CONCENTRATION FACTORS

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Abstract. There are many application that call for the determination of the points at which a function changes values in a discontinuous fashion and that require knowledge of the change in the function's value at such points. In this paper we present some simple examples of *concentration factors*. Concentration factors take the Fourier coefficients of a function and return a function that tends to zero at points of continuity of the original function and that tends to the height of the jumps at the location of the jumps.

In the analysis of concentration factors, we make use of many elementary results from analysis and many properties of Fourier series. The material here—except for the last section—can be presented to anyone with a reasonable knowledge of Fourier series and a decent understanding of the properties of infinite series. In the final section, we consider the effect that noisy data has on the results of the schemes we present.

Key words. concentration factors, Fourier series, harmonic series, noise

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1. Introduction.

Edge detection and the detection of discontinuities is very important in many fields. In image processing, for example, one often need to determine the boundaries of the items of which a picture is composed. (For more information about edge detection in image processing, see [7].) We consider the problem of detecting the edges present in a function when given the Fourier coefficients of the function.

There are numerical methods that estimate the Fourier coefficients of a function of interest rather than directly estimating the solution. The spectral viscosity method, a numerical method used to solve nonlinear partial differential equations (PDEs), is an example of such a method. The method approximates the Fourier coefficients of the solution of a PDE. The Fourier coefficients are then used to calculate an approximation to the solution. The accurate reconstruction of the solution requires that the position of the discontinuities of the solution be known[5]. In this paper we discuss techniques for using a function's Fourier coefficients to determine the location and size of the jump discontinuities of the function.

At first glance the spectral representation of the signal—the Fourier series or transform associated with the signal—does not seem to be the ideal place to look for information about discontinuities in the signal. When a signal is discontinuous the convergence of the Fourier series or transform associated with the signal is not uniform; in such cases the Gibbs phenomenon[8] appears and truncating the series after any finite number of terms always leads to $O(1)$ oscillations in the reconstructed signal.

Considering the question again, however, one realizes that if a discontinuity is characterized by a "phenomenon," then the existence of the discontinuity is indeed encoded in the coefficients. The question becomes how to effectively "decode" the discontinuity. One does not do this by directly summing the series—one uses the spectral representation in a somewhat different way to "concentrate" the function about the discontinuity. In what follows, we explain how this is done. We restrict ourselves to periodic (or compactly supported) functions and only consider Fourier

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series. (Those interested in seeing a more general theory of concentration factors are referred to [3, 4].)

Much of the information in this article is well known[3, 4]. The use of the Euler-Mascheroni constant to improve the performance of the concentration factor in §4 and the noise analysis in §6 are new to the best of our knowledge.

In the next section we give some of the background necessary for our study. In the following sections we present the classical method of finding the discontinuities, we explain its shortcomings, we present a better method and analyze its properties, and we explore the behavior of the methods in the presence of noise.

2. Some Background.

2.1. The Convergence of the Fourier Series. Let $f(t)$ be a periodic function with period T. The Fourier coefficients of $f(t)$ are:

$$c_n = \frac{1}{T}\int_x^{x+T} e^{-in\omega t} f(t)\,dt,$$

where $\omega = 2\pi/T$, where x is an arbitrary point, and where T is the period of the function $f(t)$.

In many cases, it is possible to reconstruct a function from its Fourier coefficients. We consider three different senses in which a function is represented by its Fourier series. First, consider a piecewise continuous periodic function, $f(t)$. At all points at which $f(t)$ is continuous we have:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$

At points of discontinuity, the convergence is to the mean of the values to which the function tends from the left and the right of the discontinuity[2]. Thus, we find that for piecewise continuous functions, the convergence of the Fourier series to the function is pointwise wherever the function is continuous and is to the average value of the function at the jumps in the function's value. This shows that if $f(t)$ is continuous, then the Fourier series converges to a continuous function—$f(t)$. If $f(t)$ is discontinuous, then so is the function to which the Fourier series converges.

Now suppose that $f(t)$ is periodic with period T and square summable—that $f(t) \in L^2[0,T]$. Square summability is less restrictive than piecewise continuity. Then the Fourier series converges to the function in L^2—which is a weaker form of convergence than pointwise convergence. Additionally, for square summable functions Parseval's equation states that:

$$\sum_{-\infty}^{\infty} |c_n|^2 = \frac{1}{T}\int_x^{x+T} |f(t)|^2\,dt.$$

Parseval's equation says that if a function is square summable, so are the function's Fourier coefficients.

Finally, suppose that the Fourier coefficients of $f(t)$ are absolutely summable. That is, suppose that:

$$\sum_{-\infty}^{\infty} |c_n| < \infty.$$

As $|e^{in\omega t}| = 1$, the absolute summability of the Fourier coefficients establishes the uniform convergence of the Fourier series:

$$\sum_{n=-\infty}^{\infty} c_n e^{in\omega t}.$$

As the functions $e^{in\omega t}$ are continuous and we know that the uniform limit of continuous functions is a continuous function, we find that if the Fourier coefficients are absolutely summable, then the Fourier series converges to a continuous function. As we have already seen that the Fourier series of a piecewise continuous function tends to a continuous function if and only if the function is actually continuous, we find that if the Fourier coefficients of a piecewise continuous function are absolutely summable, then the function is continuous.

2.2. Smoothness and Convergence. Two properties of the Fourier coefficients are important to us in what follows. One property concerns the relation between the smoothness of $f(t)$ and convergence of the Fourier series associated with $f(t)$. We treat this question here. The other property concerns the effect that shifting a function has on the function's Fourier coefficients and is treated in §2.3.

Let us consider the connection between the smoothness of $f(t)$ and the summability of the Fourier series. If a function is continuous and piecewise differentiable, then the Fourier coefficients of the derivative of the function must be square summable (as the derivative is itself piecewise continuous). Let $t_0, t_1, \ldots, t_{M-1}$ be the points in the interval $[0, T)$ at which the derivative of $f(t)$ changes in a discontinuous fashion, and let a_n be the Fourier coefficients that correspond to $f'(t)$. Then we find that:

$$\begin{aligned}
Ta_n &= \int_0^T e^{-in\omega t} f'(t)\,dt \\
&= \int_0^{t_0} e^{-in\omega t} f'(t)\,dt + \ldots + \int_{t_{M-1}}^{t_T} e^{-in\omega t} f'(t)\,dt \\
&\overset{\text{by parts}}{=} e^{-in\omega t} f(t)\big|_0^{t_0} + \ldots + e^{-in\omega t} f(t)\big|_{t_{M-1}}^{T} \\
&\qquad + in\omega \int_0^{t_0} e^{-in\omega t} f(t)\,dt + \ldots + in\omega \int_{t_{M-1}}^{t_T} e^{-in\omega t} f(t)\,dt \\
&\overset{\text{continuity}}{=} in\omega \int_0^{t_0} e^{-in\omega t} f(t)\,dt + \ldots + in\omega \int_{t_{M-1}}^{t_T} e^{-in\omega t} f(t)\,dt \\
&= in\omega T c_n.
\end{aligned}$$

Making use of the Cauchy-Schwarz inequality, we find that:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |c_n| &= \frac{1}{\omega}\left(\sum_{n=-\infty}^{-1} \frac{1}{n}|-in\omega c_n| + |c_0| + \sum_{n=1}^{\infty} \frac{1}{n}|-in\omega c_n|\right) \\
&\leq \sqrt{\sum_{n=-\infty}^{-1} \frac{1}{n^2}}\sqrt{\sum_{n=-\infty}^{-1} |a_n|^2} + |c_0| + \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}}\sqrt{\sum_{n=1}^{\infty} |a_n|^2} \\
&< \infty.
\end{aligned}$$

That is, the Fourier coefficients of a continuous and piecewise differentiable function are absolutely summable. Combining this result with the final result of §2.1, we find that a piecewise differentiable function is continuous if and only if its Fourier coefficients are absolutely summable. *Thus, if a piecewise continuous function, $f(t)$, can be written:*

$$f(t) = f_d(t) + f_c(t)$$

where $f_d(t)$ and $f_c(t)$ are piecewise continuous, $f_d(t)$ is discontinuous, and $f_c(t)$ is continuous and piecewise differentiable, then the Fourier coefficients of $f(t)$ can be split into two parts. The part that corresponds to $f_d(t)$ cannot have absolutely summable Fourier coefficients. The part that corresponds to $f_c(t)$ must have absolutely summable Fourier coefficients.

2.3. Shifts of a Function. The second property we are interested in concerns the effect that shifting a function has on the function's Fourier coefficients. Let the Fourier coefficients of $f(t)$ be denoted by c_n. What are the Fourier coefficients, a_n, of $f(t-\tau)$? We find that:

$$\begin{aligned} a_n &= \frac{1}{T}\int_x^{x+T} e^{-in\omega t} f(t-\tau)\,dt \\ &\overset{u=t-\tau}{=} \frac{1}{T}\int_{x-\tau}^{x-\tau+T} e^{-in\omega(u+\tau)} f(u)\,du \\ &= e^{-in\omega\tau} c_n. \end{aligned}$$

2.4. An Important Example. Without loss of generality, in the rest of this exposition we only consider functions that are periodic with period 1. Consider $k(t)$ defined by:

$$k(t) \equiv t - \frac{1}{2}, \quad 0 \le t < 1$$

in the interval $t \in [0, 1)$ and defined elsewhere by periodically extending the function. The function as defined has a jump of height 1 at every integer.

Let us calculate c_n. We find that:

$$c_n = \int_0^1 \left(t - \frac{1}{2}\right) e^{-in2\pi t}\,dt.$$

For $n = 0$, it is clear that this is 0. For $n \neq 0$, we find that:

$$\begin{aligned} c_n &= \int_0^1 \left(t - \frac{1}{2}\right) e^{-in2\pi t}\,dt \\ &\overset{\text{by parts}}{=} \left(t - \frac{1}{2}\right)\frac{e^{-in2\pi t}}{-in2\pi}\bigg|_{t=0}^1 + \frac{1}{in2\pi}\int_0^1 e^{-in2\pi t}\,dt \\ &= \frac{i}{n2\pi}. \end{aligned}$$

We find that the coefficients are square summable—as they must be—but they are not summable. We find that the Fourier series that corresponds to $k(t)$ is:

$$\sum_{n=-\infty}^{-1} \frac{i}{n2\pi} e^{in2\pi t} + \sum_{n=1}^{\infty} \frac{i}{n2\pi} e^{in2\pi t}. \tag{2.1}$$

2.5. An Interesting Sum. Using Parseval's equation on the Fourier coefficients that correspond to $k(t) = t - 1/2$, we find that:

$$\begin{aligned}\int_0^1 k^2(t)\,dt &= \int_0^1 \left(t - \frac{1}{2}\right)^2 dt \\ &= \frac{1}{12} \\ &= \sum_{n=-\infty}^{\infty} |c_n|^2 \\ &= 2\sum_{n=1}^{\infty} \frac{1}{4\pi^2 n^2} \\ &= \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}.\end{aligned}$$

Rearranging terms, we find that:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{2.2}$$

We make use of this sum in §3.

2.6. Decomposing a Function. In the sections to come, we will need to split a piecewise differentiable function into its continuous and discontinuous parts. We now consider one way to perform this decomposition. Suppose that one has a function, $f(t)$, that is piecewise differentiable but has a jump discontinuity of height h at location τ. Then the function $f(t) - hk(t - \tau)$ is piecewise differentiable and continuous. Thus, its Fourier coefficients, which we denote by b_n, are absolutely summable. Of course the Fourier coefficients of $hk(t - \tau)$ are not absolutely summable. In fact the Fourier coefficients of $hk(t - \tau)$, which we denote by a_n, are:

$$a_n = \begin{cases} h\frac{ie^{-in2\pi\tau}}{n2\pi} & n \neq 0 \\ 0 & n = 0 \end{cases}.$$

By the linearity of the calculation of the Fourier coefficients, we find that:

$$c_n = a_n + b_n.$$

That is, the Fourier coefficients of $f(t)$ can be written as the sum of an absolutely summable set of coefficients and set of coefficients that is not absolutely summable. In §3 this procedure is extended to functions with multiple jump discontinuities.

3. The Classical Approach—the Hilbert Transform. In order to determine where the edges of the data are in a "minimally invasive way," we want to find a transformation of the Fourier coefficients that changes the Fourier coefficients as little as possible, but that causes the partial sums of the Fourier series of a discontinuous function to grow at the discontinuities but not elsewhere. Note that the reason that the Fourier series (2.1) does not diverge at $t = 0$ is that at that point the exponentials corresponding to $\pm n$ cancel one another.

Consider the Fourier coefficients of a function that is piecewise continuous but not continuous. Let us denote its Fourier coefficients by r_n. Furthermore, following the example of §2.6, let us decompose the function into its discontinuous and continuous parts, and let us denote their Fourier coefficients by a_n and b_n respectively.

We now consider a transformation of the sequence r_n. We define s_n, the transformed sequence, by the equation:

$$s_n = \begin{cases} -ir_n & n \geq 1 \\ 0 & n = 0 \\ ir_n & n \leq -1 \end{cases} .$$

This transformation is known as the *Hilbert transform*[6]. Note that the transformed version of the b_n is still absolutely summable, while the transformed version of the a_n is not absolutely summable. Thus, the continuous part of the function is transformed into a continuous function by the Hilbert transform while the Hilbert transform of the discontinuous part is still—at the very least—discontinuous. (If the coefficient of the constant term of the original function is zero, then the Hilbert transform is an l^1 isometry. If the coefficient of the constant term is non-zero, the Hilbert transform is an l^1 contraction.)

Let us consider the function that one recovers from the sum using the transformed coefficients of the discontinuous function $k(t-\tau)$. We find that the function one recovers is:

$$g(t) = \sum_{n=-\infty}^{-1} \frac{-1}{n2\pi} e^{in2\pi(t-\tau)} + \sum_{n=1}^{\infty} \frac{1}{n2\pi} e^{in2\pi(t-\tau)} = \sum_{n=1}^{\infty} \frac{\cos(2\pi n(t-\tau))}{n\pi}.$$

Note that at $t = \tau + k$ this gives us the harmonic series and $g(t+k)$ diverges (and at $t = \tau + k + 1/2$ the function $g(t)$ evaluates to the alternating harmonic series and converges conditionally $\ln(2)/\pi$).

To proceed with our analysis we must analyze the partial sums:

$$g_N(t) = \sum_{n=1}^{N} \frac{\cos(2\pi n(t-\tau))}{n\pi}$$

more carefully. To this end, we consider the properties of the *Dirichlet kernel*[10], $D_N(\xi)$—of the sum:

$$D_N(\xi) \equiv \sum_{n=-N}^{N} e^{in2\pi\xi} = 1 + 2\sum_{n=1}^{N} \cos(n2\pi\xi), \quad \xi \equiv t - \tau.$$

This is a finite geometric series whose sum is:

$$\begin{aligned} D_N(\xi) &= \sum_{n=-N}^{N} e^{in2\pi\xi} \\ &= e^{-in2\pi\xi} \sum_{n=0}^{2N} e^{in2\pi\xi} \\ &= e^{-in2\pi\xi} \frac{1 - e^{i(2N+1)2\pi\xi}}{1 - e^{i2\pi\xi}} \\ &= \frac{\sin((2N+1)\pi\xi)}{\sin(\pi\xi)} \end{aligned}$$

Clearly

$$|D_N(\xi)| \leq \frac{1}{|\sin(\pi(t-\tau))|}.$$

Consider the partial sum:

$$g_N(t) = \sum_{n=1}^{N} \frac{\cos(2\pi n\xi)}{n\pi}.$$

again. This sum can be written as:

$$g_N(t) = \frac{\cos(2\pi\xi)}{\pi} + \sum_{n=2}^{N} \frac{D_n(\xi) - D_{n-1}(\xi)}{2n\pi}.$$

Rewriting this, we find that:

$$g_N(t) = \frac{\cos(2\pi\xi)}{\pi} + \frac{D_N(\xi)}{2N\pi} - \frac{1}{2\pi}\sum_{n=2}^{N} D_{n-1}(\xi)\left(\frac{1}{n} - \frac{1}{n-1}\right) - \frac{D_1(\xi)}{2\pi}$$

which, using the definition of $D_1(\xi)$, can be further simplified to:

$$g_N(t) = \frac{D_N(\xi)}{N\pi} + \frac{1}{2\pi}\sum_{n=2}^{N} \frac{1}{n(n-1)} D_{n-1}(\xi) - \frac{1}{2\pi}.$$

Considering our previous bound on $|D_n(\xi)|$, we find that:

$$\left|\sum_{n=1}^{N} \frac{\cos(2\pi n\xi)}{n\pi}\right| \leq \frac{1}{|\sin(\pi\xi)|}\left(\frac{1}{N\pi} + \frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{1}{n(n-1)}\right) + \frac{1}{2\pi}.$$

Note that:

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Making use of (2.2) we find that:

$$|g_N(t)| \leq \left|\sum_{n=1}^{N} \frac{\cos(2\pi n\xi)}{n\pi}\right| \leq \frac{1}{|\sin(\pi\xi)|}\left(\frac{1}{2N\pi} + \frac{1}{2\pi}\frac{\pi^2}{6}\right) + \frac{1}{2\pi}.$$

This shows that as long as ξ is not a whole number—as long as $t \neq \tau + k$, the sum is bounded, and the dependence of the bound on ξ is known.

This leaves us in the position of knowing that the partial sums diverge like the harmonic series at $t = \tau + k$ and converge elsewhere. We take advantage of this fact by dividing the partial sum by the (approximate) value of the partial sum of the (divergent) harmonic series. This causes the partial sum to tend to 1 at the point at which the discontinuity occurred and to tend to zero elsewhere.

When $t = \tau + k$ the partial sum is:

$$g_N(t) = \sum_{n=1}^{N} \frac{1}{n\pi} = \frac{1}{\pi}\sum_{n=1}^{N}\frac{1}{n}.$$

We would like to develop a closed form estimate of this sum. We note that:

$$\int_n^{n+1} \frac{1}{x}\,dx \le \frac{1}{n} \le \int_{n-1}^{n} \frac{1}{x}\,dx.$$

Thus:

$$1 + \int_2^{N+1} \frac{1}{x}\,dx \le \sum_{n=1}^{N} \frac{1}{n} \le 1 + \int_1^N \frac{1}{x}\,dx.$$

We find that:

$$1 + \ln(N+1) - \ln(2) \le \sum_{n=1}^{N} \frac{1}{n} \le 1 + \ln(N)$$

Dividing both sides by $\ln(N)$ and taking the limit as $N \to \infty$, we find that:

$$\frac{\sum_{n=1}^{N} \frac{1}{n}}{\ln(N)} \to 1.$$

If the Fourier coefficients of the function whose edges we would like to find are c_n, then the sum that we consider is:

$$\text{edge}_1(t;N) \equiv \frac{\pi}{\ln(N)} \left(\sum_{n=-N}^{-1} ic_n e^{i\omega nt} + \sum_{n=1}^{N} -ic_n e^{i\omega nt} \right).$$

This sum is our first edge detector, and it has two important properties. As $N \to \infty$ the value of the sum tends to the height of the jumps in the original function at the points at which the jumps occur. At all other points, the sum tends to zero.

To prove this, consider a piecewise differentiable function $v(t)$ with m jumps at the locations $t_1, \ldots, t_m$ with the heights $h_1, \ldots, h_m$. Clearly the function:

$$w(t) = v(t) - \sum_{n=1}^{m} h_n k(t - t_n)$$

is continuous and piecewise differentiable, and:

$$v(t) = w(t) + \sum_{n=1}^{m} h_n k(t - t_n).$$

Let b_n be the Fourier coefficients of $w(t)$. Let $a_{i,j}$ be the Fourier coefficients of $h_i k(t - t_i), i = 1, \ldots, m$. Note that in what follows we make use of the fact that such a decomposition is possible. We do not need to actually decompose the function ourselves; the linearity of the edge detector takes care of that for us.

Because of the linearity of all the operations we perform, we can consider the effect of the operation on each set of Fourier coefficients. As the b_n are absolutely convergent, it is clear that:

$$\frac{\pi}{\ln(N)} \left(\sum_{n=-N}^{-1} ib_n e^{i\omega nt} + \sum_{n=1}^{N} -ib_n e^{i\omega nt} \right) = O\left(\frac{1}{\ln(N)}\right).$$

For any fixed i the set of coefficients $a_{i,j}$ are just the coefficients that correspond to $h_i k(t - t_i)$. We have seen that for this function the normalized partial sums converge to h_i at t_i and to zero (like $O(1/\ln(N))$) elsewhere.

We find that after properly normalizing the Hilbert transform of a discontinuous function, the heights and locations of the jumps become clear. We note, however, that the convergence is $O(1/\ln(N))$.

4. Shortcoming of the Technique. Consider the function $k(t) = t - 1/2$ and let us see how well our edge detector, $\text{edge}_1(t; N)$, works. In Figure 4.1, $k(t)$ is approximated using the Fourier series with $N = 1000$, and in Figure 4.2 we see the output of $\text{edge}_1(t; 1000)$.

Upon looking at Figure 4.2, two points are immediately obvious . First of all, the measured value of the jumps—which should be exactly 1—is about 1.08. Second of all, even though N is rather large, the points away from the jump are not particularly close to zero.

The second point is *the* fundamental problem with this method. Because we divide a finite number by $\ln(N)$, and because $\ln(N)$ does not increase quickly, we need a very large value of N in order to force the points away from the jumps to zero.

The first problem, however, is curable. Let us consider the partial sums that correspond to the harmonic series again. We substituted $\ln(N)$ for the partial sum. It is well known that:

$$\lim_{N\to\infty}\left(\left(\sum_{k=1}^{N}\frac{1}{k}\right)-\ln(N)\right)\equiv\gamma=0.577215\ldots.$$

Furthermore, it has been shown[9] that:

$$\frac{1}{2(N+1)}<\left(\sum_{k=1}^{N}\frac{1}{k}\right)-\ln(N)-\gamma<\frac{1}{2N}.$$

The constant γ is known as the Euler-Mascheroni constant. Rather than dividing the sum by $\ln(N)$, divide it by $\ln(N)+\gamma$. This defines a second, improved, edge detector, $\text{edge}_2(t; N)$:

$$\text{edge}_2(t;N)\equiv\frac{1}{\ln(N)+\gamma}\sum_{n=-N}^{N}s_n e^{i\omega nt}.$$

The improved edge detector returns Figure 4.3. Here the jump is indeed measured as one unit, but the convergence away from the jumps is still very slow.

5. A Better and Simpler Technique. If one's goal is to determine the location of the discontinuities of a function, there is no reason to require that the processing of the Fourier coefficients only minimally affect the coefficients. The problem with the previous method was that we were dividing a bounded function that we wanted to force to zero by $\ln(N)$. This caused the decrease towards zero away from the jumps to be very slow. It would be better to divide the bounded part by something larger, if possible.

Let us consider the following method of transforming the Fourier coefficients of our data. If r_n are the coefficients of the function, let the transformed coefficients, s_n, be:

$$s_n = -inr_n.$$

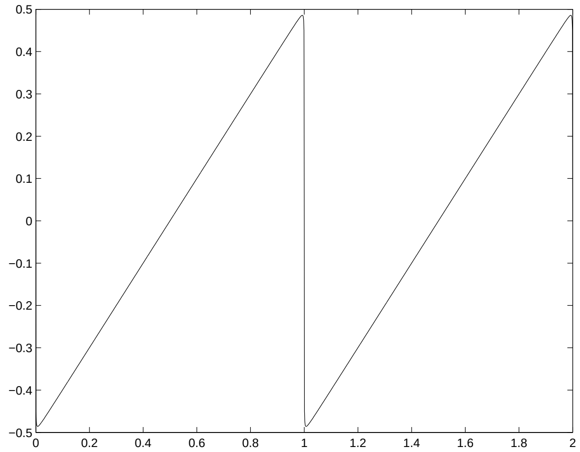

FIG. 4.1. *The function $k(t)$ as reproduced from its Fourier series with $N = 1000$.*

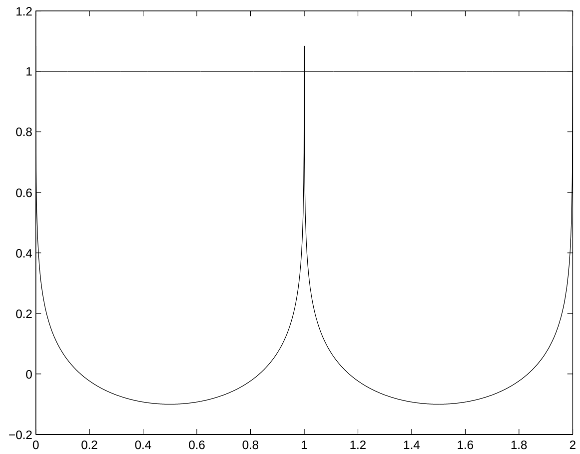

FIG. 4.2. *The edges of the function as detected by* $\text{edge}_1(t; 1000)$.

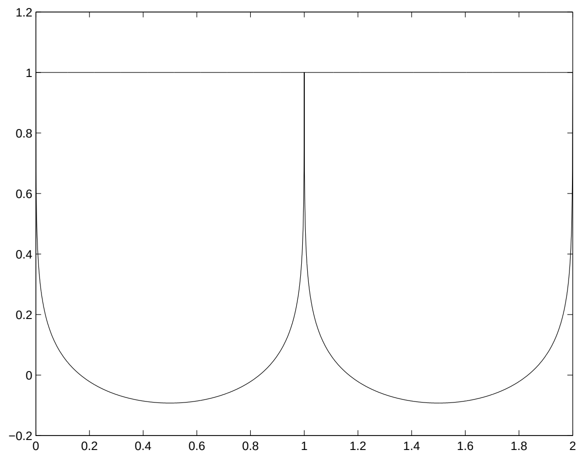

FIG. 4.3. *The edges of the function as detected by* $\text{edge}_2(t; 1000)$.

Let us consider the function that one recovers when one starts with the coefficients that correspond to the shifted sawtooth wave:

$$c_n = \begin{cases} \frac{ie^{-in2\pi\tau}}{n2\pi} & n \neq 0 \\ 0 & n = 0 \end{cases}.$$

We find that the partial sums of the transformed coefficients are:

$$g_N(t) = \sum_{n=-\infty}^{\infty} s_n e^{in2\pi t} = \frac{1}{\pi}\sum_{n=1}^{N} \cos(2\pi n(t-\tau)).$$

Considering the partial sums, $g_N(t)$, one finds that:

$$g_N(t) = \frac{1}{\pi}\frac{D_N(\xi)-1}{2} = \frac{\sin((2N+1)\pi\xi)}{2\pi\sin(\pi\xi)} - \frac{1}{2\pi}.$$

We see that the partial sum are bounded as long as ξ is not an integer. When ξ is an integer, the sums equals N/π.

Note that the transformation performed on the coefficients causes the series associated with the discontinuous part to diverge like N/π. The coefficients of the continuous part, on the other hand, will not diverge as quickly. In fact, using arguments similar to those of §2 it is easy to show that for a sufficiently smooth continuous part the sum will be absolutely and uniformly convergent.

Therefore, we can produce an effective edge detector by considering:

$$\text{edge}_3(t;N) \equiv \frac{\pi}{N}\sum_{n=-N}^{N} s_n e^{in2\pi t}.$$

The discontinuous piece contributes a component that converges to the height of the jump at the location of the jump and tends to zero like $1/N$ away from the jump. The, continuous piece, if it is smooth enough, will decay as $1/N$ as well. This technique is superior to the preceding one (except insofar as it requires that the l^1 norm of the coefficients be greatly altered). In Figure 5.1 we see the output of $\text{edge}_3(t;100)$ for $k(t) = t - 1/2$, and in Figure 5.2 we see the output of $\text{edge}_3(t;1000)$ for the same input. The latter detector performs just as one would hope, and even the former gives reasonable results.

6. Performance in Noise. The edge detectors that we have constructed can be thought of as filters that affect each input frequency in a particular way. If r_k is the Fourier coefficient of $e^{-2\pi ikt}$ at the input to the edge detector—at the input to the filter—and h_k is the value by which the filter multiplies r_k, then the Fourier coefficient of $e^{-2\pi ikt}$ at the output of the filter is $h_k r_k$. Parseval's equation says that the power in the input at the frequency k is $|r_k|^2$ and the power at the output of the filter is $|h_k|^2|r_k|^2$. The total power is the sum of the power at each frequency.

All this is easy to prove when the input is a periodic deterministic signal. In the case of a periodic stationary random signal, the proof is a bit more involved and the usual way the result is presented is a bit different. Let v_k be the power at frequency k at the input to filter and w_k be the power at frequency k at the output of the filter. Then:

$$w_k = |h_k|^2 v_k.$$

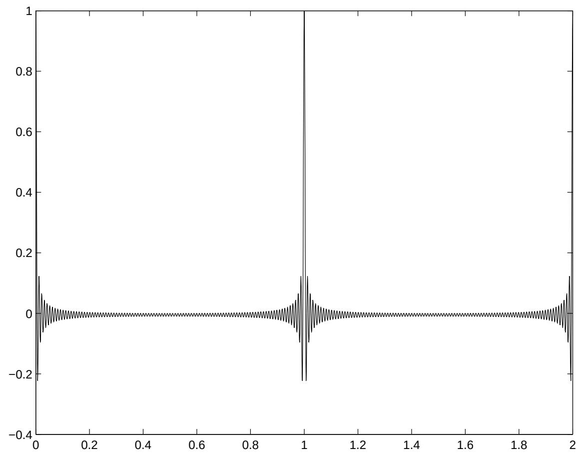

FIG. 5.1. *The edges of the function as detected by* $\text{edge}_3(t; 100)$.

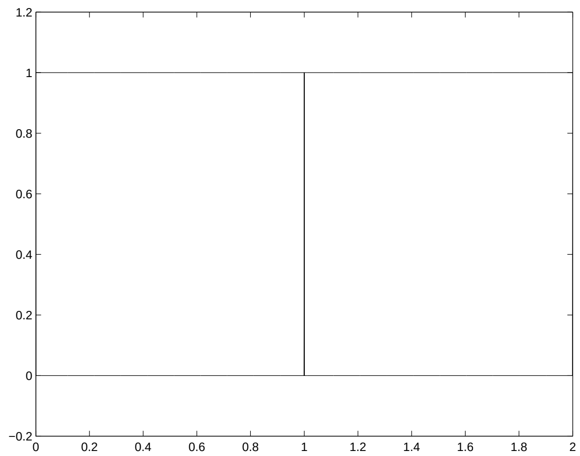

FIG. 5.2. *The edges of the function as detected by* $\text{edge}_3(t; 1000)$.

Because we are (by assumption) dealing with a stationary random input, we consider the expected value of the the power in the signal. A simple calculation shows that:

$$E\left(\frac{1}{T}\int_0^T f^2(t)\,dt\right) = \frac{1}{T}\int_0^T E(f^2(t))\,dt \overset{\text{stationarity}}{=} \frac{1}{T}\int_0^T E(f^2(0))\,dt = E(f^2(0)).$$

The expected value of the power is just the expected value of the square of the signal.

It is interesting to consider the effects of white noise on the filters we have designed. White noise, by definition, has the same power at each frequency (just as white light has the same power at each frequency). Let us take $v_k = \eta$.

Let us consider our original filter first. There, the coefficients of the filter are:

$$h_k = \frac{-i\pi \text{sgn}(k)}{\ln(N) + \gamma}, \qquad |k| \leq N$$

and zero otherwise. Thus,

$$w_k = \frac{\eta\pi^2}{(\ln(N)+\gamma)^2}, |k| \leq N, \qquad k \neq 0,$$

and $w_k = 0$ otherwise. As the power in the noise is just the sum of the power at each frequency, we find that the power in the noise is:

$$\text{power} = E((\text{noise contribution})^2) = \frac{2N\eta\pi^2}{(\ln(N)+\gamma)^2}.$$

The RMS amplitude of the noise is the square root of the power (or the standard deviation of the noise contribution):

$$\text{RMS amplitude} = \frac{\sqrt{2N\eta}\pi}{\ln(N)+\gamma}.$$

This means that the amplitude of the contribution due to noise grows as $\sqrt{N}/\ln(N)$.

Now let us consider our the improved filter. There, the coefficients of the filter are:

$$h_k = -\frac{\pi i k}{N}, |k| \leq N$$

and zero otherwise. As the input to the system is white noise, $v_k = \eta$. Thus, $r_k = \pi^2 k^2 \eta / N^2, |k| \leq N, k \neq 0$, and $r_k = 0$ otherwise. As the power in the noise is the sum of the coefficients, we find that the power in the noise is just:

$$\text{power} \approx \frac{2\pi^2 N^3 \eta}{3N^2}.$$

The RMS amplitude of the noise is:

$$\text{RMS amplitude} \approx \frac{\sqrt{2N\eta}\pi}{3}.$$

Note that as N grows the contribution of the noise—in either scheme—grows without bound. Suppose that our input has several jumps. Then as N grows the output of the filter due to the signal tends to the jump heights at the locations at which the jumps take place and tends to zero elsewhere. The contribution due to the noise grows without bound for either filter. Thus, if the signal is corrupted by white noise, it is important *not to choose too large a value of* N. Large values of N allow the noise contribution to grow without materially improving the output due to the signal.

Considering the effect noise has on the output of the two filters, we find that for large values of N the $\text{edge}_2(t;N)$ is marginally better than $\text{edge}_3(t;N)$. In the case of $\text{edge}_2(t;N)$, the contribution due to the noise grows as $\sqrt{N}/\ln(N)$, while in the improved technique, $\text{edge}_3(t;N)$, the noise contribution grows as $\sqrt{N}$. This is a second advantage (albeit a minor one) of the original technique over the "improved" technique. For a different technique for handling noise when using concentration factors, see [1].

In Figure 6.1, we consider the output of $\text{edge}_3(t;N)$ for $N = 10, 100$ and ,1000 when the input to the edge detector is $k(t)$ corrupted by white noise. When $N = 10$,

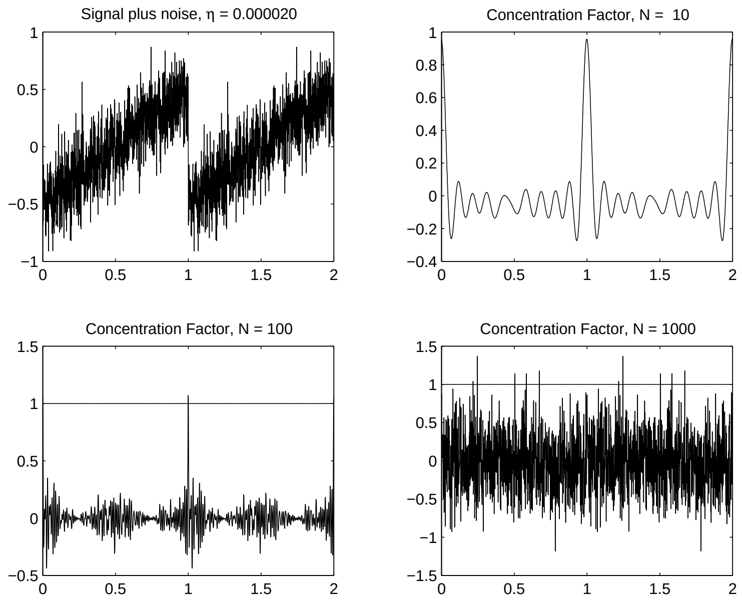

FIG. 6.1. *The effect of noise on* $\mathrm{edge}_3(t; N)$ *for* $N = 10, 100,$ *and* 1000.

the output of the edge detector is almost unaffected by noise. However, because N is small, the output of the edge detector is not close to the ideal output of an edge detector. The output is large in a large region near the location of the jump, and it does not stay near zero far from the jump either. When $N = 100$ the output of the edge detector due to the jump is, as seen in Figure 5.1, nearly ideal. In Figure 6.1 we see that when in addition to the jump there is noise, the output is not as nearly ideal, but the location and size of the jump are still clearly visible in the edge detector's output. When $N = 1000$, the output of $\mathrm{edge}_3(t; N)$ when the input is "pure signal" is, as seen in Figure 5.2, very nearly perfect. In the presence of white noise, however, the output of the edge detector is useless. We find that it is counterproductive to take N to be too large.

The value of N one should use depends on the size of the jumps that one would like to detect and the amount of noise present in the signal. In Figure 6.1 the noise level is high and N must be kept small. In cases where there is less noise, N can be larger, and the accuracy with which the height and location of jumps in the signal can be known is greater.

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