A Fast Empirical Mode Decomposition Technique for Nonstationary Nonlinear Time Series

Christopher D. Blakely

Center for Scientific Computation and Mathematical Modeling, University of Maryland, College Park MD 20740 USA

Abstract

The purpose of this paper is to introduce a fast analysis technique for decomposing time-series into a set of intrinsic mode functions (IMFs) and a residual trend. This decomposing technique, known as the empirical mode decomposition, traditionally uses cubic-splines in the decomposing process thus creating the need to solve a system of equations, albeit well conditioned, at each step. This new method being proposed takes advantage of the theory of matrix-free moving least-squares approximation to construct discrete reproducing kernels capable of interpolation to a near natural cubic-spline fit without the need for solving a system of equations. A class of compactly supported radial functions used in constructing the reproducing kernels is also given along with numerical examples validating the robustness of the fast algorithm.

Key words: Empirical Mode Decomposition, Time Series Analysis, Moving Least-Squares

1 Introduction

As an adaptive nonlinear decomposition technique referred to as *Empirical Mode Decomposition* (EMD), the wide ranging applications this method has been applied to the past few years have varied from analyzing climatology data for climate variability to the study of white noise characteristics in biological data. Being derived from the simple assumption that any data consists of simple unique oscillating modes instrinsic to the data, the EMD is completely

Email address: cblakely@cscamm.umd.edu (Christopher D. Blakely).

a posteriori in regards to the decomposition of the data into intrinsic mode functions (IMFs) and does not assume anything about the data, contrary to Fourier methods where data is assumed linear and stationary. Because of the adaptive nature of EMD, this method has been shown numerically to better describe temporal patterns in nonstationary nonlinear time series than traditional methods such as Wavelet and Fourier methods [5]. Furthermore, coupled with the Hilbert transform applied to the resulting IMFs (Hilbert-Huang transform), this decomposition method is well localized in the time-frequency domain and reveals important characteristics of the signal. Despite the success over the past few years of this analysis tool, it still lacks the speed of traditional Wavelet and Fourier methods which have become standards in the mathematical, statistical, and engineering industry partly due to their associated 'fast' transform algorithms and 'black-box' style implementations.

In this paper, we introduce a new computational algorithm for EMD which relies on the theory of reproducing kernels constructed in a moving-least squares sense [3]. As will be shown in the analysis of the method, the main motivation for such a new computational algorithm is a large gain in computational speed. This new technique for EMD is purely matrix-free and thus no linear systems need to be solved, in contrast to the original formulation of EMD by Huang et al.

In order to present this fast algorithm for empirical mode decomposition in an efficient and clear manner, two recently developed approximation tools need presentation. We thus organize the paper as follows. Before briefly reviewing the EMD algorithm, the next two sections present a matrix-free moving least squares reproducing kernel builder which will be used in the computation of the EMD algorithm. The next section deals with a review of the EMD algorithm as developed by Huang in [5]. We then propose a fast algorithm for extracting the IMFs in the EMD algorithm by applying the the approximation methods presented in sections 2 and 3. Finally, we will then conclude with numerical examples of the fast algorithm applied to a sample time series.

2 Matrix-free Moving Least Squares

Matrix-free moving least-squares (MLS) reproducing kernels were developed recently with the intention of eliminating the computational complexity of standard moving least-squares. They have mainly been used in recent years in scattered data approximation and in meshless approximation methods for partial differential equations. In this paper, we apply matrix-free MLS to build reproducing kernels for the empirical mode decomposition algorithm. Here, we review a method developed by Fasshauer [2] for constructing discrete reproducing kernels in the moving least-squares sense without the need of solving

any systems of equations. As discussed in [2], this matrix-free formulation offers a computationally efficient MLS approximation without the computational burden of having to solve many linear systems as in standard MLS, which is the main drawback of and reason why the latter has not become a popular approximation tool. We begin with the so-called Backus-Gilbert approach for MLS approximation first described by Bos and Salkauskas which considers a reproducing kernel of the form

$$Pf(\mathbf{x}) = \sum_{i=1}^{N} f(\mathbf{x_i})\Psi_i(\mathbf{x}). \tag{1}$$

where $\mathbf{f} = [f(\mathbf{x}_1), ..., f(\mathbf{x}_N)]^T$ represents the given data. This reproducing kernel $\Psi_i(\mathbf{x})$ is constructed to be minimized subject to some approximation space reproduction constraints, namely

$$\frac{1}{2}\sum_{i=0}^{N} \Psi_i^2(\mathbf{x})\frac{1}{W(\mathbf{x}, \mathbf{x}_i; \beta)} \tag{2}$$

subject to

$$\sum_{i=0}^{N} \phi(\mathbf{x}_i)\Psi_i(\mathbf{x}) = \phi(\mathbf{x}), \quad \text{for all } \phi \in \mathcal{U}. \tag{3}$$

$\mathcal{U}$ is an approximation space of dimension m and the $W(\mathbf{x}, \mathbf{x}_i)$ are positive weight functions with respect to an important shape parameter β that controls the interpolation qualities of this method. The closeness of the discrete reproducing kernel $\Psi_i(\mathbf{x})$ to the pointwise cardinal functions, (i.e., $\Psi_i(\mathbf{x}_j) = \delta_{i,j}, i, j = 1, ...N$) determines how well the summation in (1) approximates the function $f(\mathbf{x})$. Since (1) is itself a function, we can evaluate it at any point on the given domain, this leads to a quadratic minimization problem for an evaluation point $\mathbf{x}$. To this end, defining $\Psi(\mathbf{x}) = [\Psi(\mathbf{x}, \mathbf{x}_1), ..., \Psi(\mathbf{x}, \mathbf{x}_N)]^T$ and

$$\frac{1}{2}\Psi(\mathbf{x})^T Q(\mathbf{x})\Psi(\mathbf{x})$$

with

$$Q(\mathbf{x}) = \text{diag}(\frac{1}{W(\mathbf{x}, \mathbf{x}_1; \beta)}, ..., \frac{1}{W(\mathbf{x}, \mathbf{x}_N; \beta)})$$

The approximation space reproduction can be written as

$$A\Psi(\mathbf{x}) = \mathbf{p}(\mathbf{x}),$$

with $A_{ji} = \phi_j(\mathbf{x}_i), i = 1, ...N, j = 1, ...m$ and $\phi = [\phi_1, ..., \phi_m]^T$, the basis of $\mathcal{U}$. Using Lagrangian multipliers to solve this MLS problem leads to a system

where we solve for the Langrangian multipliers and the reproducing kernel functions and has the form

$$\begin{bmatrix} \mathbf{Q}(\mathbf{x}) & -\mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Psi(\mathbf{x}) \\ \lambda(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \phi(\mathbf{x}) \end{bmatrix} \tag{4}$$

Since $\mathbf{A}$ has full rank and $\mathbf{Q}$ is invertible, we can apply Gaussian elimination to the block matrix and arrive at

$$\lambda(\mathbf{x}) = (AQ^{-1}(\mathbf{x})A^T)^{-1}\phi(\mathbf{x}) \tag{5}$$

$$\mathbf{\Psi}(\mathbf{x}) = (Q^{-1}(\mathbf{x})A^T\lambda(\mathbf{x})). \tag{6}$$

But this just means that the Lagrangian multipliers $\lambda(\mathbf{x})$ at the evaluation point $\mathbf{x}$ are given as the solution to the Gram system

$$G(\mathbf{x})\lambda(\mathbf{x}) = \phi(\mathbf{x}). \tag{7}$$

where the entries of G are the weighted l_2-inner products, namely

$$G(\mathbf{x})_{jk} = \langle \phi_j, \phi_k \rangle_{W(\mathbf{x},\mathbf{x}_i;\beta)} = \sum_{i=1}^{N} \phi_j(\mathbf{x}_i)\phi_k(\mathbf{x}_i)W(\mathbf{x}, \mathbf{x}_i; \beta), \quad j,k = 1, \ldots, m. \tag{8}$$

We note the the Gram matrix is symmetric and positive definite because of the positive weights and the basis being linearly independent.

This gives an explicit form of the generating functions which are

$$\mathbf{\Psi}_i(\mathbf{x}) = W(\mathbf{x}, \mathbf{x}_i; \beta) \sum_{j=1}^{m} \lambda_j(\mathbf{x}) p_j(\mathbf{x}_i), \quad i = 1, \ldots, N. \tag{9}$$

Therefore, after determining the Lagrangian multipliers, we can determine the explicit form of the reproducing kernel. With this formulation, however, an $m \times m$ Gram system must be solved at each evaluation point $\mathbf{x}$.

In order to arrive at a more efficient computation of MLS where no linear systems need to be solved, we must restrict the approximation space $\mathcal{U}$ to shifted monomials $p_i(\mathbf{x}) = (\mathbf{x}_i - \mathbf{x})^\alpha$ for $\mathbf{x} \in \mathbb{R}$, $0 \leq \alpha \leq q$ and $\mathbf{x}_i$ an interpolation point. Although our approximation space is now restricted, we still have the freedom in choosing the weight function $W(\mathbf{x}_i, \mathbf{x}; \beta)$ which can already have certain approximation properties built in it.

The main advantage in choosing such an approximation space is that we now have the ability to solve for λ_j in the Backus-Gilbert approach analytically. This is done by choosing the dimension of the shifted-monomial space $\mathcal{U}$ to be relatively small (e.g. $m \leq 4$). This way, the resulting reproducing kernels can be constructed analytically and can be directly programmed into computer code. To illustrate, we give a couple of examples from [2] to construct a one-dimensional reproducing kernel. We first define the moments

$$\mu_\alpha = \sum_i^N (\mathbf{x}_i - \mathbf{x})^\alpha W(\mathbf{x}_i, \mathbf{x}; \beta) \;\; \mathbf{x} \in \mathbb{R} \;\; \mathbf{x}_i \in \mathcal{I}_x$$

where $\mathcal{I}_x$ is a set of N distinct points in the domain of interest $[0, T]$.

Example 1
If we let $m = 2$, the polynomial space becomes $\mathcal{U} = \text{span}\{p_1(s) = 1, p_2(s) = s - x\}$. Choosing $W(\mathbf{x}_i, \mathbf{x}; \beta)$ to be a positive weight function, then

$$\lambda_1(\mathbf{x}) = \frac{\mu_2}{\mu_0\mu_2 - \mu_2^2}, \;\; \lambda_2(\mathbf{x}) = \frac{\mu_1}{\mu_1^2 - \mu_0\mu_2}. \tag{10}$$

We can then use the λ_js and weight function $W(\mathbf{x}_i, \mathbf{x}; \beta)$ to create the discrete reproducing kernel as

$$\Psi_i(\mathbf{x}) = [\lambda_1(\mathbf{x}) + \lambda_2(\mathbf{x})(\mathbf{x}_i - \mathbf{x})]W(\mathbf{x}_i, \mathbf{x}; \beta), \;\; i = 1, \ldots, N. \tag{11}$$

Example 2
If we let $m = 3$, the polynomial space becomes $\mathcal{U} = \text{span}\{p_1(s) = 1, p_2(s) = s - \mathbf{x}, p_3(s) = (s - \mathbf{x})^2\}$. Choosing $W(\mathbf{x}_i, \mathbf{x}; \beta)$ to be a positive weight function, then

$$\lambda_1(\mathbf{x}) = \frac{\mu_2\mu_4 - \mu_3^2}{D}, \;\; \lambda_2(\mathbf{x}) = \frac{\mu_2\mu_3 - \mu_1\mu_4}{D}, \;\; \lambda_2(\mathbf{x}) = \frac{\mu_1\mu_2 - \mu_2^2}{D} \tag{12}$$

where $D = 2\mu_1\mu_2\mu_3 - \mu_0\mu_3^2 - \mu_2^3 - \mu_1^2\mu_4 + \mu_0\mu_2\mu_4$ Now we can then use the λ_js and weight function $W(\mathbf{x}_i, \mathbf{x}; \beta)$ to create the discrete reproducing kernel as

$$\Psi_i(\mathbf{x}) = [\lambda_1(\mathbf{x}) + \lambda_2(\mathbf{x})(\mathbf{x}_i - \mathbf{x}) + \lambda_3(\mathbf{x})(\mathbf{x}_i - \mathbf{x})^2]W(\mathbf{x}_i, \mathbf{x}; \beta), \;\; i = 1, \ldots, N. \tag{13}$$

3 Empirical Mode Decomposition

As an innovative time series analysis tool, EMD developed by Huang et al. in [5] has proven to be an important alternative to traditional methods for

analyzing time series such as wavelet methods, Fourier methods, and empirical orthogonal functions. For an in-depth description of EMD along with its coupled Hilbert transform for studying the notion of instantaneous frequency, the reader is referred to the original paper by Huang et al. In this section we briefly describe the computations of the traditional EMD algorithm and then show how to apply matrix-free MLS reproducing kernels and approximate approximation for an overall faster computational approach to this novel decompostion tool.

The goal of EMD is to decompose a time series into a finite number of intrinsic mode functions plus a residual which is conventionally termed as the time series trend. In [6], a good analysis of properties of IMFs is given along with a description of why they are important in *a posteriori* data decomposition.

To briefly summarize, an IMF is a function that satisfies two conditions: (1) The number of extrema and zero-crossings of the function along the domain of interest must be equal or differ by no more than one; (2) The mean at any point of the envelope defined by the local maxima and local minima is zero. It is also interesting to note that a larger class of functions to IMFs, namely functions that only satisfy property (1), are solutions to the self-adjoint ordinary differential equation

$$\frac{d}{dx}\Big(P\frac{du}{dx}\Big) + Qu = 0, \quad x \in (a,b)$$

where $P > 0$ and P, Q are both continuous functions. For further details on solutions to self-adjoint ODEs, the reader is referred to ([6]) and references therein.

Let $f(t)$ be a time series defined on an interval $[0, T]$. We wish to decompose $f(t)$ into a set of IMFs $\psi_n(t)$ and a residual trend $R(t)$ such that $f(t) = \sum_n \psi_n(t) + R(t)$. Locating the IMFs of $f(t)$ is accomplished by the following steps:

Initiate Set $n := 0$, $f_0 = f(t)$

- **Step 1** Set $h_0 := f_n$ and $k := 0$
- **Step 2** *Construct the upper and lower envelopes for* h_k
  This is done by locating all local maxima and minima and then interpolating the extrema via cubic splines to use as the upper envelope $U(t)$ and the lower envelope $L(t)$.
- **Step 3** *Sifting process*
  Since $L(t) \leq f_k(t) \leq U(t)$ we can define the mean of the envelopes as $m_k(t)$. The $k+1$th component is then defined to be $h_{k+1} := h_k - m_k$. h_{k+1} is then tested for IMF critera. If h_{k+1} is not an IMF, increment k and

repeat from **Step 2**. Otherwise set $\psi_n := h_{k+1}$ and set $f_{n+1} := f_n - \psi_n$. If convergence criteria are met for f_{n+1}, stop. Otherwise, increment n and return to **Step 1**.

A few further details of the algorithm and its terminating criteria are now in order. First, when testing h_{k+1} for IMF criteria in the sifting process, two tests must be passed. The number of extreme and zero-crossings must not differ by more than one. The second criterion is that the mean between the upper and lower envelopes must be close to zero according to some criterion. In [4], an approach to choosing this criteria was proposed to guarantee globally small fluctuations in the mean while taking into account locally large excursions. This is accomplished by introducing at each sifting iteration an amplitude $a(t) = (U(t) - L(t))/2$ and an evalutation function $\sigma(t) = \left|\frac{m(t)}{a(t)}\right|$. According to two thresholds, θ_1 and θ_2, we then iterate until $\sigma(t) < \theta_1$ for a certain fraction γ of the iterations and then until $\sigma(t) < \theta_2$ for the other fraction of iterations.

The second set of criteria for algorithm termination that has to be met is with regards to f_{n+1}. For each iteration n of the algorithm, the time series $f_n(t)$ loses extrema after subtracting the IMF. Therefore, after a finite number of iterations, f_{n+1} will tend to a monotonic or constant function and the EMD algorithm terminates.

As described in [5], the sifting process is meant to eliminate riding waves and extract a more symetric wave profile in h_k, eventually smoothing uneven waves. However, although a powerful method, EMD must be used cautiously when extracting the IMFs. As mentioned in [4] when locating the extrema of the time series at each sifting process, the end points (boundary conditions) of the time series are to be treated differently in order to minimize error propagations due to finite observations in length. There are a variety of techniques that have been used in past studies on EMD, and [4] offers one of the simplest yet very robust by mirrorizing the extrema at the time series boundary conditions.

3.1 A Fast Empirical Mode Decomposition Technique

We now propose a technique that can be used to yield faster computational results in decomposing a nonstationary nonlinear time series into a set of IMFs. The main motivation in seeking a faster computational technique stems from the fact that for large time series data sets (e.g. $\mathcal{O}(10^3)$), the cubic spline interpolation procedure in constructing the envelopes can be a computational burden, since a linear system must be solved at every sifting iteration. In this version of the algorithm, at each sifting iteration, we apply a matrix-free MLS reproducing kernel to the extrema of the time series, which is a linear $\mathcal{O}(K)$ operation where K is the number of evaluation points of the envelopes. We

also note that much of the methods speed comes from the ability to reuse the discrete reproducing kernels if the amount of extrema has not changed. This greatly speeds up the computational time of the sifting process in **Step 3**. Despite the method's speed and accuracy in interpolation, special care must be considered when choosing the weight function $W(\mathbf{x}_i, \mathbf{x}; \beta)$ of the kernel and its shape parameter β. When chosen correctly, the resulting interpolation with the reproducing kernel will be relatively close to the cubic spline interpolation.

In addition to proposing the fast technique, we also propose a class of compactly supported radial weight functions known for their superior approximation abilities and fast summation properties that can be used in constructing the reproducing kernel. For the shape parameter, we devise a remedy which will automatically select a near optimal shape parameter for the reproducing kernel that depends strictly on the amount of extrema needing interpolation.

Initiate Set $n := 0$, $f_0 = f(t)$

- **Step 1** Set $h_0 := f_n$ and $k := 0$
- **Step 2** *Construct the upper and lower envelopes for* h_k
  - Locate all N_{max} local maxima $h_{k,i}^{max}$ and N_{min} local minima $h_{k,j}^{min}$, $t_i, t_j \in [0, T]$ of h_k.
  - Construct a sub-grid on $[0, 1]$ of N_{max} uniformly spaced points and another sub-grid on $[0, 1]$ of N_{min} uniformly spaced points while placing normalized values of $h_{max}(t_i)$ and $h_{min}(t_j)$ on their respective sub-grids.
  - If number if extrema for iteration k is different from $k-1$, construct new matrix-free MLS reproducing kernels $\Psi_i^{max}(t), \Psi_j^{min}(t)$ with respect to the two sub-grids of N_{max} and N_{min} points. Calculate the upper and lower envelopes $U(t)$ and $L(t)$ by applying the reproducing kernels as

$$U(t) = \sum_{i=0}^{N_{max}} h_{k,i}^{max} \Psi_i^{max}(t) \;\; t \in [0, 1] \tag{14}$$

    and

$$L(t) = \sum_{j=0}^{N_{min}} h_{k,j}^{min} \Psi_j^{min}(t) \;\; t \in [0, 1] \tag{15}$$

  - Rescale uniform grid points back onto $[0, T]$ along with the upper and lower envelopes $U(t)$ and $L(t)$.
- **Step 3** *Sifting process*

  Since $L(t) \leq f_k(t) \leq U(t)$ we can define the mean of the envelopes as $m_k(t)$. The $k+1$th component is then defined to be $h_{k+1} := h_k - m_k$. Next, h_{k+1} is tested for IMF critera. If h_{k+1} is not an IMF, increment k and repeat from **Step 2**. Otherwise set $\psi_n := h_{k+1}$ and set $f_{n+1} := f_n - \psi_n$ and rebuild reproducing kernels $\Psi_i^{max}(t)$, Ψ_j^{min}. If convergence criteria is met for f_{n+1}, stop. Otherwise, increment n and return to **Step 1**.

The normalization of the extrema data and placement on the uniform subgrid are necessary in order to ensure proper handling of the shape parameter for the weight function $W(\mathbf{x}_i, \mathbf{x}; \beta)$. To this end, we propose a class of weight functions in the following section on numerical examples called Wendland functions which are radial with compact support and have been shown in many papers such as [7] to yield powerful approximation qualities and fast summations due to their compact support. With these compactly supported functions, the shape parameter β allows us to adapt the support radius which can ensure that there are enough points $\mathbf{x}_i$ covered in the summation for $U(t)$ and $L(t)$ while at the same time leaving out the points outside the radius. This property contributes even further to the fast computation time of the algorithm since one needs only to sum up on the points inside the radius of influence. For more information on compactly supported radial functions of minimal degree, the reader is referred to Wendland's work in [8] and references therein.

4 Numerical Examples

In this section, we aim at showing the robustness of this fast computational method by investigating the L_1 error of the envelopes at the extrema of several time series examples. In order to minimize this error and produce an interpolant for the envelopes which resembles a cubic-spline fit, optimal shape parameters for the weight functions must be chosen. To this end, we give formulas for near optimal parameters for several different weight functions. Furthermore, in order to validate this approach to the EMD algorithm, closeness of the matrix-free moving least squares reproducing kernel envelope fit to the cubic-spline envelope fit must be shown.

As discussed in [4], the key importance in the EMD algorthm is obtaining estimates of the IMFs of the time series in a robust manner. Errors in interpolating the extrema can tend to increase the amount of sifting iterations which ultimately "over-decomposes" the signal by spreading out their components over adjacent nodes. Another issue in the interpolation of the extrema is minimizing error propagations due to finite observation lengths. To this end, [4] suggests mirrorizing the extrema at the ends of the time series which is the method used throughout these numerical examples.

We first begin by listing four examples of weight functions that can be used and their associated near optimal shape parameters that should be used for the fast EMD algorithm which were approximated by numerical tests. Let N_{ext} be the number of local maxima or minima in the time series, $r = |\mathbf{x}_i - \mathbf{x}|/\beta$, and $|\Omega|$ be the length of the time series. The table below shows the 4 commonly used radial functions for matrix-free MLS which were used throughout testing

the fast computational method presented in this paper. For theory on these radial functions, the reader is referred to [1] and references therein.

Name	$W(\mathbf{x}_i, \mathbf{x}; \beta)$	β
$\mathcal{O}(h^4)$ Gaussian	$\frac{1}{\sqrt{\pi}}(\frac{3}{2} - r^2)e^{-r^2}$	$\lvert\Omega\rvert/4 + 10^{-4} \times N_{ext}$
$\mathcal{O}(h^6)$ Gaussian	$\frac{1}{\sqrt{\pi}}(\frac{15}{8} - \frac{5}{2}r^2 + \frac{1}{2}r^4)e^{-r^2}$	$\lvert\Omega\rvert/4 + 10^{-4} \times N_{ext}$
$\mathcal{O}(h^2)$ Wendland	$(1-r)_+^3(3r+1)$	$2\lvert\Omega\rvert + 10^{-4} \times N_{ext}$
$\mathcal{O}(h^4)$ Wendland	$(1-r)_+^5(8r^2+5r+1)$	$2\lvert\Omega\rvert + 10^{-4} \times N_{ext}$

Table 1
Weight Functions and their approximated optimal shape parameters

In our first numerical example of the enveloping process using the matrix-free MLS reproducing kernel method, we show the L_1 errors of interpolating the extrema in a"toy" time series given by

$$f(t) = \sin(t)\cos(t)(1.354t) + 10\sin(2.3t + 10\cos(t)) \tag{16}$$

The weight function that we use throughout these examples is the 2nd order compactly supported Wendland function. The figures below show the envelopes generated by the reproducing kernel method for 170, 200, and 230 sampled points with the amount of extrema increasing for an increasing number of sample points. We used the formula for the optimal shape parameter given in the above table.

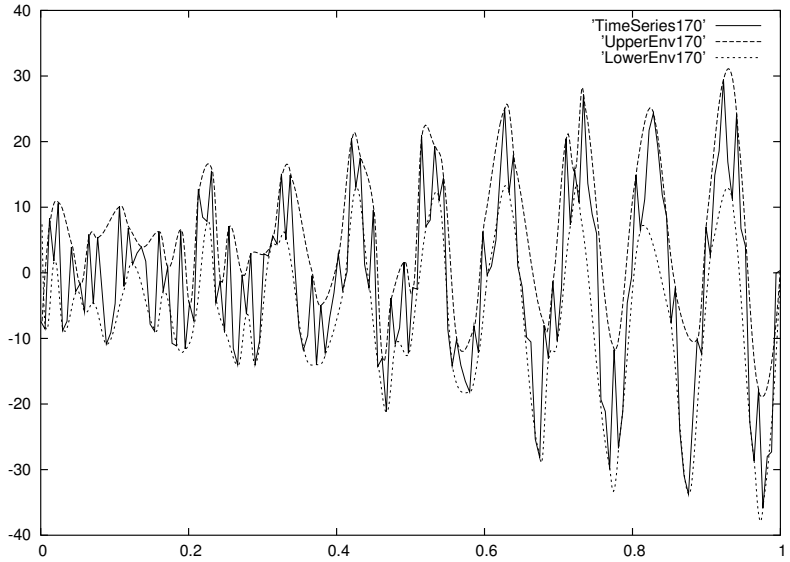

Fig. 1. Plot of envelopes for 170 points

We remark that when the number of sampling points in the time series changes, a new reproducing kernel does not necessarily need to be constructed. Only when the number of extrema or the placement of the extrema changes does

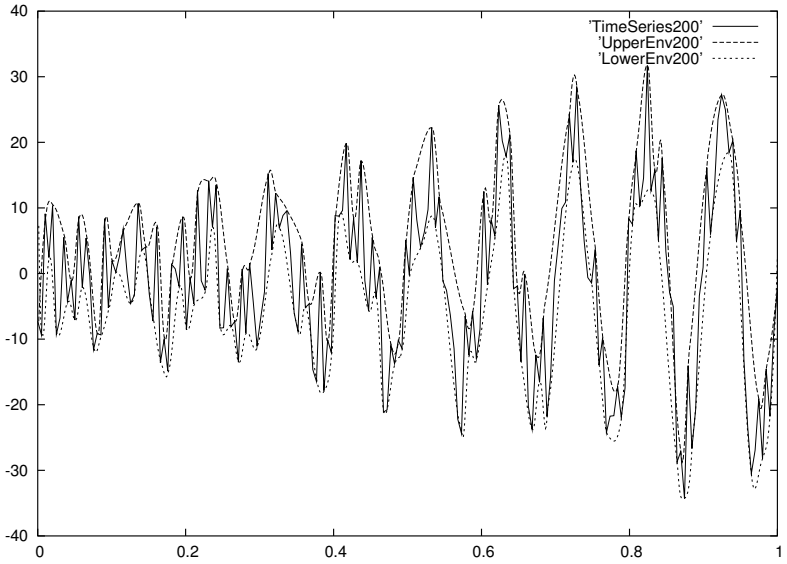

Fig. 2. Plot of envelopes for 200 points

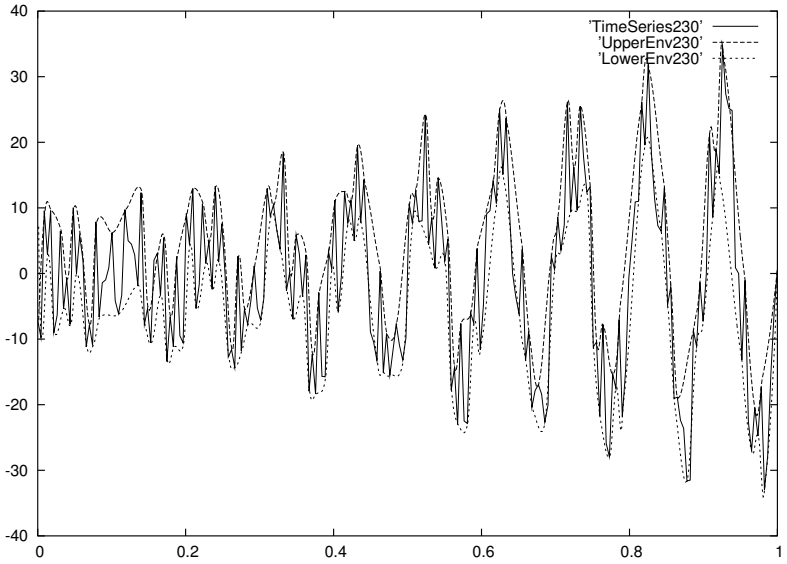

Fig. 3. Plot of envelopes for 230 points

the $\mathcal{O}(N_{ext})$ process of building a kernel for extrema interpolation need to take place.

Although the reproducing kernel method does not exactly interpolate the extrema as cubic splines do, the errors at the extrema introduced during interpolation are small enough not to introduce unnecessary oscillatory effects which could perturb the sifting process and ultimately lead to an "over-domposition" of the time series. Table (2) shows the L_1 errors of the interpolation at the extrema (local maxima) points N_{max} for an increasing amount of sampled points in the "toy" time series (16). Since the distribution of local maxima becomes more dense when the number of sampled points N increases, we expect

the interpolation error results to decrease since more data information for the matrix-free moving least-squares reproducing kernel construction is available.

L_1 error	N_{max}	L_1 error	N_{max}
5.37156e-02	10	8.27118e-05	60
3.57931e-03	22	4.53544e-05	71
1.58906e-03	28	9.26301e-06	80
9.12094e-04	41	5.11399e-06	88
4.22903e-04	48	3.1146e-06	92

Table 2
L_1 errors at envelope interpolation points for time series sampled at N points

In our final numerical example, we show the closeness of the matrix-free MLS constructed reproducing kernel interpolation of the extrema to the cubic-spline fit. To facilitate the comparison, we chose to sample the time series at 60 and 100 points and plot the upper part of the envelopes for the cubic-spline and kernel interpolation results. Figures (4) and (5) show the closeness of the two different upper envelopes for the sampled time series. It is easy to see that the closeness of the MLS reproducing kernel fit to the cubic-spline fit improves for an increasing amount of extrema.

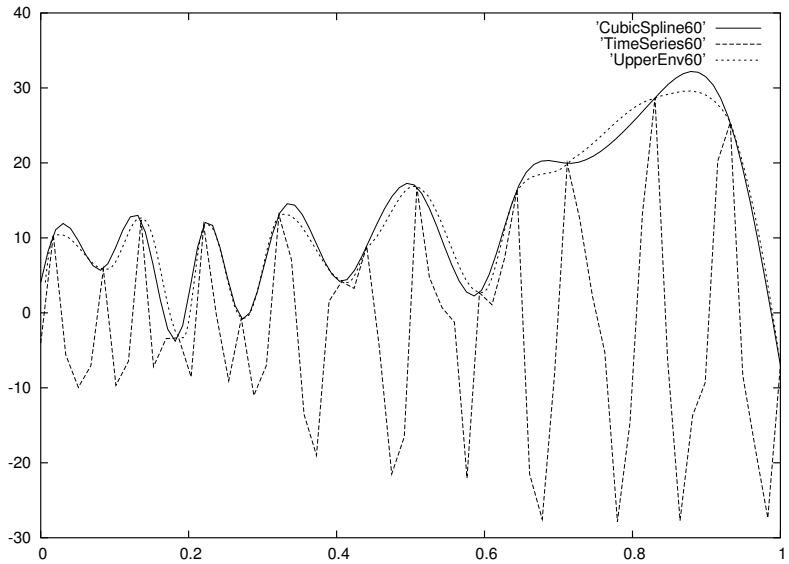

Fig. 4. Comparison of cubic-spline and reproducing kernel upper envelopes for 60 points

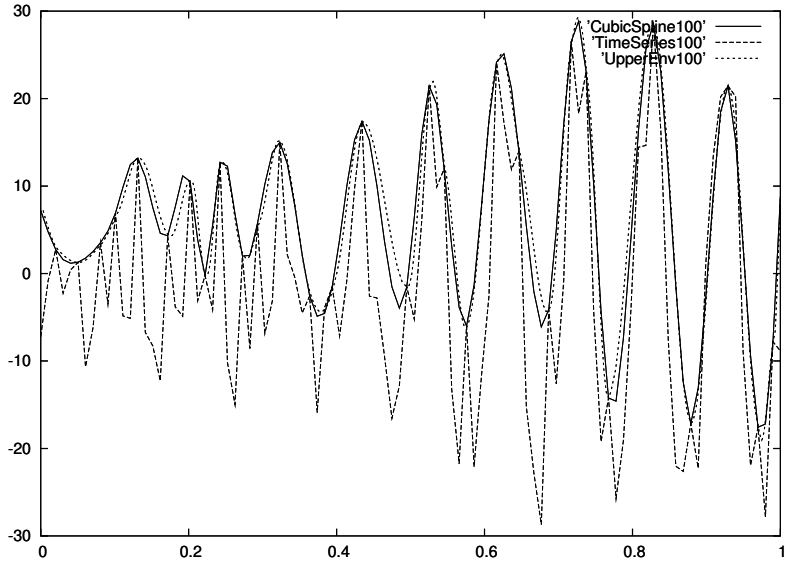

Fig. 5. Comparison of cubic-spline and reproducing kernel upper envelopes for 100 points

5 Conclusion

In this paper we attempted to introduce a fast computational algorithm for EMD which relies on the theory of MLS reproducing kernels. Although theoretical work for convergence rates of the matrix-free MLS reproducing kernel method still remains a difficult research domain for applications such as partial differential equations due to the dependence on the shape parameter for the weight functions and dimension of problem, the method has been shown in this paper to yield promising results for extrema interpolation in extracting IMFs in the empirical mode decomposition algorithm. Since the EMD method deals uniquely with 1-D time-series that can scale the time series extrema to a sub-grid, providing a near optimal shape parameter for the weight function can be accomplished with relative ease and should not pose a threat to the efficiency of the method. Furthermore, due to its robust fast computational time, the method provides an attractive alternative to the traditional cubic-spline approach which suffers from slow computation time due to the necessity of solving a linear system.

Future work of this computational method includes embedding it into the Hilbert-Huang transform to produce a fast method for obtaining well-localized time-frequency representations of time series. More information on the Hilbert-Huang transform can be found in [5]. Lastly, adapting this method for use in 2-D signals is also currently being investigated.

Acknowledgements. The author would like to thank Norden Huang of NASA Goddard Space Flight Center for introducing me to empirical mode decompo-

sition and the Hilbert-Huang transform. The author would also like to thank Ward Kingkade, Gregory Spencer, Kelly Holder, and Jason Devine of the the U.S. Census Bureau for their kind and encouraging support during the work of this paper.

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