
&= \frac{\pi}{16}\lambda_2^2\lambda_4\int_0^\infty (2r^3 - r)\rho_2^2\rho_3 dr. \qquad (3.30)
\end{aligned}$$

Consequently,

$$\begin{aligned}
I_2^{\pm} &= J_1 + J_2 + J_3 \\
&= \frac{\pi}{16}\lambda_2\lambda_3\lambda_4 \int_0^{\infty} \left[(4 + 2\frac{\lambda_3}{\lambda_2})r^3 - r\right] \rho_2^2\rho_3 dr \\
&\qquad + \frac{\pi}{16}\lambda_2^2\lambda_4 \int_0^{\infty} (2r^3 - r)\rho_2^2\rho_3 dr \\
&= \frac{\pi}{32}\lambda_2^2\lambda_4\kappa \int_0^{\infty} \left[(4+\kappa)r^3 - r\right] \rho_2^2\rho_3 dr \\
&\qquad + \frac{\pi}{16}\lambda_2^2\lambda_4 \int_0^{\infty} (2r^3 - r)\rho_2^2\rho_3 dr \\
&= \frac{\pi}{32}\lambda_2^2\lambda_4(\kappa+2)\left[(\kappa+2)K_1 - K_2\right], \qquad (3.31)
\end{aligned}$$

where,

$$K_1 = \int_0^{\infty} r^3\rho_2^2\rho_3 dr, \quad \text{and} \quad K_2 = \int_0^{\infty} r\rho_2^2\rho_3 dr.$$

We evaluate,

$$\begin{aligned}
K_1 &= \frac{64}{\lambda_2^2}\int_0^{\infty} \frac{r^3(1+r^{2N+2})}{(1+r^2)^{4+\kappa}} dr \\
&= \frac{32}{\lambda_2^2}\left[\int_0^{\infty} \frac{t}{(1+t)^{4+\kappa}} dt + \int_0^{\infty} \frac{t^{N+2}}{(1+t)^{4+\kappa}} dt\right] \\
&= \frac{32}{\lambda_2^2}\left[\frac{1}{(3+\kappa)(2+\kappa)} + \frac{(N+2)!}{(3+\kappa)(2+\kappa)\cdots(1+\kappa-N)}\right],
\end{aligned} \qquad (3.32)$$

and

$$\begin{aligned}
K_2 &= \frac{64}{\lambda_2^2}\int_0^{\infty} \frac{r(1+r^{2N+2})}{(1+r^2)^{4+\kappa}} dr \\
&= \frac{32}{\lambda_2^2}\left[\int_0^{\infty} \frac{1}{(1+t)^{4+\kappa}} dt + \int_0^{\infty} \frac{t^{N+1}}{(1+t)^{4+\kappa}} dt\right] \\
&= \frac{32}{\lambda_2^2}\left[\frac{1}{3+\kappa} + \frac{(N+1)!}{(3+\kappa)(2+\kappa)\cdots(2+\kappa-N)}\right].
\end{aligned} \qquad (3.33)$$

Substituting (3.32) and (3.33) into (3.31), we obtain

$$\begin{aligned}
I_2^{\pm} &= \pi(\kappa+2)\lambda_4\left[\frac{1}{3+\kappa}+\frac{(N+2)!}{(3+\kappa)(1+\kappa)\kappa\cdots(1+\kappa-N)}\right.\\
&\qquad\qquad\qquad\left.-\frac{1}{3+\kappa}-\frac{(N+1)!}{(3+\kappa)(2+\kappa)\cdots(2+\kappa-N)}\right]\\
&= \frac{\pi(\kappa+2)\lambda_4(N+1)![(N+2)(2+\kappa)-(1+\kappa-N)]}{(3+\kappa)(2+\kappa)\cdots(1+\kappa-N)}\\
&= \frac{\pi\lambda_4(N+1)!(N+1)}{(1+\kappa)\kappa\cdots(1+\kappa-N)}.
\end{aligned}$$

This completes the proof of Lemma 3.2. □

Proof of Proposition 2.3: Given $f=(f_1,f_2,f_3)\in(X_\alpha)^3$, we need to show the solvability in ${Y_\alpha}^3\times\mathbb{C}^2$ of the linear equation:

$$A[v_1,v_2,v_3,\alpha,\beta]=f. \tag{3.34}$$

Equivalently,

$$\begin{aligned}
&L_1v_1+\lambda_1\left[-4\rho_1w_1+\frac{1}{2}\rho_2\rho_3\right](\varphi_{1,+}\alpha_1+\varphi_{1,-}\alpha_2)\\
&\qquad -\lambda_1(\frac{\lambda_3}{\lambda_2}+1)\rho_2\rho_3(\varphi_{2,+}\beta_1+\varphi_{2,-}\beta_2)=f_1,
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
&L_2v_2-2\lambda_3\lambda_4\rho_2\rho_3(\varphi_{1,+}\alpha_1+\varphi_{1,-}\alpha_2)-2\lambda_4\Delta[\rho_3(\varphi_{1,+}\alpha_1+\varphi_{1,-}\alpha_2)]\\
&\quad -4\left[\lambda_2\rho_2w_2-\lambda_3\lambda_4(\frac{\lambda_3}{\lambda_2}+1)\rho_2\rho_3\right](\varphi_{2,+}\beta_1+\varphi_{2,-}\beta_2)\\
&\qquad -4\frac{\lambda_4\lambda_3}{\lambda_2}\Delta[\rho_3(\varphi_{2,+}\beta_1+\varphi_{2,-}\beta_2)]=f_2,
\end{aligned} \tag{3.36}$$

$$\begin{aligned}
&\Delta v_3+\lambda_3\rho_2v_2-\frac{\lambda_1}{2}\rho_1v_1+\left[2\lambda_1\rho_1w_1-\frac{2\lambda_3}{\varphi_0^2}\rho_2\rho_3\right](\varphi_{1,+}\alpha_1+\varphi_{1,-}\alpha_2)\\
&\quad -\left[4\lambda_3\rho_2w_1-\frac{4\lambda_3}{\varphi_0^2}(\frac{\lambda_3}{\lambda_2}+1)\rho_2\rho_3\right](\varphi_{2,+}\beta_1+\varphi_{2,-}\beta_2)=f_3.
\end{aligned} \tag{3.37}$$

By the orthogonality property of the system $\{\varphi_{1,\pm},\varphi_{2,\pm}\}$ and Lemma 3.2, we can explicitly determine,

$$\alpha_1=-\frac{\lambda_2(\kappa+1)}{2\pi\lambda_1}\int_{\mathbb{R}^2}f_1\varphi_{1,+},\quad \alpha_2=-\frac{\lambda_2(\kappa+1)}{2\pi\lambda_1}\int_{\mathbb{R}^2}f_1\varphi_{1,-}$$

in (3.35) in order to verify

$$(L_1 v_1, \varphi_{1,\pm})_{L^2} = 0. \tag{3.38}$$

Similarly by (3.26) we can choose β_1, β_2 in (3.36) so that

$$(L_2 v_2, \varphi_{2,\pm})_{L^2} = 0. \tag{3.39}$$

With such choice of α_1, α_2 and β_1, β_2 we are in position to use (3.12), to obtain $v_1, v_2 \in Y_\alpha$, solution respectively to (3.35) and (3.36). At this point, set

$$\begin{aligned} g = & -\lambda_3 \rho_2 v_2 + \frac{\lambda_1}{2}\rho_1 v_1 - \left[2\lambda_1 \rho_1 w_1 - \frac{2\lambda_3}{\varphi_0^2}\rho_2\rho_3\right](\varphi_{1,+}\alpha_1 + \varphi_{1,-}\alpha_2) \\ & + \left[4\lambda_3\rho_2 w_1 - \frac{4\lambda_3}{\varphi_0^2}(\frac{\lambda_3}{\lambda_2}+1)\rho_2\rho_3\right](\varphi_{2,+}\beta_1 + \varphi_{2,-}\beta_2) + f_3 \in X_\alpha, \end{aligned}$$

and observe that (3.37) is solvable in Y_α with corresponding solution given by

$$\begin{aligned} v_3(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} & \log(|x-y|)g(y)dy \\ & + C \end{aligned} \tag{3.40}$$

for any constant $C \in \mathbb{R}$. So the operator $\mathcal{A}$ is onto. Furthermore, $Ker\mathcal{A}$ can be determined by letting $f_1 = f_2 = f_3 = 0$ in the above argument, which leads to $\alpha_1 = 0 = \alpha_2$ and $\beta_1 = 0 = \beta_2$ and $v_3 = -\frac{1}{2}v_1 + \frac{\lambda_3}{\lambda_2}v_2 + C$ with $v_j \in KerL_j$, $j = 1, 2$ and any constant $C \in \mathbb{R}$ (see Proposition 3.1 part (i)). Therefore the desired conclusion (3.24) follows by taking into account Proposition 3.2. $\square$

Proof of Theorem 2.1: We decompose $(Y_\alpha)^3 \times \mathbb{C}^2 = U_\alpha \oplus Ker\mathcal{A}$ with $U_\alpha = (Ker\mathcal{A})^\perp$, so that

$$\mathcal{A} = P'_{(u_1,u_2,u_3,a,b)}(0,0,0,0,0,0) : U_\alpha \to (X_\alpha)^3$$

defines an isomorphism. The standard implicit function theorem (see e.g. [9], [16]), applies to the operator $P : U_\alpha \times (-\varepsilon_0, \varepsilon_0) \to (X_\alpha)^3$, for sufficiently small ε_0, and implies that there exists $\varepsilon_1 \in (0, \varepsilon_0)$ and a continuous function:

$$\varepsilon \mapsto \psi_\varepsilon = (u^*_{1,\varepsilon}, u^*_{2,\varepsilon}, u^*_{3,\varepsilon}, a^*_\varepsilon, b^*_\varepsilon)$$

from $(-\varepsilon_1, \varepsilon_1)$ into a neighborhood of the origin in U_α such that,

$$P(u^*_{1,\varepsilon}, u^*_{2,\varepsilon}, u^*_{3,\varepsilon}, a^*_\varepsilon, b^*_\varepsilon, \varepsilon) = 0, \quad \text{for all } \varepsilon \in (-\varepsilon_1, \varepsilon_1),$$

and $u^*_{j,\varepsilon=0} = 0$ for every $j = 1, 2, 3$, and $a^*_{\varepsilon=0} = 0 = b^*_{\varepsilon=0}$. Consequently,

$$\begin{aligned}
u(z) &= \log \rho^I_{\varepsilon,a^*_\varepsilon}(z) + \varepsilon^2 w_1(\varepsilon z) + \varepsilon^2 u^*_{1,\varepsilon}(\varepsilon z) \\
\eta(z) &= \log \rho^{II}_{\varepsilon,b^*_\varepsilon}(z) + \varepsilon^2 w_2(\varepsilon z) + \varepsilon^2 u^*_{2,\varepsilon}(\varepsilon z) \\
v(z) &= \log \rho^{III}_{\varepsilon,a^*_\varepsilon,b^*_\varepsilon}(z) + \varepsilon^2 w_3(\varepsilon z) + \varepsilon^2 u^*_{3,\varepsilon}(\varepsilon z)
\end{aligned} \tag{3.41}$$

defines a solution for the system (2.3)-(2.5), $\forall \varepsilon \in (-\varepsilon_1, \varepsilon_1)$, $\varepsilon \neq 0$.

Furthermore, from Proposition 3.1 we have that,

$$|u^*_{j,\varepsilon}(x)| \leq C\|u^*_{j,\varepsilon}\|_{Y_\alpha}(\log^+|x| + 1) \leq C\|\psi_\varepsilon\|_{U_\alpha}(\log^+|x| + 1), \quad j = 1, 2, 3,$$

with

$$\|\psi_\varepsilon\|_{U_\alpha} \to 0, \qquad \text{as } \varepsilon \to 0.$$

Therefore,

$$\sup_{\mathbb{R}^2} \frac{|u^*_{j,\varepsilon}(\varepsilon x)|}{1 + \log^+|x|} = o(1) \tag{3.42}$$

as $\varepsilon \to 0$. Since (2.17) holds, then the explicit form of $\rho^I_{\varepsilon,a^*_\varepsilon}(z)$, $\rho^{II}_{\varepsilon,b^*_\varepsilon}(z)$, $\rho^{III}_{\varepsilon,a^*_\varepsilon,b^*_\varepsilon}(z)$, together with the asymptotic behaviors of w_1, w_2, w_2 described in Lemma 3.1 and (3.42) imply that the solution $(u^\varepsilon, \eta^\varepsilon, v^\varepsilon)$ in (3.41) satisfies also the boundary condition (1.14). This completes the proof of Theorem 2.1.□

Final Remarks:

(i) By a complete application of the Implicit Function Theorem(e.g. [9]), we can actually claim the existence of a family of solutions depending on a number of parameters that equals the dimension of $ker\mathcal{A}$.

(ii) By a minor modification of the proof presented above, we can actually include equality in (2.17). In this case the image of the operator P is mapped into the space $(X_{\alpha-\delta_0})^3$ for suitable δ_0 sufficiently small. Notice that, according to Lemma 3.1 the function $w_j, j = 1, 2, 3$ are bounded in this case, while w_3 diverges at infinity with logarithmic growth. As a consequence the resulting string solution no longer admits finite energy in this case. It is an interesting question to know whether or not problem (2.3),(2.4) and (2.5) admits a solution when (2.17) is violated, or more precisely,

$$\frac{2\lambda_3}{\lambda_2} < N + 1. \tag{3.43}$$

By our discussion, it seems reasonable to expect an existence result to hold under the assumption: $\frac{2\lambda_3}{\lambda_2} > N$. However, under (3.43) we see that the function w_3 admits a power growth at infinity, and so it fails to belong to Y_α. Therefore a modified functional framework is required in order to handle this situation. On the other hand, by the above discussion also follows that, as far as selfgravitating Electroweak solutions are concerned, (1.13) seem to occur also as a necessary condition in order to guarantee the finite energy property (1.15).

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